

\[ \text{Introduction} \]

1. Introduction

In this section, we introduce the concept of entanglement and its role in quantum information processing. Entanglement is a fundamental resource in quantum information theory, enabling tasks that are impossible in classical systems.

- In quantum mechanics, two or more particles can be entangled in such a way that their states are correlated, even when separated by large distances.
- Entangled states can be used to perform quantum computations, teleportation, and quantum cryptography.

We investigate multipartite entanglement in relation to the quantized processes of quantum state exchange.

Multipartite entanglement and quantum state exchange
that quantum state exchange can be used to generate an
entangled pure state for two groups of \( N \) trapped ions in
two spatially separated ion traps. In Subsection II C
we present a detailed summary of the remainder of the
paper. In Sections III and IV we investigate the nature
of the state’s 2\( N \)-way entanglement. That is, the nature
of its entanglement that spans across all 2\( N \) ions. In
particular, in Subsections III A and IV we qualitatively
explore this entanglement by presenting a novel neces-
sary and sufficient condition for the presence of \( M \)-way
entanglement for \( M \)-partite pure states and then showing
that the state satisfies it. In Section IV we quantitatively
explore the state’s 2\( N \)-way entanglement by first preznt-
ing a novel multipartite entanglement measure for pure
states in Subsection IV A. This measure is based on the
von Neumann entropies [2, 30] for all the reduced density
certices obtainable from some pure state of interest by
tracing over some of the subsystems for the state. After
defining the measure, we then use it to calculate lower
bounds on the amount of 2\( N \)-way entanglement in the
state in Subsection IV B. Finally, we discuss our results
in Section V.

II. BACKGROUND THEORY

A. Quantum state exchange

Perhaps the simplest system in which quantum state
exchange can occur [13] involves a two-level atom con-
fined within a harmonic trap which, in turn, lies inside
a linearly damped optical cavity with one lossy mirror
and one ideal one. The atom’s vibrational motion is
described by the annihilation operators \( b_x, b_y \) and \( b_z \). The
two-level atom, which has a transition frequency of \( \omega_c \),
couples to both an intracavity electromagnetic field mode of
frequency \( \omega_c \) described by the annihilation operator \( a \) and
an external laser of frequency \( \omega_L \). The cavity and
external laser frequencies are chosen so as to drive Raman
transitions that couple adjacent atomic vibrational levels.
The cavity’s axis coincides with the \( x \)-axis whilst the
external laser’s beam is perpendicular to this axis. Lastly,
we assume that the harmonic trap is centered on a
cavity-field mode and thus a schematic diagram of this
system is as in Fig. 1.

The system’s Hamiltonian is, in a reference frame ro-
tating at frequency \( \omega_L \),

\[
H^{\text{single}}_{\text{total}} = H_{\text{res}} + \kappa (a R^\dagger + a^\dagger R) + H_{\text{sys}},
\]

where \( H_{\text{res}} \) is the free Hamiltonian of the reservoir
coupled to the cavity mode, \( \kappa \) is a damping rate, \( R \) is
a reservoir operator and \( H_{\text{sys}} \) is the Hamiltonian for the
cavity-atom system which is

\[
H_{\text{sys}} = \sum_{j=x,y,z} \hbar \omega_j (b_j^\dagger b_j + 1/2) + \hbar \alpha a^\dagger a + \hbar \Delta \sigma^+ \sigma^- \\
+ \hbar \left[ E_L(y, z, t) \sigma^+_k + E_L^*(y, z, t) \sigma^-_k \right] \\
+ \hbar g_L \sin(kx) \left( a^\dagger \sigma^- + a \sigma^+ \right),
\]

where \( \nu_x, \nu_y \) and \( \nu_z \) are the harmonic-oscillator frequen-
cies along the trap’s \( x, y \) and \( z \) axes, \( \sigma^+ \) and \( \sigma^- \) are
atomic raising and lowering operators for the two-level
atom, \( \Delta = \omega_c - \omega_L, \Delta = \omega_c - \omega_L \). \( E_L \) is the complex
amplitude of the external laser field, \( k = 2\pi / \lambda \), where
\( \lambda \approx \pi / \omega_c \), \( x = \sqrt{\hbar / 2 m_v \nu_x (b_x + b_x^\dagger)} \), where \( m \) is
the mass of the two-level atom, and \( y_L (y_L \in \mathbb{R}) \) is the
coupling constant for the atom-field interaction. Observe
that \( \omega_c - \omega_L = \nu_x \).

The following reasonable assumptions are made about
the system so as to make calculations involving it more
tractable:

1. The cavity field and external laser frequencies are
approximately detuned from \( \omega_c \) and the two-level
atom is initially in the ground state. Thus, the ex-
cited internal state is sparsely populated and sponta-
nous emission effects are negligible and can be
ignored.

2. Vibrational decoherence occurs over a timescale
much longer than that of the interaction producing
quantum state exchange, as is the case in an ion-
trap realization of the system [13]. Consequently, it
has a minimal effect over our timescale of interest
and is ignored.

3. The trap dimensions are small compared to the
cavity mode wavelength and thus \( \sin(kx) \ll 1 \). It
follows from this that \( \sin(kx) \approx \eta_x (b_x + b_x^\dagger) \), where
\( \eta_x = k \sqrt{\hbar / 2 m_v \nu_x} \). It also follows that we can
arrange things so that the \( y \) and \( z \) dependence of the
external laser field is negligible and thus, assuming
\( E_L \) is time independent, that \( E_L(y, z, t) \approx E e^{-i \omega L t} \),
where \( E \) is a real time-independent amplitude.

FIG. 1: Schematic diagram of a simple system in which quantum
state exchange can occur. A two-level atom lies within a
harmonic trap that is itself inside an optical cavity. The
cavity, which is aligned along the \( x \)-axis, supports a mode of
frequency \( \omega_c \) and has one lossy mirror (with damping con-
stant \( \kappa \)) and one ideal one. An external laser of frequency \( \omega_L \)
is incident from a direction perpendicular to the \( x \)-axis.
4. The damping parameter κ is such that $\nu_s \gg g_\theta \frac{\nu}{\Delta}$.

For the system just outlined, it has been shown [13] that, in the steady-state regime, the vibrational state of the atom in the $x$ direction is solely determined by the input field (i.e., the light field entering the cavity) such that

$$\tilde{b}_x(\omega) = \frac{\sqrt{2}T}{\kappa \nu - \Gamma} \tilde{a}_m(\omega),$$  \hspace{1cm} (3)

where $\tilde{b}_x(\omega) = \sqrt{\frac{2}{\kappa \nu}} \int_{\nu - \infty}^{\nu + \infty} dB_x(t) e^{i(\nu - \omega)t} \frac{\kappa \nu - \Gamma}{\sqrt{2\pi} \nu} \int_{\nu - \infty}^{\nu + \infty} dt e^{-\kappa \nu t} \tilde{a}_m(t)$, where $e(\omega')$ is the value of the reservoir annihilation operator for the frequency $\omega'$ at time $t = 0$. The proportionality between $\tilde{b}_x(\omega)$ and $\tilde{a}_m(\omega)$ present in Eqn (3) denotes that the “statistics of the input field have been written onto the state of the oscillator” [13] (p. 498) (i.e., onto the atom’s vibrational state in the $x$ direction). We thus say that quantum state exchange has taken place when this equation holds.

**B. State of interest**

In this paper we use quantum state exchange to generate a particular state involving two groups of $N$ trapped ions which we later show contains interesting multilevel entanglement. The system that we employ to produce this state comprises of firstly, a subthreshold nondegenerate optical parametric amplifier (NOPA) [18, 19, 20] for which the two external output fields first pass through Faraday isolators and then feed into a different linearly damped optical cavity for which one mirror is perfect and the other is lossy. The axes of both cavities coincide with the $x$-axis. Each cavity supports a cavity mode of frequency $\omega_c$ that is described by the annihilation operator $a_j$, where $j$ enumerates the cavities. Within the $j^{th}$ cavity $N$ identical two-level ions of mass $M$ charge $Z$ and internal transition frequency $\omega_i$. These ions are trapped in a linear configuration that is parallel to the $x$-axis by a harmonic potential (a linear ion trap[21]) and so are tightly confined in the $y$ and $z$ directions. Both traps are centred on nodes of the corresponding cavity fields. The annihilation operator $b^{(m)}_j$ describes the vibrational motion of the $m^{th}$ ion in the $j^{th}$ trap in the $x$ direction. Finally, external lasers of frequency $\omega_L$ whose beams are perpendicular to the $x$-axis are incident on the first ions of both traps and thus Fig. 2 illustrates the system under consideration.

The Hamiltonian for the $j^{th}$ optical cavity and the ions within it is

$$H_j^{\text{total}} = H_j^{\text{ion}} + H_j + H_j^{\text{ion-ion}} + H_j^{\text{int}},$$  \hspace{1cm} (4)

where $H_j^{\text{ion}}$ is the free Hamiltonian for the cavity field and the ions’ internal states and $H_j^{\text{ion-ion}}$ is the free Hamiltonian for the vibrational states of the $N$ ions. The term $H_j^{\text{ion-ion}}$ is the interaction Hamiltonian for the electromagnetic coupling between ions and $H_j^{\text{int}}$ is the interaction Hamiltonian describing the Raman process involving the cavity field, the external laser and the first ion. To be more specific, $H_j^{\text{ion-ion}} = h\nu_L \sum_{m=1}^{N} (b^{(m)}_j b^{(m)}_j + \frac{1}{2})$, where $\nu_L = (\nu_s - \omega_L)$ is the frequency of the harmonic trap along the $x$-axis which is the same for both traps. The Hamiltonian $H_j^{\text{int}}$ is, in a frame rotating at frequency $\omega_L$,

$$H_j^{\text{int}} = \hbar \delta a_j \dot{a}_j + \hbar \Delta \sum_{m=1}^{N} \sigma^{(m)}_j \sigma^{(m)}_j \dot{a}_j \dot{a}_j,$$  \hspace{1cm} (5)

where $\delta = \omega_L - \omega_c$, $\Delta = \omega_L - \omega_i$ and $\sigma^{(m)}_j$ and $\sigma^{(m)}_j$ raising and lowering operators for the internal states of the $m^{th}$ ion in the $j^{th}$ trap. The Hamiltonian $H_j^{\text{ion-ion-ion}}$ is [22]

$$H_j^{\text{ion-ion-ion}} = \sum_{m,n=1}^{N} \sigma^{(m)}_j \sigma^{(n)}_j \sigma^{(m)}_j \sigma^{(j)}_j \frac{2}{\pi \epsilon_0 |x_{jn}(t) - x_{jm}(t)|},$$  \hspace{1cm} (6)

where $\epsilon_0$ is the permittivity of free space and $x_{jn}$, for $l = 1 \ldots N$, is the position of the $l^{th}$ ion in the $j^{th}$ trap. The term $H_j^{\text{int}}$ is

$$H_j^{\text{int}} = \hbar \Delta \sum_{m=1}^{N} \sigma^{(m)}_j \sigma^{(m)}_j + \hbar \Delta \sum_{m=1}^{N} \sigma^{(m)}_j \sigma^{(m)}_j \frac{2}{\pi \epsilon_0 |x_{jn}(t) - x_{jm}(t)|},$$  \hspace{1cm} (7)

where $\sigma_L$ is the complex amplitude for both external lasers, $k$ is as defined in Subsection 2A and $g_\theta (g_\theta \in \mathbb{R})$.
is the coupling constant for the ion-field interaction. Finally, $H_{\text{res}}$ is the Hamiltonian for the external reservoir that couples to the $j^{th}$ cavity, for which $R_j$ is a reservoir annihilation operator and $\kappa$ is a damping constant.

The feasible assumptions below are made about the system in order to simplify calculations for it and to focus on its most important aspects:

1. All ions are so cold that they only move from their mean position $x_j^0$ by a small amount and so we can approximate $x_j(t)$ by $x_j^0 + q_j(t)$, where $q_j(t)$ is a small displacement.

2. The cavity field and external laser frequencies are appreciably detuned from $\omega_s$ and all two-level ions are initially in the ground state. Thus, the excited internal states are sparsely populated and spontaneous emission effects are negligible and can be ignored.

3. Vibrational dephasing occurs over a timescale much longer than that of the interactions producing quantum state exchange and consequently can be ignored.

4. The wavelength of the cavity mode is much greater than the distances that the first ions in both traps stray from the centres of their traps. Thus, $\sin(kx_{j1}) \approx kx_{j1} << 1$. This allows us to arrange things so that the $y$ and $z$ dependence of the external laser fields are negligible and thus, assuming $\mathcal{E}_L$ is time independent, that $\mathcal{E}_L(y, z, t) \approx \mathcal{E}_0 e^{-i\omega_L t}$, where $\mathcal{E}$ is a real time-independent amplitude.

5. The damping parameter $\kappa$ is such that $\nu_x \gg \kappa \gg \gamma_0 \eta \mathcal{E}/\sqrt{N \Delta}$.

6. For each trap, the frequencies of different normal or collective modes [22] in the $x$-direction are well-separated from each other and so only centre-of-mass modes in this direction couple to cavity fields.

Given assumptions 1, 4, and 6. above, calculations in [22] show that we can write $H_{\text{total}}$ in terms of normal-mode creation and annihilation operators as

$$H_{\text{total}} = \hbar \sum_{m=1}^{N} \nu_m \left( B_{jx}^{(m)} \left( B_{jx}^{(m)} + \frac{1}{2} \right) + H_{j0} \right) + h \left[ \mathcal{E}_L(y, z, t) \sigma_j^{(1)} + \mathcal{E}_L(y, z, t) \sigma_j^{(1)} \right] + \frac{\hbar \nu_x}{\sqrt{N}} \left( B_{jx}^{(1)} + B_{jx}^{(1)} \right) \left( a_j^{\dagger} \sigma_j^{(1)} + a_j \sigma_j^{(1)} \right) + \kappa \left( a_j^{\dagger} R_j + a_j R_j^\dagger \right) + H_{\text{res}},$$

where $B_{jx}^{(m)}$ is the annihilation operator for the $m^{th}$ normal mode for the $j^{th}$ trap in the $x$ direction. For example, $B_{jx}^{(1)}$ is a centre-of-mass mode annihilation operator which is $B_{jx}^{(1)} = 1/\sqrt{N} \left( b_{jx}^{(1)} + b_{jx}^{(2)} + \ldots b_{jx}^{(N)} \right)$ whilst $B_{jx}^{(2)}$ is the annihilation operator for the breathing mode which is $B_{jx}^{(2)} = 1/\sqrt{N} \left( b_{jx}^{(1)} + b_{jx}^{(2)} \right)^2 \text{ when } N = 2$.

Comparing Eqn (8) to Eqn (2), we see that $B_{jx}^{(1)}$ in Eqn (8) plays an almost identical role to that of $b_x$ in Eqn (2). Given that $\sin(kx) \approx \nu_x (b_x + \tilde{b}_x)$ in Eqn (2), the only difference between the forms in which the two operators appear results from the factor of $1/\sqrt{N}$ in Eqn (8). As a consequence, $B_{jx}^{(1)}$ in Eqn (8) couples to its cavity mode identically --- aside from the factor of $1/\sqrt{N}$ --- to the manner in which $b_x$ couples to $a$. It follows that as quantum state exchange takes place in the system described by $H_{\text{total}}^\text{single}$ with information about an input electromagnetic field being transferred to $b_x$, it also occurs in the system described by $H_{\text{total}}$ due to the correspondence between the two system’s Hamiltonians. Thus, in the latter system, information about the input field is transferred to the centre-of-mass mode for the trapped atoms in the $x$ direction just as if this mode was a vibrational mode for a single harmonically trapped atom. The only difference between the $N$-ion case and one described by $H_{\text{total}}^\text{single}$ is that the effective coupling in the former case is reduced by a factor of $1/\sqrt{N}$.

In [14], it was shown that we can transfer the intra-cavity steady-state for the subthreshold nondegenerate parametric amplifier which is

$$|\psi\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \tanh^N r |n\rangle_1 |n\rangle_2,$$

where the subscripts 1 and 2 denote the two output modes and $r$ is a real squeezing parameter, into the vibrational states in the $x$ direction for two single trapped atoms in different harmonic traps. Using the correspondence between the quantum state exchange processes involving a single harmonically trapped atom and $N$ harmonically trapped ions demonstrated above, it follows that in the system illustrated in Fig. 2 we can transfer $|\psi\rangle$ into the centre-of-mass modes in the $x$ direction for the two sets of $N$ trapped ions thus producing, in the steady state,

$$|\psi_{\text{CM}}\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \tanh^N r |N\rangle_1 |N\rangle_2,$$

where $|N\rangle_j$ denotes the centre-of-mass vibrational number state for the $x$ direction with eigenvalue $N$ for the $j^{th}$ trap.

Importantly, the process of creating $|\psi_{\text{CM}}\rangle$ just outlined does not seem to be overly experimentally feasible. This is so as optical cavities and nondegenerate optical parametric amplifiers have been widely realized quantum optical laboratories for some time. In addition, experiments in which a single harmonically trapped ion has been placed within an optical cavity have been conducted [23]. Relatedly, neutral atoms have also been confined within a standing-wave dipole-force trap that, in turn, lies within an optical cavity [24].
C. Summary

In this paper, we explore multipartite entanglement in relation to quantum state exchange and in Section III, follow on from a multipartite entanglement condition implicit in work by Dür and Cirac [25] by presenting a novel condition. The satisfaction of this novel condition implies that any pure state comprising of M subsystems is M-way entangled. Here, an M-way entangled state is one possessing entanglement that spans across M subsystems as does the generalized GHZ state \( |\psi\rangle = |0\rangle^\otimes M + |1\rangle^\otimes M \). After presenting this condition, we then use it to show qualitatively that \( |\psi_{CM}\rangle \) is 2-N-way entangled. In Section IV, we quantitatively consider the entanglement in \( |\psi_{CM}\rangle \). We introduce a novel multipartite entanglement measure for pure states we call the entanglement of minimum bipartite entropy or \( E_{\text{MEE}} \), which is the minimum of the von Neumann entropies of all the reduced density matrices obtained from some pure state of interest by tracing over all of the subsystems for the state.

After this, we use \( E_{\text{MEE}} \) to calculate a lower bound for the amount of 4-way, 6-way and 8-way entanglement in \( |\psi_{CM}\rangle \) for \( N = 2, 3, 4 \) respectively for a range of \( r \) values. Finally, we discuss the nature of our results.

It is interesting to investigate the nature \( |\psi_{CM}\rangle \)'s 2-N-way entanglement for a number of reasons. Firstly, it has been claimed — but not demonstrated — that a state identical to \( |\psi_{CM}\rangle \) but for 2N neutral atoms as opposed to charged ions is an “entangled state of all \( 2^N \) ... atoms” [13]. If true, then, because of the similarity between the two states, this implies that \( |\psi_{CM}\rangle \) is also such a state, i.e. it is 2N-way entangled. It is thus interesting to investigate \( |\psi_{CM}\rangle \)'s 2-N-way entanglement in order to see if this implied claim is true. Secondly, it is interesting to investigate \( |\psi\rangle \)'s 2-N-way entanglement as it is a massive-particle state which is important as, to date, mostly massless photons have been used to experimentally investigate entanglement. Thirdly, if the implied claim is true, then it means that \( |\psi_{CM}\rangle \) is a state consisting of 2N entangled harmonic oscillators, each possessing an infinite-dimensional Hilbert space as opposed to the two-dimensional Hilbert space of a qubit, that is 2 N-way entangled.

III. QUALITATIVE RESULTS

A. Negative partial transpose sufficient condition

Assume that, for a certain state \( \rho \), we wish to know the answer to the question “Does \( \rho \) contain at least some \( M \)-way entanglement?” Whilst answering this question does not tell us everything about the nature of \( \rho \)'s \( M \)-way entanglement, it nevertheless tells us something of interest. One way to answer it, provided that \( \rho \) consists of qubits, is to use a condition that can be readily derived from work by Dür and Cirac [25]. This condition involves negative partial transposes (NPTs) [26, 27, 28] and thus we name it the NPT sufficient condition. It is sufficient for the presence of \( M \)-way entanglement for all \( \rho \)'s consisting of \( P \) qubits, where \( P \geq M \), and is based on generalizing the notions of separability and inseparability to many-qubit systems. Before stating the condition, it is first useful to mention two things. Firstly, we define an \( M \)-partite split of \( \rho \) [25] to be a division or split of \( \rho \) into \( M \) parts which each consist of one or more subsystems. Secondly, we observe that \( \rho \) can always be converted to a state that is diagonal in a certain basis by a “depolarization” process consisting of particular local operations [25]. This basis consists of \( M \)-qubit generalized GHZ states of the form \( |\psi\rangle = 1/\sqrt{2} (|0\rangle^\otimes |j\rangle^\otimes |1\rangle^\otimes |N-j\rangle^\otimes |1\rangle^\otimes |N-j\rangle) \), where \( j \) is a natural number that we write in binary as \( M - 1 \) bits, i.e. \( j = j_1 j_2 \ldots j_{M-1} \), where \( j_x \) is the \( x \)th bit in \( j \)'s binary representation. Given these two things, the NPT sufficient condition states that \( \rho \) is \( M \)-way entangled for a given \( M \)-partite split if the diagonal state that it depolarizes to is such that all bipartite splits that contain the \( M \)-partite split have negative partial transposes. A bipartite split is one that divides a system into two parts, i.e. a 2-partite split. Also, a bipartite split that contains an \( M \)-partite split is one that does not separate members of any of the \( M \) subsystems onto two different sides of the bipartite split. That is, that does not cross any of the divisions created by the \( M \)-partite split.

B. Result

Following on from the NPT sufficient condition, we propose a necessary and sufficient condition for the existence of \( M \)-way entanglement for \( M \)-partite pure states. Our condition is based on the traces of the squares of reduced density matrices obtained by tracing over some of the subsystems constituting our system of interest. After formulating it, we then use it to demonstrate that \( |\psi_{CM}\rangle \) contains some \( 2 \)-N-way entanglement. Our motivations for employing our condition instead of the NPT sufficient condition are that i) it seems to be mathematically simpler to calculate whether or not our condition is satisfied and ii) as we are concerned with a pure state, our condition is stronger than the NPT sufficient condition in the sense that it is both necessary and sufficient as opposed to just being sufficient.

Our \( M \)-way entanglement condition utilizes the fact that when a pure state \( |\psi\rangle \) for \( M \) subsystems is \( M \)-way entangled then we cannot write it as \( |\psi\rangle = |\phi_1\rangle_{Q_1} \otimes |\phi_2\rangle_{Q_2} \), where \( |\phi_1\rangle_{Q_1} \) and \( |\phi_2\rangle_{Q_2} \) are the states for the subsystems denoted by \( Q_1 \) and \( Q_2 \) respectively and both \( Q_j \) and \( \bar{Q}_j \) denote at least one subsystem. To put this another way, when \( |\psi\rangle \) is \( M \)-way entangled then there is no way to represent it as the tensor product of two pure states. Consequently, excluding all such possibilities suffices to show, and is also, in general, necessary to show, that \( |\psi\rangle \) is \( M \)-way entangled. This can be done by first checking that no single-subsystem state can be factored out from the state of the remaining \( M - 1 \) subsystems.
We do this by checking that the traces of the squares of all the reduced density matrices obtainable from $|\psi\rangle$ by tracing over one subsystem are less than one. That is, if $Tr(\rho_{Q_j})^2 < 1$, where $\rho_{Q_j}$ is the reduced density operator obtained from $|\psi\rangle$ by tracing over the subsystem denoted by $Q_j$ for all $Q_j$ denoting just one subsystem. We can then repeat this procedure, considering all $Q_j$s corresponding to all pairs of subsystems, then all triples and so forth until we have considered all $Q_j$'s corresponding to all sets of $R$ subsystems, where $R = [M/2]$, where $[x]$ is the largest integer less than or equal to $x$. It is sufficient to only consider sets of up to 3 corresponding to $[M/2]$ subsystems as a necessary condition for being able to factor out any larger number of subsystems from $|\psi\rangle$ is the ability to also factor out $[M/2]$ or fewer subsystems. Underlying the process just described is that of seeing whether or not we can exclude all the ways that $|\psi\rangle$ could fail to be $M$-way entangled.

Our condition can be formalized as Definition 1 which is as follows:

**Definition 1**: For a pure state $|\psi\rangle$ for $M$ subsystems, consider the set $Q$ whose members $Q_j$ are themselves sets of subsystems for the system corresponding to $|\psi\rangle$. This set $Q$ contains all sets of $P$ subsystems for this system, where $1 \leq P \leq [M/2]$. Given this, $|\psi\rangle$ is $M$-way entangled iff, for all $Q_j$, $Tr(\rho_{Q_j})^2 < 1$, where $\rho_{Q_j}$ is the reduced density operator obtained by beginning with $|\psi\rangle$ and tracing over the subsystems $Q_j$.

To illustrate Definition 1 consider, for example, the GHZ state $|\psi\rangle_{\text{GHZ}} = 1/\sqrt{2}(|000\rangle_{123} + |111\rangle_{123})$, where the subscripts 1, 2 and 3 denote subsystems of $|\psi\rangle_{\text{GHZ}}$. The parameter $P = [3/2] = 1$ and consequently the set $Q$ comprises of all sets of one subsystem and thus $Q = \{\{1\}, \{2\}, \{3\}\}$, where the numbers again denote subsystems for $|\psi\rangle_{\text{GHZ}}$. For the element $\{1\}$, for example, $Tr(\rho_{\{1\}})^2 = 1/2$. Calculating $Tr(\rho_{\{1\}})^2$ for all of $Q$'s other elements, we find that it is $1/2$ in all three cases. Thus, $|\psi\rangle_{\text{GHZ}}$ satisfies Definition 1 and hence is said to be 3-way entangled, as is the case.

To further explain Definition 1, we now apply it to determining whether the following four-party states are $4$-way entangled:

1.) $|\psi_4^{(1)}\rangle = 1/\sqrt{2}(|0000\rangle_{1234} + |1111\rangle_{1234})$.

2.) $|\psi_4^{(2)}\rangle = 1/\sqrt{2}|0100\rangle_1 \otimes (|0000\rangle_{234} + |1111\rangle_{234})$.

3.) $|\psi_4^{(3)}\rangle = |\phi^+\rangle_{12} \otimes |\phi^+\rangle_{34}$.

Turning to 1.), we see that upon tracing over any single subsystem, we produce a reduced density matrix of the form $\rho_{Q_j} = 1/2(\rho_{Q_j} | \rho_{Q_j} | 1\langle 1 | \rho_{Q_j} \rangle)$ for which $Tr(\rho_{Q_j})^2 = 1/2$. Similarly, tracing over any two subsystems produces a density matrix of the form $\rho_{Q_j} = 1/2(\rho_{Q_j} | \rho_{Q_j} | 1\langle 1 | \rho_{Q_j} \rangle)$ for which, again, $Tr(\rho_{Q_j})^2 = 1/2$. Thus, Definition 1 gives the correct result that $|\psi_4^{(1)}\rangle$ is 4-way entangled. For 2.), tracing over the first subsystem produces $|\psi_4^{(2)}\rangle = 1/\sqrt{2}(|0000\rangle_{23} + |1111\rangle_{234})$, which is a pure state and hence $Tr(\rho_{Q_j})^2 = 1$ for the corresponding $j$. Consequently, Definition 1 tells us that $|\psi_4^{(2)}\rangle$ is not 4-way entangled, as is the case. For 3.), tracing over any one subsystem produces the mixed state $\rho = 1/2(|\phi^+\rangle \langle \phi^+| + |\phi^+\rangle \langle \phi^+|)$ and so we might be tempted to infer that $|\psi_4^{(3)}\rangle$ is 4-way entangled. However, when we trace over subsystems 1 and 2 or subsystems 3 and 4 we produce the pure state $|\phi^+\rangle$ for which $Tr(\rho_{\{j\}} | \rho_{\{j\}} | 1\langle 1 | \rho_{\{j\}} \rangle) = 1$. Hence, Definition 1 correctly tells us that $|\psi_4^{(3)}\rangle$ is not 4-way entangled.

In applying Definition 1 to $|\psi_{\text{CM}}\rangle$, we first write $|\psi_{\text{CM}}\rangle$ in terms of vibrational number states for the $2N$ ions involved as we wish to see if they are $2N$-way entangled. As a step towards doing so, upon observing that $|N\rangle_j = \left(\frac{\hbar_j}{2}\right)^{N/2} |0\rangle_j$, we express $|N\rangle_j$ in terms of vibrational number states in the $x$ direction for individual ions as

$$|N\rangle_j = \sum_{n_c(\{\theta_n\})} c(\tilde{n}, N)|\tilde{n}\rangle_j,$$  (11)

where $\tilde{n}$ is the $N$-component vector $(n_1, n_2, \ldots, n_N)$, the state $|\tilde{n}\rangle_j = \bigotimes_{i=1}^N |n_i\rangle_j$ denotes a number state for the $k$th ion in the $j$th trap and

$$c(\tilde{n}, N) = \left(\begin{array}{c} N \\ \ N-n_1 \\ \ N-n_2 \\ \cdots \\ \ N-n_{N-1} \\ \ N-n_N \end{array}\right) \times \sqrt{N!} \times \sqrt{N!} \times \cdots \times \sqrt{N!}$$  (12)

Thus, $\sum_{n_c(\{\theta_n\})}$ denotes the sum over all combinations of $n_1, n_2, \ldots, n_N$ such that $\sum_{j=1}^N n_j = N$ [31]. Using Eqn (11) to represent $|\psi_{\text{CM}}\rangle$ in terms of vibrational number states for individual ions, we obtain

$$|\psi_{\text{CM}}\rangle = \frac{1}{\cosh r} \sum_{N'=0}^{\infty} \tan N' r \left(\sum_{\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N} c(\tilde{n}, N)|\tilde{n}\rangle_1 \right) \otimes \left(\sum_{\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N} c(\tilde{n}, N)|\tilde{n}\rangle_2 \right).$$  (13)

We now show that the right-hand side of Eqn (13) satisfies Definition 1 and thus that $|\psi_{\text{CM}}\rangle$ is $2N$-way entangled. We do this by first writing $|\psi_{\text{CM}}\rangle$ as the most general bipartite state possible involving vibrational number states for individual ions. Next, we show that, upon tracing over the ions in the half of the bipartite split containing the lesser number of ions and then finding the trace of the square of the resulting reduced density matrix, that this is less than one. It follows that, for all $j$, $Tr(\rho_{Q_j})^2 < 1$. Hence, we satisfy Definition 1 and so $|\psi_{\text{CM}}\rangle$ is $2N$-way entangled.
Dividing the ions in \(|\psi_{CM}\rangle\) into subsystems \(A\) and \(B\) containing, respectively, \(R\) and \(2N-R\) ions \((R \neq 0)\), we can write \(|\psi_{CM}\rangle\) as

\[
|\psi_{CM}\rangle = \sum_{i=0}^{\infty} c_i |f_i\rangle_A \otimes |g_i\rangle_B,
\]

where \(||f_i|| = ||g_i|| = 1\) and the \(|f_i\rangle_A\) but not necessarily the \(|g_i\rangle_B\) are mutually orthogonal. (As we can always write \(|\psi_{CM}\rangle\) in biorthogonal form \([29]\), there exist \(|g_i\rangle_B\) that are mutually orthogonal. However, we are not concerned with this form in the current calculation and so do not consider such a decomposition of \(|\psi_{CM}\rangle\).) To give an example, when \(N = 2\) and \(A\) contains the first ion in the first trap

\[
|\psi_{CM}\rangle = \frac{1}{\cosh r} |0\rangle_A \otimes \left( |000\rangle_B + \frac{\tanh r}{2} |101\rangle_B + \frac{\tanh r^2}{4} |202\rangle_B + \frac{\sqrt{2} \tanh^2 r}{4} |211\rangle_B + \frac{\tanh^2 r}{2} |220\rangle_B + \ldots \right) + \frac{1}{\cosh r^2} |1\rangle_A \otimes \left( |\tanh r - \frac{2}{2}|001\rangle_B + \frac{\tanh^2 r}{4} |012\rangle_B + \frac{\sqrt{2} \tanh^2 r}{2} |111\rangle_B + \frac{\tanh^2 r}{2} |120\rangle_B + \ldots \right) + \frac{1}{\cosh r^2} |2\rangle_A \otimes \left( |\tanh^2 r^2 - \frac{2}{2}|002\rangle_B + \frac{\sqrt{2} \tanh^2 r^2}{2} |011\rangle_B + \frac{\tanh^2 r^2}{2} |020\rangle_B + \ldots \right) + \ldots,
\]

where \(|x\rangle = |n_1 = x\rangle_A\) and \(|x_1 x_2 x_3\rangle_B = |n_2 = x_1, m_1 = x_2, m_2 = x_3\rangle_B\). Here, for example, \(c_0 = 1/\cosh r\), \(|f_0\rangle_A = |0\rangle_A\), \(c_1 = 1/\cosh r\), \(|f_1\rangle_A = |1\rangle_A\),

\[
|g_0\rangle_B = \frac{1}{\sqrt{M_0}} \left( |000\rangle_B + \frac{\tanh r}{2} |101\rangle_B + \frac{\tanh^2 r}{4} |202\rangle_B + \frac{\sqrt{2} \tanh^2 r}{4} |211\rangle_B + \frac{\tanh^2 r}{2} |220\rangle_B + \ldots \right),
\]

and

\[
|g_1\rangle_B = \frac{1}{\sqrt{M_1}} \left( \frac{\tanh r}{2} |001\rangle_B + \frac{\tanh r}{2} |010\rangle_B + \frac{\sqrt{2} \tanh^2 r}{2} |011\rangle_B + \frac{\tanh^2 r}{2} |020\rangle_B + \ldots \right).
\]

where \(M_0\) and \(M_1\) normalize \(|g_0\rangle_B\) and \(|g_1\rangle_B\). Upon tracing over \(A\) in Eqn (14) and squaring the resulting reduced density operator \(\rho_{QA}\), we obtain

\[
[\rho_{QA}]^2 = \sum_{i,j=0}^{\infty} c_i^2 c_j^2 |g_i\rangle_B \langle g_j|g_j\rangle_B.
\]

Calculating the trace of \([\rho_{QA}]^2\) yields

\[
Tr ([\rho_{QA}]^2) = \sum_{i,j=0}^{\infty} c_i^2 c_j^2 |d_{ij}|^2,
\]

where \(d_{ij} = \langle g_i|g_j\rangle\). As the trace of a density matrix is always one, we know that

\[
\sum_{i,j=0}^{\infty} c_i^2 c_j^2 = \left( \sum_{i=0}^{\infty} c_i^2 \right) \times \left( \sum_{j=0}^{\infty} c_j^2 \right) = 1.
\]

It thus follows from Eqn (17) that, as \(c_i \neq 0\) for all \(i\), if \(|d_{ij}|^2 < 1\) for at least one \(d_{ij}\) then \(Tr ([\rho_{QA}]^2) < 1\).

As the centre-of-mass state \(|N\rangle\) in \(2N\) has an even number of centre-of-mass phonons in total \((2N)\), when we express it as a sum of vibrational number states for individual ions, these states all contain an even number of individual phonons in total. Furthermore, because \(|\psi_{CM}\rangle\) contains the state \(|N = 0\rangle|N = 0\rangle\), one \(|f_i\rangle_A\) in Eqn (13), which we denote by \(|f_i^{(cm)}\rangle_A\), is a tensor product of ground states for some of the \(2N\) ions in \(|\psi_{CM}\rangle\). For example, in Eqn (15), \(|f_i^{(cm)}\rangle_A = |0\rangle_A\). Given that, in general, \(|f_i^{(cm)}\rangle_A\) contains zero individual phonons, only states with an even number of individual phonons in total are present in the \(|g_i\rangle_B\) with the same index \(i\), which we denote by \(|g_i^{(cm)}\rangle_B\). This is so as we require the total number of individual phonons in \(|f_i^{(cm)}\rangle_A \otimes |g_i^{(cm)}\rangle_B\) to be even.

In addition to \(|f_i^{(cm)}\rangle_A\), because \(|\psi_{CM}\rangle\) includes the term \(|N = 1\rangle|N = 1\rangle\), there also exists an \(|f_i\rangle_A\) in Eqn (13) containing just one individual phonon which we denote as \(|f_i^{(cm)}\rangle_A\). For example, in Eqn (15) \(|f_i^{(cm)}\rangle_A = |1\rangle_A\). In general, the \(|g_i\rangle_B\) with the same index \(i\) as \(|f_i^{(cm)}\rangle_A\), which we denote by \(|g_i^{(cm)}\rangle_B\), comprises of states with an odd number of individual phonons in total as dictated by the requirement that the total number of individual phonons for \(|f_i^{(cm)}\rangle_A \otimes |g_i^{(cm)}\rangle_B\) is even. Thus, \(|g_i^{(cm)}\rangle_B\) is orthogonal to \(|g_i^{(cm)}\rangle_B\) and the corresponding \(|d_{ij}|^2 = \langle g_i^{(cm)}|g_j^{(cm)}\rangle^2 = 0\). Returning to the right-hand side of Eqn (17), this means that \(Tr ([\rho_{QA}]^2) < 1\) for \(Q_A\) and thus that Definition 1 is satisfied. This allows us to infer that \(|\psi_{CM}\rangle\) is \(2N\)-way entangled and consequently we have verified the implied assertion that \(|\psi_{CM}\rangle\) is an “entangled state of all \(2N\) ... atoms” — except, of course, when \(r = 0\).
IV. QUANTIFYING THE AMOUNT OF 2N-WAY ENTANGLEMENT IN $|\psi_{CM}\rangle$

A. Theory

In the previous subsection we presented a qualitative result which showed that $|\psi_{CM}\rangle$ possessed some 2N-way entanglement. However, we would also like to know how much 2N-way entanglement $|\psi_{CM}\rangle$ contains. For this reason, we present a novel quantitative measure of M-way entanglement for M-partite pure states, for arbitrary M. This measure is based on the von Neumann entropies of reduced density operators produced by considering all bipartite splits for some state of interest. We call it the entanglement of maximum bipartite entropy $E_{M_{BE}}$, which we soon define. After all, we then argue that it is a plausible measure and finally use it to calculate a lower bound on the amount of 2N-way entanglement in $|\psi_{CM}\rangle$.

For a pure state $|\psi\rangle$ with M subsystems, $E_{M_{BE}}$ is

$$E_{M_{BE}}(|\psi\rangle) = \min(S_{all}),$$

where $S_{all}$ is the set containing the von Neumann entropies of all the reduced density operators obtained from $|\psi\rangle$ by tracing over a set of P subsystems in $|\psi\rangle$, where $1 \leq P \leq \lfloor M/2 \rfloor$. The function $\min(X)$ returns the smallest element of the set X. Thus, as the von Neumann entropy of both sides of any bipartite split of $|\psi\rangle$ are equal [2] (p. 513), $S_{all}$ contains the von Neumann entropies of all the reduced states that we can generate from $|\psi\rangle$. For example, when $|\psi\rangle = |\psi\rangle_{GTH} = 1/\sqrt{2} (|000\rangle_{123} + |111\rangle_{123})$, the sets of subsystems containing P members that we trace over in obtaining $S_{all}$ are $[1]$, $[2]$ and $[3]$, where the numbers denote either $[1]$’, $[2]$ or $[3]$ subsystems of $|\psi\rangle_{GTH}$. As the von Neumann entropy of the state $\rho = S(\rho) = -Tr(\rho \log \rho)$ [2, 30], the von Neumann entropy for the reduced density operator generated from $|\psi\rangle_{GTH}$ upon tracing over the subsystem denoted by any one of these sets is 1. Hence $S_{all} = \{1, 1, 1\}$ and so $E_{M_{BE}}(|\psi\rangle_{GTH}) = 1$. We thus say that $|\psi\rangle_{GTH}$ has 1 unit of 3-way entanglement.

To provide some insight into $E_{M_{BE}}$, it is now shown that it can be thought of as a distance-based measure of M-way entanglement. That is, as measuring the distance between $|\psi\rangle$ and the closest pure state with zero M-way entanglement given a certain metric. To understand this, observe that, naively, it seems reasonable to think that there exists a pure state $|\psi_{zero}\rangle$ with zero M-way entanglement that has an identical $S_{all}$ to $|\psi\rangle$‘s except for one element. This element corresponds to the smallest element of $S_{all}(|\psi\rangle)$ and is zero. The next step in comprehending the distance-based nature of $E_{M_{BE}}$ is representing $S_{all}(|\psi\rangle)$ and $S_{all}(|\psi_{zero}\rangle)$ by points A and B respectively in a co-ordinate space for which each co ordinate denotes the possible values of an element of either $S_{all}(|\psi\rangle)$ or $S_{all}(|\psi_{zero}\rangle)$. That is, a space that graphically represents $S_{all}(|\psi\rangle)$ and $S_{all}(|\psi_{zero}\rangle)$.

To further highlight the plausibility of $E_{M_{BE}}$, consider the following analogy. Imagine an ordinary chain with M links. If $M - 1$ of these are strong and the other one is weak, then the chain is close to breaking and so only has a small amount of “nonbroken-ness” — even though all but one of the links are solid. This is so as nonbroken-ness is a wholistic property that is a manifestation of the nature of all M links. Relating this to $E_{M_{BE}}$ just as nonbroken-ness is a wholistic property, so $E_{M_{BE}}$ measures a wholistic property, namely M-way entanglement, that relates to the nature of all M subsystems of M-partite states. In analogy with a chain with just one weak link, an M-partite pure state for which all members
of $S_{\text{all}}$ are large, except for one, is very close to possessing no $M$-way entanglement. In this way, we see that $E_{\text{MBE}}$ and, in particular, the presence of the min function in it seem plausible.

Another positive feature of $E_{\text{MBE}}$ is that it satisfies three well-known desiderata for bipartite entanglement measures [32], as we now show. (It seems plausible that these should also be desiderata for multipartite entanglement measures.) They are:

1.) The proposed entanglement measure is zero for all product states.

2.) The proposed entanglement measure is invariant under local unitaries.

3.) The proposed entanglement measure does not increase on average under local operations, classical communication (LOCC) and division into subensembles.

Beginning with 1.), if the state of interest is a product state, where we define a product state to be one for which we can factor out the state of at least one of the subsystems, then at least one member of $S_{\text{all}}$ is zero and so $E_{\text{MBE}}$ is also zero, as we desire. Turning to 2.), we note that for a general bipartite split, the von Neumann entropy of the reduced density matrix obtained by tracing over the subsystems on the side of the split with the lesser number of particles is invariant under unitary transformations which act on only one subsystem. Consequently, if we define local unitaries to be those which act just on a single subsystem, then $E_{\text{MBE}}$ satisfies 2.)

In considering 3.), it is important to remember that $E_{\text{MBE}}$ is only for pure states and thus we ignore local operations that convert $|\psi\rangle$ to a mixed state. For example, we do not consider local operations that transform $|\psi\rangle$ to a state that is close to a maximally mixed state and thus has large values for the von Neumann entropies of all its reduced states. We choose this example as such local operations increase the value of $\min(S_{\text{all}})$ for a system of interest. However, they manifestly do not increase its $M$-way entanglement but instead transform its state into one for which $E_{\text{MBE}}$ is not applicable. With this constraint in mind, we define a local operation to be one that involves just one subsystem, such as a one-dimensional measurement on a single subsystem. Given this definition, it can be shown that for bipartite pure states, LOCC and division into subensembles is additive over tensor products [32]. However, it can be shown that $E_{\text{MBE}}$ is superadditive. That is, that the $M$-way entanglement of a combined state generated from two states with $a$ and $b$ units of $M$-way entanglement can be greater than $a + b$ (but, importantly, not when $M = 2$). It is an open question as to whether or not multipartite entanglement is additive and so we do not know if the superadditivity of $E_{\text{MBE}}$ represents a flaw.

For $E_{\text{MBE}}$ to be a reasonable measure, it ought to reduce to the standard pure state bipartite entanglement measure of the entropy of entanglement. For $E_{\text{MBE}}$, when $N = 1$, we have $E_{\text{MBE}} = \min(S_{\text{all}}) = S_{11}$, where $S_{11}$ is the von Neumann entropy for the reduced density operator $\rho_{Q_{1}} = \text{Tr}_{1}(|\psi\rangle\langle\psi|)$, and so we recover the desired measure, namely the entropy of entanglement. Finally, $E_{\text{MBE}}$ seems to be plausible as for $|\psi\rangle = \sqrt{\frac{1}{N}}\sum |i\rangle^{\otimes N}$, we have $E_{\text{MBE}} = -c \log_{2} 2 = -c \log_{2} (1 - c)$. This expression increases monotonically in the interval $c \in [0, 1]$ and attains its maximum value of one for $c = 1/2$. Such behaviour seems reasonable.

### B. Results

In this subsection we use $E_{\text{MBE}}$ to calculate lower bounds on the amount of $2N$-way entanglement present in $|\psi_{\text{CM}}\rangle$ for $N = 2, 3, 4$, for a range of $r$ values. We obtain these lower bounds by first, calculating $\text{Tr}(|\rho_{Q_{j}}|^{2})$ for a general $Q_{j}$. Next, we determine the linear entropy $S_{L}(\rho_{Q_{j}})$ [33] from the relation $S_{L}(\rho_{Q_{j}}) = 1 - \text{Tr}(|\rho_{Q_{j}}|^{2})$ and then use the fact that $S_{L}(\rho) \log_{2} e \leq S(\rho)$ to obtain our lower bounds. We calculate a lower bound rather than $E_{\text{MBE}}$ itself as it is computationally infeasible to calculate $E_{\text{MBE}}$ due to the fact that it is computationally infeasible to calculate the required von Neumann entropies of reduced density matrices given the infinite-dimensional bases of the harmonic oscillators comprising $|\psi_{\text{CM}}\rangle$. This is so as these are generally calculated by first diagonalizing $\rho$ and it is computationally infeasible to do this, in general, when $\rho$ is a square matrix of infinite dimension.

We begin with the initial density operator $\rho_{\text{CM}} = |\psi_{\text{CM}}\rangle\langle\psi_{\text{CM}}|$ which can be written in the centre-of-mass number-state basis as

$$\rho_{\text{CM}} = \sum_{N'N} f(N, N') |N\rangle_{1}^{N'} \langle N|_{1}^{N'} |N'\rangle_{2} \langle N'|_{2}^{N'} ,$$

(20)

where $f(N, N') = \tanh^{N + N'/2} r / \cosh^{2} r$. To obtain a general $\rho_{Q_{j}}$, we trace over the first $T$ ions in the first trap and the first $V$ in the second one, arriving at

$$\rho_{Q_{j}} = \sum_{N'N} \sum_{P} f(N, N') \langle \tilde{P} | N\rangle_{1}^{N'} \langle N'|_{1}^{N'} \langle \tilde{P}' | N\rangle_{2} \langle N'|_{2}^{N'} | \tilde{P}' \rangle ,$$

(21)

where $\tilde{P}$ is a dummy variable given by $\tilde{P} = (P_{1}^{(1)}, P_{2}^{(1)}, ..., P_{T}^{(2)}, P_{1}^{(2)}, ..., P_{V}^{(2)})$, where $P_{a}^{(j)}$ denotes a vibrational number state for the $a^{th}$ ion in the $j$ direction.
in the $j$th ion trap, $\vec{\alpha} = (\theta_{0(1)}, \theta_{0(2)}, \theta_{0(3)}...0_{T+V})$ and $\vec{\alpha'} = (\infty_{0(1)}, \infty_{0(2)}, \infty_{0(3)}...\infty_{T+V})$, where a bracketed subscript enumerates the elements of $\vec{\alpha}$ or $\vec{\alpha'}$. We adopt a notation such that a sum of the form $\sum_{X=0}^{\infty} \sum_{Y=0}^{\infty} f(N,N')f(M,M') \langle \vec{\alpha}|N\rangle\langle N'|\vec{\alpha'} \rangle$, we also assume that a state of the form $\langle \vec{X}|$ denotes the state $|X_1\rangle|X_2\rangle...|X_F\rangle$. Note that due to an exchange symmetry for ions in the same group of ions, it is sufficient to just consider the reduced density operators denoted by Eqn (21) to deal with all possible $\rho_{Q_i}$’s. That is, we do not need to consider, say, tracing over the first and third ions in the first trap and the second one in the second trap. This is so as the $\rho_{Q_i}$ this yields is identical to that produced by tracing over the first two ions in the first trap and the first one in the second trap.

We now find $[\rho_{Q_i}]^2$ and then trace over the remaining $2N - (T + V)$ atoms, producing

$$Tr([\rho_{Q_i}]^2) = Tr \left( \sum_{\vec{\alpha}} \sum_{\vec{\alpha'}} \sum_{N,N',M,M'} f(N,N')f(M,M') \langle \vec{\alpha}|N\rangle\langle N'|\vec{\alpha'} \rangle \times \langle \vec{\alpha}|M\rangle\langle M'|\vec{\alpha'} \rangle \right).$$

(22)

where, in analogy with $\vec{\alpha}$, $\vec{\alpha'}$ is a dummy variable given by $\vec{\alpha'} = (P_1^{(1)}, P_2^{(1)}, ..., P_T^{(1)}, P_1^{(2)}, ..., P_V^{(2)})$ where $P_{\alpha}^{(j)}$ denotes a vibrational number state in the $x$ direction for the $\alpha$th ion in the $j$th trap.

Using Eqn (22), we now numerically determine $S_L(\rho_{Q_i})$ for arbitrary $T$ and $V$ particular values of $N$ and $r$. Our results provide lower bounds for $S(\rho_{Q_i})$ as $S(\rho) = \log_2 e \leq S(\rho)$ as can be verified by considering a power series expansion for $S(\rho)$. Hence, knowing $S_L(\rho_{Q_i})$ for all bipartite splits of $|\psi_{CM}\rangle$ allows us to infer a lower bound for $\min(S_{all})$ and hence one for $E_{MBE}$. We thus calculate $S_L(\rho_{Q_i})$ for $N = 2, 3, 4$ for a range of $r$ values numerically using straightforward C++ code. These results are then used to place lower bounds on $E_{MBE}(|\psi_{CM}\rangle)$ for 4-way, 6-way and 8-way entanglement which appear in Figs 4(a) and (b).

As $|\psi_{CM}\rangle$ is the sum of an infinite number of state-vectors, to calculate $S_L$ in practice, we truncate the sum over $N$ in the definition of $|\psi_{CM}\rangle$ at a finite value. This induces errors in our lower bounds for $E_{MBE}(|\psi_{CM}\rangle)$ for which upper bounds can be derived. For all data points in Figs 4 (a) and (b), the errors on our lower bounds for $E_{MBE}(|\psi_{CM}\rangle)$ have been calculated to be less than $10^{-3}$ and hence are negligible.

Two interesting features of Figs 4 (a) and (b) are that, firstly, for a given $r$ value our lower bound on $E_{MBE}$ decreases for increasing $N$. It is possible that we can understand this behaviour by observing that for constant $r$ we initially have a fixed entanglement resource, namely the entangled output of the NO PA. It is conceivable that the decrease under consideration results from this fixed resource being spread amongst a larger number of subsystems as we increase $N$ thus, perhaps, causing it to distribute less bipartite entanglement to any given bipartite split of $|\psi_{CM}\rangle$. In turn, this may decrease the $S_L$ of both halves of an arbitrary split, thus explaining the decrease in our lower bound for $E_{MBE}$ for increasing $N$. The second interesting feature of Figs 4 (a) and (b) is that as we increase $r$, $E_{MBE}$ increases as expected given that an increased $r$ means that we have more centre-of-mass entanglement.
V. DISCUSSION

A. State of interest

One noteworthy feature of the quantum state exchange process that creates $|\Phi_{\text{CM}}\rangle$ is that the effective coupling $g_0/\sqrt{N}$ between the centre-of-mass vibrational states in the $x$ direction and their cavity fields becomes small for large $N$. This is undesirable as it means that the timescale over which we come close to achieving quantum state exchange, and thus generate a state that has a high fidelity with $|\Phi_{\text{CM}}\rangle$, is long for large $N$. In turn, this makes ideal quantum state exchange difficult to experimentally realize for large $N$ as it necessitates that the system maintains coherence for a long time to do so. One possible solution to this problem is to simultaneously couple all $N$ ions in both traps to their cavity fields by having external lasers incident on all ions. This strategy has been shown to produce an effective coupling of $g_0\sqrt{N}$ for a case of quantum state exchange involving $N$ neutral atoms in a single harmonic trap and so may prove useful for the system described in Subsection IIIB.

B. Qualitative results

The thinking underlying Definition 1 is the same as that which underlies the NPT sufficient condition for $M$-way entanglement. However, there are significant differences between the two. Firstly, Definition 1 involves arbitrary dimensional subsystems, whereas the NPT sufficient condition deals only with qubits. Secondly, the NPT sufficient condition is a sufficient condition for $M$-way entanglement whereas the satisfaction of Definition 1 is both necessary and sufficient for pure states. Thirdly, the NPT sufficient condition uses the partial transpose to determine the presence of $M$-way entanglement, whereas Definition 1 uses the mathematically simpler entry the trace of the square of a reduced density operator. Observe that Definition 1 is narrower than the NPT sufficient condition in the sense that it only applies to pure states whilst the NPT sufficient condition is applicable to both pure and mixed states.

C. Quantitative results

A number of issues surround $E_{\text{MIE}}$, which we now discuss.

1. What does $E_{\text{MIE}}$ tell us about what quantum resource we have? Ideally, we would like to be able to relate $E_{\text{MIE}}$ to one or more quantum tasks or protocols such as distributed quantum computation [34] with $E_{\text{MIE}}$ telling us something valuable about how well we can perform these tasks. This is so as if we could do this, then it would increase $E_{\text{MIE}}$’s utility. Unfortunately, however, this has not yet been accomplished.

2. Can we tractably calculate $E_{\text{MIE}}$? For an entanglement measure to be useful, it must be tractable and able to be calculated in practice. Unfortunately, $E_{\text{MIE}}$ seems to be difficult to calculate, at least for the state considered.

Although $E_{\text{MIE}}$ has the two above negative features we note that, firstly, further research may eliminate them and, secondly, we should consider them alongside the positive features of $E_{\text{MIE}}$ which are that it is a reasonable measure and that it helps us to understand the nature of the entanglement in $|\Phi_{\text{CM}}\rangle$ and also the capabilities of quantum state exchange.

To conclude, we have shown that quantum state exchange can be used to produce the state $|\Phi_{\text{CM}}\rangle$ for two sets of trapped ions in spatially separated ion traps. We have also show that $|\Phi_{\text{CM}}\rangle$ is a 2$N$-partite entangled state and, in addition, have placed a lower bound on the amount of such entanglement that it possesses. Our results contribute to our understanding of multipartite entanglement that involves i) massive particles and ii) infinite-dimensional Hilbert spaces within a context that is not overly experimentally feasible.

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