INTRODUCTION

The photon dispersion relation is a fundamental property of electromagnetic fields. In the context of quantum field theory, the dispersion relation describes the relationship between the frequency and the wave vector of a photon. The dispersion relation is crucial for understanding the behavior of light and other electromagnetic waves.

The dispersion relation for the photon is given by the equation:

\[ c = \frac{1}{\sqrt{\varepsilon \mu}} \]

where \( c \) is the speed of light, \( \varepsilon \) is the permittivity of the medium, and \( \mu \) is the permeability of the medium.

In the case of vacuum, where \( \varepsilon = 1 \) and \( \mu = 1 \), the speed of light is given by the classical value:

\[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s} \]

This is the dispersion relation for the photon in the vacuum, and it is a cornerstone of modern physics.
nine independent form factors, and that in the long wavelength limit only three are independent, with the rest being expressed in terms of them. Using the (real-time) finite temperature field theory method, we obtain the one-loop expressions for the form factors, expressed as integrals over the distribution functions of the particles in the background. Those formulas are valid to all orders in the magnetic field and are explicitly evaluated for a variety of conditions of physical interest. As an application we determine the photon dispersion relations and calculate the Faraday rotation for plane polarized light, for various cases. They reproduce the well-known semi-classical results when the appropriate limits are taken, but remain valid for more general situations.

In Section II we give the general decomposition of the photon self-energy in terms of the nine independent form factors. We also collect there several kinematical relations that are useful in later stages of the calculations. The one-loop formulas for the photon polarization tensor are derived in Section III. These are used in Section IV to discuss the dispersion relations, focusing on the long wavelength limit as a special case. The explicit formulas for the independent form factors are given in Section V for different possible conditions of the background electron gas, and for various regimes of interest including the low frequency regime and the weak-field (linear) limit. Our conclusions are summarized in Section VI, while in Appendix A we summarise the conventions that we use regarding the Schwinger formula for the electron propagator in an external magnetic field, and Appendix B contains the derivation of an integral formula used in the calculation. Some of the details of the derivation of the low-frequency formulas, and the weak-field formulas, for the self-energy form factors are shown in appendices C and D, respectively.

II. KINEMATICS

A. General decomposition of the polarization tensor

In the absence of the magnetic field, the photon self-energy $\pi^{\mu\nu}$ depends in general on the photon momentum $q^\mu$, and on the velocity four-vector of the medium $u^\mu$. In the frame of reference in which the medium is at rest, $u^\mu$ has components given by

$$u^\mu = (1, \vec{0}),$$

and in that frame we write

$$q^\mu = (\omega, \vec{Q}).$$

In the presence of a magnetic field, but otherwise an isotropic medium, $\pi^{\mu\nu}$ depends in addition on the vector $b^\mu$ that is determined by the magnetic field. The vector $b^\mu$ is defined such that, in the frame in which the medium is at rest,

$$b^\mu = (0, \vec{b})$$

where we denote the magnetic field vector by

$$\vec{B} = B\vec{b}.$$  \hfill (2.4)

For a given photon momentum vector we define the unit vectors $\vec{e}_i$ ($i = 1, 2, 3$) by writing

$$\vec{Q} \equiv Q\vec{e}_3,$$ \hfill (2.5)

with $e_{1,2}$ chosen such that

$$\vec{e}_1 \cdot \vec{e}_3 = e_1 \cdot e_3 = 0,$$

$$\vec{e}_2 = \vec{e}_3 \times \vec{e}_1.$$ \hfill (2.6)

In addition, for the problem that we are considering in the present work, without loss of generality, we can choose the vectors $\vec{e}_{1,2}$ such that $\vec{b}$ lies in the 1, 3 plane. Thus we can write,

$$\vec{b} = \cos \theta \vec{e}_3 + \sin \theta \vec{e}_1,$$ \hfill (2.7)

where

$$\cos \theta = \frac{Q \cdot \vec{b}}{Q}.$$ \hfill (2.8)
We now introduce the vectors

\[ X_3^\mu = u^\mu - \frac{(u \cdot q) q^\mu}{q^2}, \]
\[ X_2^\mu = \epsilon^{\mu\alpha\beta\gamma} q_\alpha b_\beta u_\gamma, \]
\[ X_1^\mu = \epsilon^{\mu\alpha\beta\gamma} X_2^\alpha q_\beta u_\gamma, \] (2.9)

which satisfy

\[ q \cdot X_i = 0, \]
\[ X_i \cdot X_j = 0 \quad (i \neq j). \] (2.10)

Therefore, the \( X_i^\mu \) form a basis of vectors orthogonal to \( q^\mu \), and the nine bilinear combinations \( X_i^\mu X_j^\nu \) form a complete set of tensors in terms of which the photon self-energy can be decomposed. To exploit this more fully, it is useful to define the (normalized) vectors

\[ \epsilon_i^\mu (\vec{Q}) = \frac{X_i^\mu}{\sqrt{-X \cdot X}}, \] (2.11)

which satisfy

\[ \epsilon_i(\vec{Q}) \cdot q = 0, \]
\[ \epsilon_i(\vec{Q}) \cdot \epsilon_j(\vec{Q}) = -\delta_{ij}, \]
\[ g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} = - \sum_{i=1,3} \epsilon_{i\mu}(\vec{Q}) \epsilon_{i\nu}(\vec{Q}). \] (2.12)

and which take the form

\[ \epsilon_1^\mu(\vec{Q}) = (0, \hat{e}_1), \]
\[ \epsilon_2^\mu(\vec{Q}) = (0, \hat{e}_2), \]
\[ \epsilon_3^\mu(\vec{Q}) = \frac{1}{\sqrt{q^2}} (Q, \omega \hat{e}_3) \] (2.13)

in the rest frame of the medium.

Whence, in the most general case, the photon self-energy can be expressed as a linear combination of the bilinears \( \epsilon_i^\mu \epsilon_j^\nu \) in the form

\[ \pi^{(\text{eff})}(\omega, \vec{Q}) = - \sum_{i,j} \pi^{(ij)}(\omega, \vec{Q}) \epsilon_{i\mu}(\vec{Q}) \epsilon_{j\nu}(\vec{Q}), \] (2.14)

involving the nine independent coefficient functions \( \pi^{(ij)}(\omega, \vec{Q}) \). Furthermore, as will be seen in Section IV B, in the long wavelength limit all the \( \pi^{(ij)}(\omega, \vec{Q}) \) can be determined in terms of just three independent functions [i.e., Eq. (4.33)]. This fact will be the starting point in Section IV to obtain the polarization vectors and dispersion relations of the normal modes.

**B. Additional kinematic relations**

It will prove to be useful to introduce the following tensors that are transverse to \( u^\mu \),

\[ Q_{\mu\nu} = -b_\mu b_\nu, \]
\[ R_{\mu\nu} = q_{\mu\nu} - u_{\mu} u_{\nu} - Q_{\mu\nu}, \]
\[ P_{\mu\nu} = i \epsilon_{\mu\alpha\beta\gamma} b^\alpha u^\beta. \] (2.15)

They have the orthogonality properties

\[ QR = QP = 0, \] (2.16)
and they satisfy the multiplication rules

\begin{align}
Q_\mu^2 &= Q, \\
R_\mu^2 &= R, \\
P_\mu^2 &= R, \\
R P &= P,
\end{align}

(2.17)

as well as the normalization conditions

\begin{align}
Q_\mu^\nu &= 1, \\
R_\mu^\nu &= 2.
\end{align}

(2.18)

In addition, we define

\[ U_\mu^\nu = u_\mu b_\nu - b_\mu u_\nu, \]

(2.19)

which satisfies

\begin{align*}
U R &= RU, \\
U^\mu_\nu U_\lambda^\nu &= u^\mu u_\nu - b^\mu b_\nu, \\
Q_\mu^\nu U_\mu^\alpha U_{\nu\beta} &= -u_\alpha u_\beta \\
&= g_\mu^\nu - R_\mu^\nu, \\
\frac{1}{2} P_\mu^\nu \epsilon_\mu \alpha \beta &= i U_{\alpha \beta}.
\end{align*}

(2.20)

In any case, any four-vector \( v^\mu \) can be decomposed according to

\[ v^\mu = v^\mu_\parallel + v^\mu_\perp, \]

(2.21)

where

\[ v^\mu_\perp = R^\mu_\nu v^\nu = (0, \vec{v}_\perp), \]
\[ v^\mu_\parallel = v^\mu - v^\mu_\perp = (v^\parallel, \vec{v}_\parallel), \]

(2.22)

with \( \vec{v}_\perp \) and \( \vec{v}_\parallel \) being the three-dimensional components of \( \vec{v} \) that are perpendicular and parallel to \( \vec{b} \), respectively.

In particular, for any two such vectors,

\begin{align}
\vec{v}_\perp \cdot v'_\perp &= \vec{v} \cdot v' - \vec{b} \cdot \vec{v}' \cdot \vec{b}, \\
\vec{v}_\parallel \cdot v'_\parallel &= \vec{v} \cdot v' - \vec{b} \cdot \vec{v}' \cdot \vec{b}.
\end{align}

(2.23)

### III. Photon Self-Energy

In one-loop, the 11 element of the photon thermal self-energy is given by

\[ i \pi_{11\mu}(\omega, \vec{Q}) = -i(\epsilon)^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu i S_\epsilon(p + q) \gamma_\nu i S_\epsilon(p) \right]. \]

(3.1)

where \( S_\epsilon \) stands for the 11 component of the electron thermal propagator. It can be decomposed in the form

\[ S_\epsilon(p) = S_{F\epsilon}(p) + S_{R\epsilon}(p), \]

(3.2)

where \( S_{F\epsilon} \) is the propagator in vacuum in the presence of the magnetic field while \( S_{R\epsilon} \) incorporates the effects of the thermal background. As shown in Appendix A, \( S_F(p) \) can be written in the form

\[ i S_F(p) = \int_0^\infty d\tau G(p, s) e^{i\gamma_\Phi(p, \tau) - \gamma_\tau}, \]

(3.3)
where
\[ \Phi(p, s) = p^2 - m_c^2 + \left( s^{-\frac{1}{2}} \tan(s) - 1 \right) R_{\mu\nu} p^\mu p^\nu, \]
\[ G(p, s) = \sqrt{s} \left( p^2 + i \gamma_5 \nabla (p, s) + m_e + m_e \tan(s) i \gamma_5 \right), \]  
(3.4)

with
\[ V_{1\mu}(p, s) = \sec^2(s) p_\mu + \tan^2(s) \left( (p \cdot b) b_\mu - (p \cdot u) u_\mu \right), \]
\[ V_{2\mu}(p, s) = \tan(s) \left( (p \cdot b) u_\mu - (p \cdot u) b_\mu \right), \]  
(3.5)

and
\[ s = \epsilon_B T. \]  
(3.6)

Using the relation \( \sec^2(s) = 1 + \tan^2(s) \), Eq. (3.5) can be rewritten as
\[ V_{1\mu}(p, s) = \left[ g_{\mu\nu} + \tan^2(s) R_{\mu\nu} \right] p^\nu, \]
\[ V_{2\mu}(p, s) = \tan(s) U_{\mu\nu} p^\nu. \]
(3.7)

On the other hand, the thermal part is given by
\[ iS_{T\nu}(p) = -\eta_v(p) \int_{-\infty}^{\infty} d\tau G(p, s) e^{i\Phi(p, s) - \epsilon_\tau^1}, \]
(3.8)

where
\[ \eta_v(p) = \theta(p \cdot u) f_v(p \cdot u) + \theta(-p \cdot u) f_v(-p \cdot u), \]
(3.9)

with
\[ f_v(x) = \frac{1}{e^{\beta(x - \mu_v)} + 1}, \]
\[ f_v(x) = \frac{1}{e^{\beta(x + \mu_v)} + 1}. \]
(3.10)

Here \( \beta \) stands for the inverse temperature and \( \mu_v \) the electron chemical potential.

When Eq. (3.2) is substituted into Eq. (3.1), we obtain four different terms. Only those two that contain one factor of \( S_{T\nu} \) and \( S_{T\nu} \) each, contribute to the real part of the self-energy, which we denote by \( \pi^{(eff)}_{\mu\nu} \), and is the quantity in which we are interested. Thus
\[ \pi^{(eff)}_{\mu\nu} (\omega, \vec{Q}) = i e^2 \int \frac{d^4 p}{(2\pi)^4} \left\{ \eta_v(p) \int_{-\infty}^{\infty} d\tau e^{\lambda(p, \tau)} \int_{-\infty}^{\infty} d\tau' e^{\lambda(p, \tau')} \text{Tr} \left[ \gamma_\mu G(p + q, \tau') \gamma_\nu G(p, \tau) \right] \right\} + \eta_v(p + q) \int_{-\infty}^{\infty} d\tau e^{\lambda(p, \tau)} \int_{-\infty}^{\infty} d\tau' e^{\lambda(p + q, \tau')} \text{Tr} \left[ \gamma_\mu G(p + q, \tau') \gamma_\nu G(p, \tau) \right], \]
(3.11)

where, to simplify the notation, we have defined
\[ \lambda(p, \tau) = \epsilon T \Phi(p, s) - \epsilon T \tau. \]
(3.12)

In the term involving the factor \( \eta_v(p + q) \), we make the change of variables \( p \rightarrow p - q \), and \( \tau \leftrightarrow \tau' \). Then, using the cyclic property of the trace, we obtain
\[ \pi^{(eff)}_{\mu\nu} (\omega, \vec{Q}) = 4i e^2 \int \frac{d^4 p}{(2\pi)^4} \eta_v(p) \int_{-\infty}^{\infty} d\tau e^{\lambda(p, \tau)} \left\{ \int_{-\infty}^{\infty} d\tau' \left[ e^{\lambda(p + q, \tau')} L_{\mu\nu}(p, q) + e^{\lambda(p - q, \tau')} L_{\nu\mu}(p, q) \right] \right\}, \]
(3.13)

where
\[ L_{\mu\nu}(p, q) = \frac{1}{4} \text{Tr} \left[ \gamma_\mu G(p + q, \tau') \gamma_\nu G(p, \tau) \right]. \]
(3.14)

Using the expression for \( G \) given in Eq. (3.4), by straightforward evaluation of the trace we obtain
\[ L_{\mu\nu} = T^{(2)}_{\mu\nu} + L^{(s)}_{\mu\nu}, \]
(3.15)
\[
L^{(s)}(p, q) = V_1 \mu V_1^\nu + V_1^\mu V_1^\nu - V_1 \cdot V_1^\mu g_{\mu\nu} - V_2 \cdot V_2^\mu g_{\mu\nu} - V_2^\mu V_2^\nu + V_2 \cdot V_2^\nu g_{\mu\nu},
\]
\[
+ m_2^2 g_{\mu\nu} + m_2^2 \tan(s) \tan(s') (g_{\mu\nu} - 2 u_\mu u_\nu + 2 b_\mu b_\nu),
\]
\[
L^{(s)}(p, q) = \epsilon_{\mu\nu\alpha\beta} V_1^\mu V_2^\nu + \epsilon_{\mu\nu\alpha\beta} V_2^\alpha V_2^\beta + m_2^2 [\tan(s) - \tan(s')] \epsilon_{\mu\nu\alpha\beta} u^\alpha u^\beta,
\]
with the exception that \(\epsilon^{0123} = 1\), and using the notation \( s' = |q| B s' \), \( V_1^\mu = V_1^\mu(p + q, s') \), and similarly for \( V_2^\mu \).

Using Eq. (3.7), \( L^{(s)}(\mu, \nu) \) can be written in the form
\[
L^{(s)}(p, q) = 2 m_2^2 \tan(s) \tan(s') R_{\mu\nu} + m_2^2 [1 - \tan(s) \tan(s')] (g_{\mu\nu} - g_{\mu\nu} [1 + \tan(s) \tan(s')] (p^2 + p \cdot q)
\]
\[
+ \{ g_{\mu\alpha} + \tan(s') R_{\mu\alpha} \} [g_{\nu\beta} + \tan(s) \tan(s')] - \tan(s) \tan(s') U_{\mu\alpha} U_{\nu\beta} + (\mu \leftrightarrow \nu) \} p^\alpha (p + q) \beta,
\]
\[
L^{(s)}(p, q) = \epsilon_{\mu\nu\alpha\beta} [\tan(s) g^{\alpha\lambda} U_{\beta\lambda} + \tan(s') g^{\alpha\lambda} U_{\beta\lambda} + \tan(s) \tan(s') R_{\alpha\lambda} U_{\beta\lambda} + \tan(s') \tan(s') U_{\alpha\lambda} R_{\beta\lambda}] p_\lambda (p + q) \beta,
\]
\[
+ m_2^2 \tan(s) - \tan(s') \epsilon_{\mu\nu\alpha\beta} u^\alpha u^\beta.
\]

where \( p^\alpha_\perp \) and \( q^\beta_\perp \) are defined according to Eq. (2.22). It is seen from Eq. (3.17) that
\[
L^{(s)}(\mu, \nu) = L^{(s)}(\mu, \nu)(p, q).
\]

Using this and the fact that \( \lambda(-p, s) = \lambda(p, s) \), Eq. (3.13) can be written in the form
\[
\sigma^{(s)}(\omega, \vec{Q}) = 4i e^2 \int_{|p| \geq 0} \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} d\tau e^{\lambda(p, \tau)} \int_{\tau}^{\infty} d\tau' \left\{ \left[ \lambda(p + q, \tau') \right] L^{(s)}(p, q)[f_\sigma(p, u) + f_\sigma(p, u)] + \left[ q \rightarrow -q \right] \right\}.
\]

The integrals over \( \tau, \tau' \) can be expressed in the form
\[
\int_{-\infty}^{\infty} d\tau e^{\lambda(p, \tau)} \int_{\tau}^{\infty} d\tau' e^{\lambda(p + q, \tau')} L^{(s)}(p, q) = -2m_2^2 K_1(p) J_1(p + q) R_{\mu\nu} + m_2^2 g_{\mu\nu} [K_0(p) J_0(p + q) + K_1(p) J_1(p + q)]
\]
\[
+ \{ g_{\mu\alpha} + \tan(s') R_{\mu\alpha} \} [g_{\nu\beta} + \tan(s) \tan(s')] - \tan(s) \tan(s') U_{\mu\alpha} U_{\nu\beta} + (\mu \leftrightarrow \nu) \} p^\alpha (p + q) \beta,
\]
\[
\int_{-\infty}^{\infty} d\tau e^{\lambda(p, \tau)} \int_{\tau}^{\infty} d\tau' e^{\lambda(p + q, \tau')} L^{(s)}(p, q) = \frac{i}{\mu_0} [K_1(p) J_1(p + q) g^{\alpha\lambda} U_{\beta\lambda} + K_0(p) J_0(p + q) U^{\alpha\lambda} g_{\beta\lambda}]
\]
\[
- K_1(p) J_1(p + q) R^{\alpha\lambda} g_{\beta\lambda} + K_0(p) J_0(p + q) U^{\alpha\lambda} R_{\beta\lambda} \} p_\lambda (p + q) \beta,
\]
\[
+ \frac{i}{\mu_0} \frac{m_2^2}{\mu_0} u^\alpha u^\beta [K_1(p) J_0(p + q) - K_0(p) J_1(p + q)]
\]
where
\[
J_\mu(p) = \frac{(-i)^n}{|q| B} \int_{\tau}^{\infty} dse^{is\omega} \frac{1}{|s| B} \tan(s) \tan(s'),
\]
with
\[
a_\mu \equiv \frac{-p^\mu_\parallel}{|q| B} \frac{\vec{p}^2_\perp}{|q| B^2},
\]
\[
\psi \equiv \frac{1}{|q| B} \left( p^\mu_\parallel - m^2 + iv \right) = \frac{1}{|q| B} \left( (p^\mu_\parallel - p^\mu_\parallel - m^2 + iv) \right).
\]

The \( K_n \) are defined by the same integrals as in Eq. (3.21), but with the limits of integration being \(-\infty < s < \infty\) instead. Therefore, the following relations are easily obtained
\[
K_n(p) = J_n(p) + J'_n(p).
\]
Furthermore, from Eq. (3.21) it is easily seen that

$$J_{n+1}(p) = \frac{\partial J_n(p)}{\partial a_p}, \quad \cdots \tag{3.24}$$

and therefore, only $J_0$ needs to be evaluated explicitly. As shown in Appendix B, it can be expressed in the form

$$J_0(p) = \frac{i}{|A|B} \sum_{\ell=0}^{\infty} \frac{\nu_p}{p_{\ell}^2 - 2\ell}, \quad \cdots \tag{3.25}$$

where the $D_n(a_p)$ are given in terms of the Laguerre polynomials by

$$D_n(a) \equiv (-1)^n e^{-a} \left[ L_n(2a) - L_{n-1}(2a) \right], \quad \cdots \tag{3.26}$$

with the convention

$$L_{-1} \equiv 0. \quad \cdots \tag{3.27}$$

From the relations in Eq. (3.24) we then obtain

$$J_1(p) = \frac{i}{|A|B} \sum_{\ell=0}^{\infty} \frac{\nu_p}{p_{\ell}^2 - 2\ell}, \quad \cdots \tag{3.28}$$

$$J_2(p) = \frac{i}{|A|B} \sum_{\ell=0}^{\infty} \frac{\nu_p}{p_{\ell}^2 - 2\ell}, \quad \cdots \tag{3.29}$$

Furthermore, from Eq. (3.23), it follows that

$$K_0(p) = 2\pi \sum_{n=0}^{\infty} D_n(a_p) \delta(p^2_{\parallel} - m^2) - 2m|A|B, \quad \cdots \tag{3.30}$$

$$K_1(p) = 2\pi \sum_{n=0}^{\infty} D'_n(a_p) \delta(p^2_{\parallel} - m^2) - 2m|A|B, \quad \cdots \tag{3.30}$$

$$K_2(p) = 2\pi \sum_{n=0}^{\infty} D''_n(a_p) \delta(p^2_{\parallel} - m^2) - 2m|A|B. \quad \cdots \tag{3.30}$$

Substituting Eq. (3.20) into (3.19), and using Eq. (3.30) we then obtain

$$\pi^{(\nu \nu')}(\omega, \vec{Q}) = 4ie^2 \sum_{n=0}^{\infty} \int \frac{d^3p}{(2\pi)^3 2E_n} \left\{ \left[ M_{\nu \nu'}^{(\nu)}(p, q)[f_{\nu}(E_n) + f_{\nu}(E_n)] + (q \rightarrow -q) \right] + \left[ N_{\nu \nu'}^{(\nu)}(p, q)[f_{\nu}(E_n) - f_{\nu}(E_n)] - (q \rightarrow -q) \right] \right\}. \quad \cdots \tag{3.31}$$

where

$$p^0 = E_n, \quad \cdots \tag{3.32}$$

with

$$E_n = \sqrt{p^2_{\parallel} + m^2} + 2n|A|B. \quad \cdots \tag{3.33}$$
and

\[
M^{(n)}_{\mu\nu}(p, q) = -2m^2 \nabla^\mu a_\nu J_1(p + q) R_{\mu\nu} + m^2 \left[ D_n(ad_p) J_0(p + q) + D_n^\prime(ad_p) J_1(p + q) \right] \eta_{\mu\nu}
\]

\[-g_{\mu\nu} \left[ D_n(ad_p) J_0(p + q) - D_n^\prime(ad_p) J_1(p + q) \right] \left( p^2 + q^2 \right)\]

\[-g_{\mu\nu} \left[ -D_n(ad_p) J_0(p + q) - D_n^\prime(ad_p) J_2(p + q) \right] + D_n^\prime(ad_p) J_2(p + q) \]

\[+ g_{\mu\nu} \left[ D_n(ad_p) J_0(p + q) + D_n^\prime(ad_p) J_1(p + q) \right] \left( p^2 + q^2 \right) \]

\[-D_n(ad_p) J_2(p + q) - R_{\mu\alpha} R_{\nu\beta} D_n^\alpha(a_p) J_2(p + q) \]

\[+ U_{\mu\alpha} U_{\nu\beta} D_n(a_p) J_1(p + q) + \left( \mu \leftrightarrow \nu \right) p^\alpha(p + q)^\beta,\]

\[
N^{(n)}_{\mu\nu}(p, q) = -g_{\mu\nu} \left[ D_n(ad_p) J_0(p + q) + D_n^\prime(ad_p) J_1(p + q) \right] \eta_{\mu\nu}
\]

\[-D_n(ad_p) J_2(p + q) - R_{\mu\alpha} R_{\nu\beta} D_n^\alpha(a_p) J_2(p + q) \]

\[+ U_{\mu\alpha} U_{\nu\beta} D_n(a_p) J_1(p + q) + \left. \left( \mu \leftrightarrow \nu \right) p^\alpha(p + q)^\beta \right\}

\[m^2 \left[ D_n(ad_p) J_0(p + q) + D_n^\prime(ad_p) J_1(p + q) \right] . \quad (3.34)\]

Using Eq. (3.31) as the starting point, the form factors that enter in the dispersion relations of the propagating modes can be computed, as we show next.

**IV. DISPERSION RELATIONS OF THE NORMAL MODES**

**A. General case**

As already argued in Section II, the photon self-energy can be expressed as shown in Eq. (2.14). The nine independent coefficients \( \pi^{(ij)}(\omega, Q) \) are then determined to one-loop order by

\[
\pi^{(ij)}(\omega, \bar{Q}) = -c_i^j(\bar{Q}) \eta^{\prime}(\bar{Q}) \pi^{(ef)}(\omega, \bar{Q}), \quad (4.1)
\]

with \( \pi^{(ef)}(\omega, \bar{Q}) \) given by Eq. (3.31). While \( c_1^j(\bar{Q}) \) and \( c_2^j(\bar{Q}) \) are independent of \( Q \), \( c_3^j(\bar{Q}) \) is not. We will denote by \( c_i^j(0) \) the basis vectors for \( Q = 0 \), and therefore

\[
c_1^j(0) = (0, 0), \quad c_2^j(0) = (0, 0), \quad (4.2)
\]

It is useful to note that

\[
c_3^j(\bar{Q}) = \frac{Q}{\sqrt{q^2}} u^\mu + \frac{\omega}{\sqrt{q^2}} c_3^j(0), \quad (4.3)
\]

and by using

\[
q^\mu = \omega u^\mu + Q c_3^j(0), \quad (4.4)
\]

we can also write

\[
c_3^j(\bar{Q}) = \frac{Q}{\sqrt{q^2}} u^\mu + \frac{\sqrt{q^2}}{\omega} c_3^j(0). \quad (4.5)
\]

The polarization vectors of the various propagating modes can be expanded in the form

\[
\xi^\mu = \sum_{i=1, 3} a_i(\bar{Q}) c_i^p(\bar{Q}), \quad (4.6)
\]

where the coefficients \( a_i(\bar{Q}) \), and the corresponding dispersion relations, are determined by solving the equation

\[
(q^2 - \Pi) a = 0. \quad (4.7)
\]

In matrix notation, Eq. (4.7) can be written in the form

\[
(q^2 - \Pi) a = 0, \quad (4.8)
\]
\[ \alpha = \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \end{pmatrix}, \quad (4.9) \]

and \( \Pi \) is the matrix formed by the coefficients \( \pi^{(ij)} \). The formulas for the functions \( \pi^{(ij)} \) are obtained by applying Eq. (4.1). For that purpose, it is useful to note that we can make the replacement

\[ \epsilon^{ij}_s(\vec{Q}) \rightarrow \frac{\sqrt{\pi}}{\omega} \epsilon^{ij}_0(0), \quad (4.10) \]

which follows by using Eq. (4.5) and the transversality condition satisfied by \( \pi^{(eff)}_{\mu\nu} \). Thus we obtain

\[
\pi^{(ab)}(\omega, \vec{Q}) = -\pi^{(eff)\, ab}(\omega, \vec{Q}) \\
\pi^{(a3)}(\omega, \vec{Q}) = -\frac{\sqrt{\pi}}{\omega} \pi^{(eff)\, a3}(\omega, \vec{Q}) \\
\pi^{(3a)}(\omega, \vec{Q}) = -\frac{\sqrt{\pi}}{\omega} \pi^{(eff)\, 3a}(\omega, \vec{Q}) \\
\pi^{(33)}(\omega, \vec{Q}) = -\frac{\sqrt{\pi}}{\omega} \pi^{(eff)\, 33}(\omega, \vec{Q}), \quad (4.11)
\]

where the indices \( a, b \) can take the values between 1 and 2. While Eq. (4.11) allows us to read the elements \( \pi^{(ij)} \) off Eq. (3.19) by inspection, the problem of finding the dispersion relations in the general case is a formidable one. For this reason we now consider in some detail the so-called long wavelength limit, which is a particularly important case that holds in a variety of physical applications.

The point to stress here is that the nine elements \( \pi^{(ij)} \) determined by Eq. (4.11), form a complete set of independent functions that parametrize the photon self-energy in the most general way that is consistent with the transversality condition. In particular, this characteristic remains valid independently of any approximation that may be used to compute the elements \( \pi^{(eff)\, ij} \) that must be substituted in the right-hand side in Eq. (4.11).

### B. Long wavelength limit

The long wavelength \((\vec{Q} \rightarrow 0)\) limit is particularly important and we consider it separately as a special case. We stress that, while we are considering this particular limiting case, we do not make any assumptions about the conditions of the gas or the magnitude or the magnetic field. Therefore, the formulas that we obtain below hold for relativistic or non-relativistic gases, whether they are degenerate or not, and for any value of the magnetic field. This limit is valid under the condition

\[ \omega \gg v_e Q, \quad (4.12) \]

where \( v_e \) stands for the average velocity of an electron in the gas. In the limit \( \vec{Q} \rightarrow 0 \), we can write

\[ v^\mu = \omega u^\mu, \quad (4.13) \]

and therefore only \( u^\mu \) and \( b^\mu \) are independent vectors. The most general form of \( \pi^{(eff)}_{\mu\nu} \) consistent with the transversality condition in this case is then

\[ \pi^{(eff)}_{\mu\nu}(\omega, \vec{Q} \rightarrow 0) = \pi^{\mu\nu}_{T}(\omega, \vec{Q} \rightarrow 0) R_{\mu\nu} + \pi^{\mu\nu}_{L}(\omega, \vec{Q} \rightarrow 0) Q_{\mu\nu} + \pi^{\mu\nu}_{P}(\omega, \vec{Q} \rightarrow 0) P_{\mu\nu}, \quad (4.14) \]

where \( R_{\mu\nu} \), \( P_{\mu\nu} \) and \( Q_{\mu\nu} \) are defined in Eq. (2.15), and the functions \( \pi_{T, L, P} \) are determined from the one-loop expression for \( \pi^{(eff)}_{\mu\nu} \) by means of the projection formulas

\[
\pi_{T}(\omega, \vec{Q}) = \frac{1}{2} R_{\mu\nu} \pi^{(eff)}_{\mu\nu}(\omega, \vec{Q}), \\
\pi_{L}(\omega, \vec{Q}) = Q_{\mu\nu} \pi^{(eff)}_{\mu\nu}(\omega, \vec{Q}), \\
\pi_{P}(\omega, \vec{Q}) = -\frac{1}{2} P_{\mu\nu} \pi^{(eff)}_{\mu\nu}(\omega, \vec{Q}). \quad (4.15)
\]
Although we are not indicating it explicitly, the functions $\pi_{T,L,P}(\omega, \vec{Q})$ depend, in addition, on the magnetic field $B$. Substituting Eq. (3.19) into (4.15) we obtain

$$\pi_{T,L}(\omega, \vec{Q}) = 4ie^2 \sum_{n=0}^{\infty} \int \frac{d^3p}{(2\pi)^3} \mathcal{E}_{n}^{(p)} \left[ f_T(E_n) + f_T(E_n) \right] \left[ I_{T,L}^{(n)}(p, q) + (q \rightarrow -q) \right],$$

$$\pi_{P}(\omega, \vec{Q}) = 4ie^2 \sum_{n=0}^{\infty} \int \frac{d^3p}{(2\pi)^3} \mathcal{E}_{n}^{(p)} \left[ f_P(E_n) - f_P(E_n) \right] \left[ I_{P}^{(n)}(p, q) - (q \rightarrow -q) \right],$$

(4.16)

where

$$I_{T}^{(n)} \equiv \frac{1}{2} R_{\mu
u} M_{\mu
u}^{(n)} ,$$

$$I_{L}^{(n)} \equiv Q_{\mu
u} M_{\mu
u}^{(n)} ,$$

$$I_{P}^{(n)} \equiv \frac{1}{2} p_{\mu
u} M_{\mu
u}^{(n)} .$$

(4.17)

By direct calculation, it is straightforward to obtain, for any $q$, the following expressions

$$I_{T}^{(n)} = \left[ D_0(a_0) J_0(p+q) - D_n(a_n) J_1(p+q) \right] h_T(p, q) ,$$

$$I_{L}^{(n)} = -\left[ D_0(a_0) J_0(p+q) + D_n(a_n) J_0(p+q) - D_n(a_n) J_0(p+q) - D_n(a_n) J_0(p+q) \right] h_L1(p, q) - \left[ D_n(a_n) J_0(p+q) + D_n(a_n) J_0(p+q) \right] h_L2(p, q) ,$$

$$I_{P}^{(n)} = \left[ D_n(a_n) J_1(p+q) - D_n(a_n) J_0(p+q) \right] h_T(p, q) ,$$

(4.18)

where

$$h_T = (p \cdot b)^2 - (p \cdot u)^2 + (p \cdot b) (q \cdot b) - (p \cdot u) (q \cdot u) + m_e^2 ,$$

$$h_L1 = p_\perp^2 + p_\perp \cdot q_\perp ,$$

$$h_L2 = (p \cdot u)^2 + (p \cdot b)^2 + (p \cdot b) (q \cdot b) + (p \cdot u) (q \cdot u) - m_e^2 .$$

(4.19)

In order to evaluate the functions $\pi_{T,L,P}$ in the $\vec{Q} \rightarrow 0$ limit, as indicated in Eq. (4.14), we proceed as follows. Remembering that $p^0 = E_n$, in this limit

$$\psi_{p+q} = \frac{1}{|eB|} \left( \omega^2 + 2 \omega E_n + 2n|e|B \right) ,$$

(4.20)

and then from Eq. (3.25)

$$J_0(p+q) = \sum_{\ell} \frac{D_\ell(a_\ell)}{\omega^2 + 2 \omega E_n + 2(n-\ell)|e|B} ,$$

(4.21)

with similar formulas for $J_{1,2}(p+q)$, but with $D_\ell(a_\ell)$ being replaced by $D_\ell(a_\ell)$ and $D_\ell(a_\ell)$, respectively. Taking the $\vec{Q} \rightarrow 0$ limit of the expressions in Eq. (4.19) and substituting them in Eqs. (4.18) we then obtain

$$I_{T}^{(n)} \bigg|_{\vec{Q} \rightarrow 0} = -i \sum_{\ell} \left( D_\ell(a_\ell) D_\ell(a_\ell) - D_n(a_n) D_\ell(a_\ell) \right) \frac{(\omega E_n + 2n|e|B)}{\omega^2 + 2 \omega E_n + 2(n-\ell)|e|B} ,$$

$$I_{L}^{(n)} \bigg|_{\vec{Q} \rightarrow 0} = i a_\ell |e|B \sum_{\ell} \left( D_\ell(a_\ell) D_\ell(a_\ell) + D_n(a_n) D_\ell(a_\ell) - D_n(a_n) D_\ell(a_\ell) - D_\ell(a_\ell) D_\ell(a_\ell) \right) \frac{(\omega E_n + 2n|e|B)}{\omega^2 + 2 \omega E_n + 2(n-\ell)|e|B}$$

$$- i \sum_{\ell} \left( D_\ell(a_\ell) D_\ell(a_\ell) + D_n(a_n) D_\ell(a_\ell) \right) \frac{(2 E_n^2 + \omega E_n - 2(m_e^2 + n|e|B))}{\omega^2 + 2 \omega E_n + 2(n-\ell)|e|B} ,$$

(4.22)

$$I_{P}^{(n)} \bigg|_{\vec{Q} \rightarrow 0} = -i \sum_{\ell} \left( D_\ell(a_\ell) D_n(a_n) - D_n(a_n) D_\ell(a_\ell) \right) \frac{(\omega E_n + 2n|e|B)}{\omega^2 + 2 \omega E_n + 2(n-\ell)|e|B} .$$
With the help of the identities,

\[
\frac{d}{dx} (L_n - L_{n-1}) = -L_{n-1},
\]

\[
\frac{dL_n}{dx} = n(L_n - L_{n-1}),
\]

the following is easily derived

\[
a(D_n(a) - D'_n(a)) = 2nD_n(a),
\]

so that the formula for \( I^{(n)}_L |_{Q \rightarrow 0} \) becomes

\[
I^{(n)}_L |_{Q \rightarrow 0} = \frac{i |B|}{a_p} \sum_{\ell} \frac{4n \ell D_n(a_p) D_{\ell}(a_p)}{\omega^2 + 2\omega E_n + 2(n - \ell) |\varepsilon| B} \left[ 2E_n^2 + \omega E_n - 2(n^2 + 2) |\varepsilon| B \right].
\]

The integrand in Eq. (4.16) depends on \( \bar{F}_n \) only through \( a_p \), so that we can replace \( d^3p \rightarrow d\eta \delta |\varepsilon| B a_p \). With the help of the normalization condition satisfied by the Laguerre polynomials

\[
\int_0^\infty dx e^{-x} L_m(x) L_n(x) = \delta_{m,n},
\]

where \( \delta_{m,n} \) is the Kronecker symbol with the convention

\[
\delta_{m,n} = 0 \quad \text{for} \quad n < 0,
\]

the following hold

\[
\int_0^\infty da D_n(a) D_m(a) = \frac{(-1)^{n+m}}{2} \delta_{n,m} - \delta_{n,-1,m} - \delta_{n,m-1} + \delta_{n-1,m-1},
\]

\[
\int_0^\infty da D'_n(a) D'_m(a) = \frac{(-1)^{n+m}}{2} \delta_{n,m} + \delta_{n,-1,m} + \delta_{n,m-1} - \delta_{n-1,m-1},
\]

\[
\int_0^\infty da D_n(a) D'_m(a) = -\frac{(-1)^{n+m}}{2} \delta_{n,m} - \delta_{n,-1,m} - \delta_{n,m-1} - \delta_{n-1,m-1},
\]

\[
\int_0^\infty da D'_n(a) D_m(a) = \frac{1}{n} \delta_{n,m}.
\]

Thus, defining

\[
\pi_X(\omega) \equiv \pi_X(\omega, \bar{Q} \rightarrow 0) \quad \text{for} \quad X = T, L, P,
\]

by substituting Eq. (4.22) in Eq. (4.16) and using the above integration formulas we obtain

\[
\pi_X(\omega) = \frac{|B|^2}{4\pi^2} \sum_{n=0}^\infty \pi_X^{(n)}(\omega),
\]

where

\[
\pi_T^{(\omega)}(\omega) = \int_{-\infty}^\infty d\eta \left[ f_T(E_n) + \frac{f_T(E_n)}{E_n} \right] \frac{\omega}{\omega^2 + 2\omega E_n - 2 |\varepsilon| B} + (\omega \rightarrow -\omega),
\]

\[
\pi_T^{(\omega')}(\omega') = \int_{-\infty}^\infty \frac{d\eta}{E_n} \left[ f_T(E_n) + \frac{f_T(E_n)}{E_n} \right] \left( \omega E_n + 2n |\varepsilon| B \right) \left[ \frac{1}{\omega^2 + 2\omega E_n - 2 |\varepsilon| B} + \frac{1}{\omega^2 + 2\omega E_n + 2 |\varepsilon| B} \right] + (\omega \rightarrow -\omega),
\]

\[
\pi_T^{(\omega)}(\omega) = \int_{-\infty}^\infty \frac{d\eta}{E_n} \left[ f_T(E_n) - \frac{f_T(E_n)}{E_n} \right] \frac{\omega}{\omega^2 + 2\omega E_n - 2 |\varepsilon| B} - (\omega \rightarrow -\omega),
\]

\[
\pi_P^{(\omega)}(\omega) = \int_{-\infty}^\infty \frac{d\eta}{E_n} \left[ f_P(E_n) - \frac{f_P(E_n)}{E_n} \right] \frac{\omega}{\omega^2 + 2\omega E_n - 2 |\varepsilon| B} + (\omega \rightarrow -\omega),
\]

\[
\pi_P^{(\omega')}(\omega') = \int_{-\infty}^\infty \frac{d\eta}{E_n} \left[ f_P(E_n) + \frac{f_P(E_n)}{E_n} \right] \frac{2m_e^2}{\omega^2 + 2\omega E_n} + (\omega \rightarrow -\omega),
\]

\[
\pi_L^{(\omega)}(\omega) = \int_{-\infty}^\infty \frac{d\eta}{E_n} \left[ f_L(E_n) + \frac{f_L(E_n)}{E_n} \right] \frac{8 \varepsilon |\varepsilon| B + 4m_e^2}{\omega^2 + 2\omega E_n} + (\omega \rightarrow -\omega),
\]

\[
\pi_L^{(\omega')}(\omega') = \int_{-\infty}^\infty \frac{d\eta}{E_n} \left[ f_L(E_n) + \frac{f_L(E_n)}{E_n} \right] \frac{8 \varepsilon |\varepsilon| B + 4m_e^2}{\omega^2 + 2\omega E_n} + (\omega \rightarrow -\omega).
\]
Since we are concerned only with the real part of the self-energy in this work, the above integrals are actually defined in the sense of their principal value part.

As we discuss next, the formulas in Eq. (4.31) allow us to study the dispersion relations in the long wavelength limit. In this case Eq. (4.8) becomes

\[ [q^2 - \Pi_0] \hat{\alpha} = 0. \]  

(4.32)

Here the matrix \( \Pi_0 \) is formed by the coefficients \( \pi^{(ij)} \) that are determined from Eq. (4.11), but using the formula for \( \pi^{(eff)}_{\mu\nu}(\omega, \vec{Q} \to 0) \) that is given by Eq. (4.14). Thus, using the kinematical relations given in Sec. II, by straightforward algebra we find

\[
\Pi_0 = \begin{pmatrix}
\pi_T(\omega) \cos^2 \theta + \pi_L(\omega) \sin^2 \theta & -i\pi_p(\omega) \cos \theta \frac{\sqrt{\omega}}{\omega} [\pi_L(\omega) - \pi_T(\omega)] \sin \theta \cos \theta \\
\pi_p(\omega) \cos \theta & \pi_T(\omega) - \pi_p(\omega) \sin \theta \frac{\sqrt{\omega}}{\omega} \pi_p(\omega) \sin \theta \frac{\sqrt{\omega}}{\omega} [\pi_T(\omega) \sin^2 \theta + \pi_L(\omega) \cos^2 \theta]
\end{pmatrix}.
\]

(4.33)

It should be remembered that in Eq. (4.32), \( \hat{\alpha} \) depends on \( \vec{Q} \). In what follows, we consider some particular cases.

1. \( Q = 0 \)

We consider first the case the zero momentum limit of the dispersion relations, which corresponds to a photon traveling with a vanishingly small momentum in an arbitrary direction. The matrix \( \Pi_0 \) in this case is given as in Eq. (4.33) but with the replacement \( q^2/\omega^2 \to 1 \). The eigenvectors of the matrix in this \( (Q = 0) \) case are found to be given by

\[
\hat{\alpha}_\pm = \begin{pmatrix}
\cos \theta \\
\pm i \\
-\sin \theta
\end{pmatrix},
\]

\[
\hat{\alpha}_L = \begin{pmatrix}
\sin \theta \\
0 \\
\cos \theta
\end{pmatrix},
\]

(4.34)

with corresponding eigenvalues \( \pi_T^0 \pm \pi_p^0 \) and \( \pi_L^0 \), respectively. Therefore, from Eq. (4.6), in the \( Q \to 0 \) limit the polarization vectors of the normal modes are given by

\[
\xi^0_{\pm}(0) = \cos \theta \xi^0_T(0) \pm i \xi^0_p(0) - \sin \theta \xi^0_L(0),
\]

\[
\xi^0_p(0) = \sin \theta \xi^0_T(0) + \cos \theta \xi^0_L(0),
\]

(4.35)

and the corresponding dispersion relations are determined by solving

\[
\omega^2 - [\pi_T \pm \pi_p] = 0,
\]

\[
\omega^2 - \pi_L = 0,
\]

(4.36)

respectively.

2. Propagation parallel to the magnetic field

This corresponds to set

\[
\theta = 0
\]

(4.37)

in Eq. (4.33). However, in contrast to the previous case, we do not set \( Q = 0 \) (although we still assume that \( Qv_e \ll \omega \), which is the condition under which Eq. (4.33) holds). In this case,

\[
\Pi_0(\omega, \vec{Q}) = \begin{pmatrix}
\pi_T^0 & -i \pi_p^0 & 0 \\
i \pi_p^0 & \pi_T^0 & 0 \\
0 & 0 & \frac{q^2}{\omega^2} \pi_L^0
\end{pmatrix},
\]

(4.38)
and therefore

\[
\tilde{\alpha}_\pm = \begin{pmatrix} 1 \pm i \\ 0 \end{pmatrix}, \quad \tilde{\alpha}_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(4.39)

with corresponding eigenvalues \( \pi_T^0 \pm \pi_P^0 \) and \( \frac{2}{\omega} \pi_L^0 \), respectively. The polarization vectors of the normal modes are given by

\[
\xi^0_T(\tilde{Q}) = \frac{1}{\sqrt{2}} \left( \xi^0_I(\tilde{Q}) \pm i \xi^0_I(\tilde{Q}) \right),
\]

\[
\xi^0_L(\tilde{Q}) = \xi^0_I(\tilde{Q}),
\]

(4.40)

and the corresponding dispersion relations are determined by solving

\[
\omega^2 - Q^2 - [\pi_T \pm \pi_P] = 0,
\]

\[
\omega^2 - \pi_L = 0,
\]

(4.41)

respectively. One of the characteristics of this solution is the fact that the longitudinal mode dispersion relation is independent of \( Q \), which is a well known feature of this case.

3. **Propagation perpendicular to the magnetic field**

In this case

\[
\theta = \frac{\pi}{2}
\]

(4.42)

and, as in the previous one, we maintain \( Q \neq 0 \). Then, from Eq. (4.33),

\[
\Pi^0(\omega, \tilde{Q}) = \begin{pmatrix} \pi_L & 0 & 0 \\ 0 & \pi_T & -i \frac{\sqrt{2}}{\pi_T} \\ 0 & i \frac{\sqrt{2}}{\pi_T} & \frac{2}{\pi_T} \end{pmatrix}.
\]

(4.43)

The dispersion relation of the longitudinal mode [with the polarization vector \( \xi^0_L(\tilde{Q}) = \xi^0_I(\tilde{Q}) \)] is obtained by solving

\[
\omega^2 - Q^2 - \pi_L = 0.
\]

(4.44)

For the transverse modes it becomes

\[
(\omega^2 - \pi_T)^2 - Q^2(\omega^2 - \pi_T) = \pi_T^2 = 0,
\]

(4.45)

which can be written in the alternate form

\[
\frac{Q^2}{\omega^2} = \frac{(1 - \frac{\pi_T}{\omega})^2 - (\frac{\pi_T}{\omega})^2}{1 - \frac{\pi_T}{\omega}}.
\]

(4.46)

The corresponding eigenvectors are linear combinations of \( \xi^0_I(\tilde{Q}) \).

For an arbitrary direction of propagation, the longitudinal and transverse polarizations mix, so that the propagating modes are neither purely longitudinal nor purely transverse. In this case, a perturbative solution, that is valid for sufficiently small \( Q \), can obtained.

V. **EXPLICIT FORMULAS**

Although the formulas in Eq. (4.31) have been obtained by taking the long wavelength \( (Q \to 0) \) limit, no approximations have been made. However, they can be calculated explicitly for a variety of situations, under certain approximations. Here we specifically consider the case in which \( \omega \) satisfies

\[
\omega \ll \sqrt{|e|B}
\]

(5.1)
and

\[ \omega \ll 2\langle E_e \rangle, \tag{5.2} \]

where \( \langle E_e \rangle \) stands for an average value of the energy of an electron in the gas. Thus, if the gas is non-relativistic, the condition holds for \( \omega \ll m_e \). If the gas is extremely relativistic, it also holds for \( \omega \gg m_e \), subject to the condition in Eq. (5.2). Together with the condition given in Eq. (4.12), this implies that we are considering a regime in which

\[ v_e Q \ll \omega \ll \sqrt{|\alpha| B} \cdot 2\langle E_e \rangle, \tag{5.3} \]

to which we will refer as the low frequency regime.

Then, under the assumption that the condition in Eq. (5.1) holds, the formulas for \( \pi^{(n)}_{T,P} \) given in Eq. (4.31) reduce to

\[ \pi^{(n)}_T(\omega) = (2 - \delta_{n,0}) \omega^2 \int_{-\infty}^{\infty} dp_{\parallel} (f_e(E_n) + f_\tau(E_n)) \frac{1}{\omega^2 E_n^2 + |\alpha|^2 B^2} \left[ E_n - \frac{n|\alpha|B}{E_n} \right], \]

\[ \pi^{(n)}_P(\omega) = (2 - \delta_{n,0}) (|\alpha| B) \int_{-\infty}^{\infty} dp_{\parallel} (f_e(E_n) - f_\tau(E_n)) \frac{1}{\omega^2 E_n^2 + |\alpha|^2 B^2} \left[ 1 - \frac{2n|\alpha|B}{E_n^2} \right], \tag{5.4} \]

and similarly,

\[ \pi^{(n)}_{\tau}(\omega) = (2 - \delta_{n,0}) \int_{-\infty}^{\infty} dp_{\parallel} (f_e(E_n) + f_\tau(E_n)) \frac{(m_e^2 + 2n|\alpha|B)}{E_n^3}, \tag{5.5} \]

assuming Eq. (5.2). We should note here that the judicious application of the low frequency regime conditions to Eq. (4.31) requires some care. The procedure we have followed to arrive at Eqs. (5.4) and (5.5) is outlined in Appendix C.

We stress once more that these formulas hold for any conditions of the electron gas, whether it is relativistic or non-relativistic, and degenerate or not, and for any value of the magnetic field, subject to Eq. (5.1). However, for specific conditions of the gas, these expressions reduce further to simpler formulas. We consider several examples separately.

### A. Non-relativistic gas

In this case the temperature and chemical potential of the electron gas are assumed to be such that

\[ \beta m_e \gg 1, \quad \frac{m_e}{\mu} \gg 1. \tag{5.6} \]

This implies that

\[ f_\tau \approx 0, \tag{5.7} \]

while for \( f_e \), only values those values of \( p_{\parallel} \) and \( n \) for which

\[ E_n \approx m_e \tag{5.8} \]

contribute significantly in the integrals because otherwise \( f_e \) becomes small. Therefore, to the lowest order in \( 1/m_e \), we can replace

\[ E_n \to m_e \tag{5.9} \]

in the integrands in Eqs. (5.4) and (5.5), and retaining the \( O(1/m_e) \) terms leads to

\[ \pi_T(\omega) = \frac{\omega^2 \Omega_0^2}{\omega^2 - \omega_B^2}, \]

\[ \pi_L(\omega) = \frac{\Omega_0^2}{\omega^2}, \]

\[ \pi_P(\omega) = \frac{\omega \Omega_B \Omega_0^2}{\omega^2 - \omega_B^2}, \tag{5.10} \]
where
\[ \omega_B \equiv \frac{|d| B}{m_e}, \tag{5.11} \]
\[ \Omega_B^2 \equiv \frac{e^2 \mu}{m_e}, \tag{5.12} \]
with
\[ n_e \equiv \frac{1}{4\pi^2} \left[ \sum_{n=0}^{\infty} (2 - \delta_{n,0}) \int_{-\infty}^{\infty} dp \| f(x) \right]. \tag{5.13} \]
The formulas in Eqs. (5.11), (5.12) and (5.13) are the standard expressions for the cyclotron frequency, the plasma frequency and total electron number density, respectively. Moreover, using the results of Section IV B, the formulas in Eq. (5.10) allows us to reproduce very simply the classic results[10] for the photon dispersion relations in the case under consideration. For example, if the photon propagates parallel to the magnetic field, the dispersion relations given by Eq. (4.41) become
\[ \omega^2 - Q^2 - \frac{\omega \Omega_B^2}{(\omega - \omega_B)} = 0, \]
\[ \omega^2 - Q^2 - \frac{\omega \Omega_B^2}{(\omega + \omega_B)} = 0, \]
\[ \omega = \Omega_B. \tag{5.14} \]
The formulas in Eq. (5.10), and whence those in Eq. (5.14), represent the leading term in powers of $1/m_e$, and they neglect entirely the momentum-dependent terms. The latter can give non-negligible corrections at higher temperatures or chemical potential. However, while the leading terms given above depend only on the total number densities and not on the shape of the distribution function, the same is not true with the momentum-dependent corrections. Thus, for example, the corrections are different if we consider the gas to be degenerate or classical.

In what follows we determine the corrections specifically for the case of a classical gas. Thus we can put
\[ E_n \propto m_e + \frac{p_\|^2}{2m_e} + n\omega_B, \tag{5.15} \]
and for the distribution function
\[ f_e \propto e^{-\beta(E_n - \mu)}. \tag{5.16} \]
We consider first $\pi_{T, p}$. Using Eq. (5.15) we write the denominators of the integrands in Eq. (5.4) in the form
\[ \frac{1}{\omega^2 E_n^2 - \mu^2 B^2} \propto \frac{1}{m_e^2(\omega^2 - \omega_B^2)} \left[ 1 - \frac{\omega^2}{\omega^2 - \omega_B^2} \left( \frac{p_\|^2}{m_e^2} + \frac{2n\omega_B}{m_e} \right) \right], \tag{5.17} \]
and similarly
\[ E_n - n|\epsilon| B = m_e \left( 1 + \frac{p_\|^2}{m_e^2} \right), \]
\[ 1 - \frac{2n|\epsilon| B}{E_n^2} = 1 - \frac{2n\omega_B}{m_e}, \tag{5.18} \]
up to terms that are most linear in $p_\|^2/m_e^2$ or $\omega B/m_e$. The integrals over $p_\|$ are carried out very simply, and for the sums over $n$ we use the formulas
\[ S_0 = \sum_{n=0}^{\infty} e^{-\beta n \omega_B} = \frac{1}{1 - e^{-\beta \omega_B}}, \]
\[ S_1 = \sum_{n=1}^{\infty} n e^{-\beta n \omega_B} = \frac{e^{-\beta \omega_B}}{(1 - e^{-\beta \omega_B})^2} = \frac{1}{4 \sinh^2 \left( \frac{\omega_B}{2} \right)}. \tag{5.19} \]
Then eliminating the chemical potential in favor of \( n_e \) by means of Eq. (5.13), which in the present case yields the relation

\[
   n_e = \frac{e^{-\beta (n_e - \mu)}}{4\pi^2} \left( \frac{|e| B}{2m} \right) \sqrt{\frac{2m \pi}{\beta}} \left( 2S \right) - 1 ,
\]

\[
   = e^{-\beta (n_e - \mu)} \left( \frac{|e| B}{4\pi^2} \right) \sqrt{\frac{2m \pi}{\beta}} \frac{1}{\tanh \left( \frac{2m \pi}{\beta} \right)} ,
\]

(5.20)

this procedure yields

\[
   \pi_T (\omega) = \frac{\omega^2 \Omega_B^2}{\omega^2 - \omega_B^2} (1 + \delta) - \frac{\omega^4 \Omega_B^2}{(\omega^2 - \omega_B^2)^2} (\delta + \gamma) ,
\]

\[
   \pi_P (\omega) = \frac{\omega \omega_B \Omega_B^2}{\omega^2 - \omega_B^2} (1 - \gamma) - \frac{\omega^2 \omega_B \Omega_B^2}{(\omega^2 - \omega_B^2)^2} (\delta + \gamma) ,
\]

(5.21)

with

\[
   \delta = \frac{1}{\beta m_e} , \quad \gamma = \frac{2\omega_B}{m_e \sinh \beta \omega_B} .
\]

(5.22)

Notice that for \( \beta \omega_B \) small, \( \delta \approx 2\delta_0 \). As \( \beta \omega_B \) becomes larger, \( \delta_1 \) gives the \( O(\beta^2 \omega_B^4 / m_e) \) corrections to \( \pi_T (\omega) \).

Turning the attention now to \( \pi_L \), we expand \( E_n \) in the integrand of Eq. (5.5) using Eq. (5.15). Retaining terms that are most linear in \( \tilde{p}_n^2 / m_e^2 \) or \( \omega_B / m_e \) and proceeding as above with the integrals and sums over \( n \),

\[
   \pi_L = \Omega_B^2 \left[ 1 - \frac{3}{2m_e \beta} - \frac{\omega_B}{m_e \sinh \beta \omega_B} \right] .
\]

(5.23)

In the limit \( \beta \omega_B \to 0 \), this formula reduces to

\[
   \pi_L = \Omega_B^2 \left[ 1 - \frac{5}{2m_e \beta} \right] ,
\]

(5.24)

which is the standard temperature correction to the plasma frequency (in the non-relativistic, classical regime). Otherwise, the last term in Eq. (5.23) gives the corrections of \( O(\beta^2 \omega_B^4 / m_e) \) due to the presence of the magnetic field.

### B. Relativistic gas

This limit corresponds to the conditions

\[
   1/\beta, \mu, \sqrt{|e| B} \gg m_e ,
\]

(5.25)

so that we can effectively set

\[
   E_n \approx \sqrt{\tilde{p}_n^2 + 2 |e| n B} ,
\]

(5.26)

and we specifically consider the situation in which

\[
   1/\beta, \mu \gg \sqrt{|e| B} .
\]

(5.27)

Under these conditions

\[
   \frac{1}{E_n} \frac{\delta E_n}{\delta n} \approx \frac{|e| B}{E_n} \ll 1 ,
\]

(5.28)

(because \( E_n \) is of order \( 1/\beta \) or \( \mu \)), and we can then carry out the sums over \( n \) by making the replacement

\[
   \sum_n \to \int \frac{d^2 \tilde{p}_n}{2 \pi |e| B} .
\]

(5.29)
where we have defined
\[ \bar{\rho}_1^2 \equiv 2\pi|\mathbf{E}B|. \] (5.30)

With this substitution, the formula in Eq. (5.13) for the electron number density, and the analogous one for the positron, reduce to the standard expressions
\[ n_{e,\tau} = 2 \int \frac{d^3p}{(2\pi)^3} f_{e,\tau}(E), \] (5.31)
where, using Eq. (5.30),
\[ E = \sqrt{\bar{\rho}_1^2 + \bar{\rho}_1^2} \pm |\mathbf{B}|. \] (5.32)

In Eq. (5.31) we have neglected the contribution from the \( n = 0 \) term, which is smaller than the dominant terms by factors of \( \beta \sqrt{|\mathbf{E}B|} \) and/or \( \sqrt{|\mathbf{E}B|/\mu} \). Similarly, substituting Eqs. (5.4) and (5.5) into (4.30),
\[ \pi_T(\omega) = 2e^2\omega^2 \int \frac{d^3p}{(2\pi)^3} (f_{e}(E) + f_{\tau}(E)) \frac{1}{\omega^2E^2 - |\mathbf{E}|^2B^2} \left[ E - \frac{\bar{\rho}_1^2}{2E} \right], \]
\[ \pi_L(\omega) = 2e^2 \int \frac{d^3p}{(2\pi)^3} \frac{\bar{\rho}_1^2}{B^2} (f_{e}(E) + f_{\tau}(E)) \right), \]
\[ \pi_P(\omega) = 2e^2\omega|\mathbf{E}B| \int \frac{d^3p}{(2\pi)^3} (f_{e}(E) - f_{\tau}(E)) \frac{1}{\omega^2E^2 - |\mathbf{E}|^2B^2} \left[ 1 - \frac{\bar{\rho}_1^2}{2E} \right], \] (5.33)

where we can replace \( \bar{\rho}_1^2 \rightarrow 2\bar{\rho}_1^2/3 \) and \( E \rightarrow |\mathbf{B}| \) in the integrand. The angular integration is trivial but the remaining integration over \( p \) cannot be carried out exactly in general. We therefore consider the classical and the degenerate limits, separately.

1. Case I - Classical Gas

In this case
\[ \frac{1}{\beta} \gg \mu, \] (5.34)
and
\[ f_{e,\tau}(E) = e^{-\beta(E \pm \mu)}. \] (5.35)

The relationship between the chemical potential and the electron charge density is then given by
\[ n_e - n_\tau = \frac{4}{\pi^2\beta^3} \sinh \beta\mu \simeq \frac{4\mu}{\pi^2\beta^3}, \] (5.36)
and in addition
\[ n_e + n_\tau = \frac{4}{\pi^2\beta^3} \cosh \beta\mu \simeq \frac{4}{\pi^2\beta^3}. \] (5.37)

The integration over \( p \) is also trivial for \( \pi_L \) and yields
\[ \pi_L = \frac{e^2\beta}{3}(n_e + n_\tau). \] (5.38)

On the other hand, for \( \pi_T, \pi_P \) the integration over \( p \) cannot be done exactly. The formulas for these quantities can be expressed in the form
\[ \pi_T = \frac{e^2\beta}{3}(n_e + n_\tau) F_3 \left( \frac{\Omega_B}{\omega} \right), \]
\[ \pi_P = \frac{e^2\beta}{6}(n_e - n_\tau) \frac{\Omega_B}{\omega} F_3 \left( \frac{\Omega_B}{\omega} \right), \] (5.39)
where
\[ F_n(z) = \int_0^\infty dx \, e^{-\pi \frac{x^n}{x^2 - z^2}}, \] (5.40)

with
\[ \Omega_B \equiv \frac{|e| B \beta}{m}. \] (5.41)

The quantity \( \Omega_B \) plays in the present case the role that \( \omega_B \) occupies in the non-relativistic case. We can obtain approximate values for the functions \( F_n(z) \) by considering the region in which \( z \gg 1 \) or \( z \ll 1 \).

The term \( e^{-\pi x^n} \) is dominated by the values of \( x \approx 1 \). For \( z \gg 1 \), the singularity at \( x = z \) then lies outside the effective region of integration, and we can substitute
\[ \frac{1}{x^2 - z^2} \approx \frac{1}{1 - z^2}. \] (5.42)

Therefore
\[ F_n(z) \approx \frac{n!}{1 - z^2}, \] (5.43)

and, from Eq. (5.39), we can write
\[ \pi_T = 2e^2\beta(n_e + n_\tau) \left( \frac{\omega^2}{\omega^2 - \Omega_B^2} \right), \]
\[ \pi_P = e^2\beta \frac{3}{3}(n_e - n_\tau) \left( \frac{\omega \Omega_B}{\omega^2 - \Omega_B^2} \right), \] (5.44)

for \( \Omega_B \gg \omega \). In the opposite regime \( z \ll 1 \), the singularity is within the effective region of integration and requires some care for finite \( z \). In the extreme limit,
\[ F_n(0) = (n - 2)! \] (5.45)

and
\[ \pi_T = e^2\beta \left( n_e + n_\tau \right), \]
\[ \pi_P = e^2\beta \frac{3}{3}(n_e - n_\tau) \frac{\Omega_B}{\omega}, \] (5.46)

for \( \Omega_B \ll \omega \). These expressions correspond to the weak-field limit used in Ref. [11].

For arbitrary values of \( z \), the functions can be calculated by substituting the equalities
\[ \frac{x^2}{x^2 - z^2} = 1 + \frac{z^2}{2} \left( \frac{1}{x - z} - \frac{1}{x + z} \right), \]
\[ \frac{x^3}{x^2 - z^2} = x + \frac{z^2}{2} \left( \frac{1}{x - z} + \frac{1}{x + z} \right), \] (5.47)
in Eq. (5.40), followed by the change of variable \( t = x - z \). In this way, \( F_2 \) and \( F_3 \) are found to be given by
\[ F_2(z) = 1 - \frac{z^2}{2} \left[ e^{-z} Ei(z) - e^z Ei(-z) \right], \]
\[ F_3(z) = 1 - \frac{z^2}{2} \left[ e^{-z} Ei(z) + e^z Ei(-z) \right], \] (5.48)
in terms of the Exponential Integral function
\[ Ei(z) = \mathcal{P} \int_{-\infty}^z \frac{e^t}{t} dt. \] (5.49)
By means of Eq. (5.39), \( \pi_{T,P} \) can then be computed for any value of \( z \). In particular, using the expansion

\[
Ei(z) = \ln(|z|) + \gamma + z + \frac{z^2}{4} + \frac{z^3}{18} + O(z^4),
\]

(5.50)

where \( \gamma = 0.577215 \) is the Euler constant, for \( z \) small we obtain

\[
F_2(z) = 1 - z^2 + z^2(\gamma + \ln(|z|)) + O(z^4),
\]

(5.51)

\[
F_3(z) = 1 - z^2(\ln(|z|) + \gamma) + O(z^4).
\]

When these expressions are substituted in Eq. (5.39) they of course reproduce the lowest order terms given in Eq. (5.46) plus the \( O(\Omega_B^2) \) corrections to them.

2. Case II - Degenerate gas

This case corresponds to the limit

\[
\mu \gg \frac{1}{\beta}.
\]

(5.52)

The distribution functions are

\[
f_e = \Theta(p_F - p),
\]

\[
f_T = 0,
\]

(5.53)

where the Fermi momentum is the same as the chemical potential, and its relationship with the electron number density is

\[
n_e = \frac{p_F^3}{3\pi^2},
\]

(5.54)

The formula for \( \pi_L \) in Eq. (5.33) yields

\[
\pi_L = \frac{e^2 n_e}{p_F},
\]

(5.55)

while the formulas for \( \pi_{T,P} \) become

\[
\pi_T = \left( \frac{e^2}{\pi^2} \right) \frac{2}{3} \int_0^{p_F} dp \frac{p^3}{p^2 - \frac{e^2 B}{\omega}}.
\]

\[
\pi_P = \left( \frac{e^2}{\pi^2} \right) \left( \frac{|eB|}{\omega} \right) \frac{1}{3} \int_0^{p_F} dp \frac{p^2}{p^2 - \frac{e^2 B}{\omega}}.
\]

(5.56)

With the instruction that the integrals in Eq. (5.56) are to be evaluated by taking the principal value part, they yield

\[
\pi_T = \left( \frac{e^2 n_e}{p_F} \right) \left( 1 + \frac{\Omega_P^2}{\omega^2} \log \left| 1 - \frac{\omega^2}{\Omega_P^2} \right| \right),
\]

\[
\pi_P = \left( \frac{e^2 n_e}{p_F} \right) \frac{\Omega_P}{\omega} \left( 1 + \frac{\Omega_P}{2\omega} \log \left| \frac{\omega - \Omega_P}{\omega + \Omega_P} \right| \right),
\]

(5.57)

where

\[
\Omega_P \equiv \frac{|eB|}{p_F}.
\]

(5.58)
C. Weak-field limit

By this we mean that $|eB|$ is sufficiently small, compared to an average electron energy $\langle E_e \rangle$ and $\omega$, so that we can literally take the $B \to 0$ limit in Eqs. (4.30) and (4.31), and keep only the term that is linear in $B$. In this case, Eqs. (5.27), (5.28) and (5.29) hold. As shown in Appendix D, using them in Eq. (4.30) and taking the $B \to 0$ limit results in

$$
\pi_T(\omega) = 4|e|^2 \int \frac{d^3p}{(2\pi)^3 2E} \left( f_r(E) + f_s(E) \right) \frac{E^3 - \frac{1}{2}p_0^2}{E^3 - \omega^2/4},
$$

$$
\pi_P(\omega) = 2|e|^2 \left( \frac{2|e|B}{\omega} \right) \int \frac{d^3p}{(2\pi)^3 2E} \left( f_r(E) - f_s(E) \right) \frac{E^3 - \frac{1}{2}p_0^2}{E^3 - \omega^2/4},
$$

$$
\pi_L(\omega) = 4|e|^2 \int \frac{d^3p}{(2\pi)^3 2E} \left( f_r(E) + f_s(E) \right) \frac{E^3 - \frac{1}{2}p_0^2}{E^3 - \omega^2/4}, \quad (5.59)
$$

where we have put $\bar{p}_0^2 \to \frac{2}{3}p_0^2$ in the integrand. For $\omega \ll 2\langle E_e \rangle$, these reduce further to

$$
\pi_T(\omega) = \pi_L(\omega) = 4|e|^2 \int \frac{d^3p}{(2\pi)^3 2E} \left( f_r(E) + f_s(E) \right) \left( 1 - \frac{p_0^2}{3E^2} \right), \quad (5.60)
$$

which is the standard expression for the plasma frequency squared, and

$$
\pi_P(\omega) = \frac{4|e|^2 B_B}{\omega} \int \frac{d^3p}{(2\pi)^3 2E} \left( f_r(E) - f_s(E) \right) \left( 1 - \frac{2p_0^2}{3E^2} \right), \quad (5.61)
$$

both of which can be evaluated explicitly for the various limiting forms of the electron distribution functions that we have considered. We do not proceed any further in this direction, but we mention that the results thus obtained agree with the corresponding ones obtained in the previous sections, in their common range of validity. That is, for example, for a degenerate distribution in the relativistic regime, from Eq. (5.61) we obtain

$$
\pi_P = \frac{|e|^2 B_B}{\omega}, \quad (5.62)
$$

which agrees with the formula that is obtained from Eq. (5.57) in the limit $\Omega_P \ll \omega$ (which corresponds to the weak-field limit there). For a non-relativistic gas, putting $f_r \approx 0$ and neglecting the term $p_0^2/E^2$ in the integrand of Eqs. (5.60) and (5.61),

$$
\pi_P(\omega) = \frac{\omega B}{\omega} \pi_T(\omega), \quad (5.63)
$$

with $\pi_T = \Omega_0^2$, which coincides with the result obtained by taking the weak-field limit ($\omega_B \ll \omega$) in Eq. (5.10). Analogous relations can be verified for other cases as well. However, we emphasize that the results of the previous sections hold for a wider range of conditions than those for which Eq. (5.59), and the relations based on it, hold. The latter are limited to situations in which retaining only the linear terms in $B$ is a valid approximation.

D. Faraday Rotation

The fact that the two transverse modes have different dispersion relations leads to the Faraday rotation effect, as is well known. After traveling a distance $L$, the direction of polarization of the wave has rotated by an angle

$$
\theta = \frac{1}{2} \omega \Delta n L, \quad (5.64)
$$

where

$$
\Delta n = (n_- - n_+), \quad (5.65)
$$

with $n_\pm(\omega)$ being the refractive indices of the left and right polarized modes, respectively. These can be computed from the dispersion relations given above by using $n(\omega) = Q/\omega$. For illustrative purposes, taking as an example the
case of a non-relativistic gas, for propagation parallel to the magnetic field from Eq. (5.14) we find
\[ n_+^2(\omega) = 1 - \frac{\Omega_B^2}{\omega(\omega - \omega_B)}, \]
\[ n_-^2(\omega) = 1 - \frac{\Omega_B^2}{\omega(\omega + \omega_B)}, \] (5.66)
and therefore
\[ n_-^2(\omega) - n_+^2(\omega) = \frac{2\Omega_B^2\omega_B}{\omega^2 - \omega_B^2} \]
Writing
\[ \Delta n = \frac{n_-^2(\omega) - n_+^2(\omega)}{n_-(\omega) + n_+(\omega)),} \] (5.67)
we then obtain
\[ \frac{\theta}{L} = \frac{\Omega_B^2\omega_B}{2n\omega^2}, \] (5.69)
where we have defined \( n(\omega) = (n_-(\omega) + n_+(\omega))/2. \)
For values of \( \omega \) such that \( \omega \gg \Omega_B, \) Eq. (5.66) tells us that \( n_- \approx n_+ \approx 1 \) and therefore
\[ \frac{\theta}{L} \approx \frac{\Omega_B^2\omega_B}{2\omega^2}. \] (5.70)
This result coincides with the one given in Ref.[16]. On the other hand, for \( \omega \gg \omega_B, \) which corresponds to the weak-field approximation[18],
\[ \frac{\theta}{L} \approx \frac{\Omega_B^2\omega_B}{2n\omega^2}, \] (5.71)
where, from Eq. (5.66),
\[ n \approx \sqrt{1 - \frac{\Omega_B^2}{\omega^2}}, \] (5.72)
For other values of \( \omega, \) the formula in Eq. (5.69) interpolates nicely between the two limiting cases we have mentioned.

More generally, for propagation parallel to the magnetic field as we have considered above, the rotation angle is given by Eq. (5.64) with
\[ n_+^2 = 1 - \frac{\pi r \pm \pi p}{\omega^2}. \] (5.73)
Thus, explicit formulas for the angle of rotation can be deduced simply for the other situations of interest using the results for \( \pi r, p \) that we have already obtained for the various cases.

VI. DISCUSSION AND CONCLUSIONS

The subject of the propagation of a photon in a background of particles embedded in an external magnetic field appears in many physical contexts. As we have mentioned in the Introduction, there exist situations of interest for which neither the semiclassical methods nor the linear field-theoretic approach are directly applicable. In these cases, a more general field-theoretic treatment that does not involve the weak field assumption is required.

With this motivation in mind, in this work we have reexamined the subject. We have given a general decomposition of the photon self-energy in a matter background that contains a magnetic field, in terms of the minimal set of tensors consistent with isotropy and the transversality condition. From this result, we have shown that the self-energy can be expressed in terms of nine independent form factors, that in the long wavelength limit reduce to three. In this limit, by applying the (real-time) finite temperature field theory method, we have calculated the one-loop formulas for the form factors. The formulas obtained in this way are valid for arbitrary distributions of the electron gas and for strong magnetic fields. They were explicitly evaluated for a variety of conditions of interest in physical applications, including the weak-field limit as a special case. From them we determined the photon dispersion relations and computed the Faraday rotation for various cases. They reproduce the well-known semi-classical results when the appropriate limits are taken, but they remain valid for more general situations, including those in which the magnetic field is not weak.
APPENDIX A: THERMAL ELECTRON PROPAGATOR

In this section we explain briefly the formula for the electron propagator that will be used in the subsequent calculations. Our starting point is the expression given in the book by Itzykson and Zuber[17]. In vacuum, but in the presence of a constant magnetic field, the electron propagator in coordinate space is given by

\[ S_A(x, x') = \left[ i \partial_x - eA(x) + m \right] (-i) \int_{-\infty}^{0} d\tau U(x, x'; \tau), \]  

(A1)

where \[ U(x, x'; \tau) = \frac{i}{16\pi^2 \tau^2} \phi(x, x') \exp \left\{ \frac{i}{4} (x - x') \epsilon \cosh(\epsilon F\tau)(x - x') + i \left( \frac{\phi}{2} \cdot F + m^2 - i\epsilon \right) \tau \right\} - \frac{i}{2} \text{Tr} \ln \left( (\epsilon F\tau)^{-1} \sinh(\epsilon F\tau) \right), \]  

(A2)

and with the convention that \( \epsilon \) is the charge of the electron. The factor \( \phi(x, x') \) is given in general by

\[ \phi(x, x') = \exp \left\{ -i e \int_{x'}^{x} d\xi^\mu \left[ A_\mu(\xi) + \frac{1}{2} F_{\mu\nu}(\xi - x')^\nu \right] \right\}, \]  

(A3)

and it depends on the gauge but is independent of the path of integration, and it has the form

\[ A_\mu = -\frac{1}{2} F_{\mu\nu} x^\nu. \]  

(A5)

The expression for the propagator in momentum space is obtained by using the formula

\[ e^{i(p - p')_\mu A_\mu(x - x')} = \left( \frac{-i}{2\sqrt{\text{Det}(A)}} \right) \int d^4p' e^{-ip'\cdot\left[ p'^\mu A'^\nu + (x - x')^\mu \right]}, \]  

(A6)

where \( \text{Det}(A) \) stands for the determinant of the matrix formed by the coefficients \( A^\mu_\nu \). For the ensuing manipulations it is useful to observe that

\[ F_{\mu\nu} = iBP_{\mu\nu}, \]  

(A7)

where \( P_{\mu\nu} \) is defined in Eq. (2.15). Using the multiplication rules \( P^2 = R \) and \( R^2 = R \), the following relations are readily derived,

\[ \cosh(\epsilon F\tau) = 1 + R \left[ \cos(s) - 1 \right], \]
\[ \left[ \cosh(\epsilon F\tau) \right]^{-1} = 1 + R \left[ \frac{1}{\cos(s)} - 1 \right], \]
\[ (\epsilon F\tau)^{-1} \sinh(\epsilon F\tau) = 1 + R \left[ \frac{\sin(s)}{s} - 1 \right]. \]  

(A8)

from which the rest of the formulas that we need can be obtained. In particular,

\[ \left[ (\epsilon F\tau) \cosh(\epsilon F\tau) \right]^{-1} = 1 + R \left[ \frac{\tan(s)}{s} - 1 \right], \]
\[ \text{Det} \left( \left[ (\epsilon F\tau) \cosh(\epsilon F\tau) \right]^{-1} \right) = \left( \frac{\tan(s)}{s} \right)^2, \]
\[ \text{Tr} \ln \left( (\epsilon F\tau)^{-1} \sinh(\epsilon F\tau) \right) = \ln \left( \frac{\sin(s)}{s} \right)^2, \]  

(A9)
where $s$ is the variable defined in Eq. (3.6).

Thus, using Eqs. (A6) in Eq. (A2), with the help of Eq. (A9) we obtain

$$U(x, x'; \tau) = \frac{1}{\cos(s)} \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-x')} e^{-i[\Phi(p, s) - \Phi(s, s) + iF + i\mathbb{F}]}.$$

(A10)

with $\Phi$ as defined in Eq. (3.4) in the text. For the gauge choice given in Eq. (A5), which implies Eq. (A4), it follows that

$$(i\partial_x - eA + m)U(x, x'; \tau) = \frac{1}{\cos(s)} \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-x')} [p - \tan(s)iP_{\mu\nu}\gamma^\nu p' + m] e^{-i[\Phi(p, s) - \Phi(s, s) + iF + i\mathbb{F}]}.$$

(A11)

Using the relation

$$e^{i\sigma F} = \cos(\epsilon B\tau) + i\gamma_5\not{b} \sin(\epsilon B\tau),$$

(A12)

which follows by noticing that

$$\frac{1}{2}\sigma \cdot F = B\gamma_5\not{b},$$

(A13)

together with the fact that $(\gamma_5\not{b})^2 = 1$, Eq. (A1) then yields

$$S_A(x, x') = \frac{1}{\cos(s)} \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-x')} S_F(k),$$

(A14)

where

$$iS_F(p) = \int_0^\infty d\tau G(p, s)e^{i[\Phi(p, s) - \epsilon\tau]},$$

(A15)

with

$$G(p, s) = [p - \tan(s)iP_{\mu\nu}\gamma^\nu p' + m][1 + i\gamma_5\not{b}\tan(s)].$$

(A16)

In writing Eq. (A15), we have made the change of variable $\tau \to -\tau$. Using the identity

$$P_{\mu\nu}\gamma^\nu k' = \gamma_5\not{b} + (k\cdot u)\gamma_5\not{b} - (k\cdot b)\gamma_5\not{b},$$

(A17)

$G$ can be expressed in the final form given in Eq. (3.4) in the text. Finally, the $11$ component of the thermal propagator is obtained from the formula

$$S_{\epsilon}(p) = \tilde{S}_F(p) - \tilde{S}_F(p)\tilde{S}_A(p)\not{1}\not{1},$$

(A18)

where $\not{1}$ is the background-dependent factor defined in Eq. (3.9) and, as usual, $\tilde{S}_F = \frac{1}{2\epsilon^2}$. This leads to the decomposition of $S_{\epsilon}$ given in Eq. (3.2), with the thermal part $S_{\epsilon}$ as defined in Eq. (3.8).

We would like to mention the following point. Strictly, the Fourier expansion given in Eq. (A10) holds only for those values of $s$ for which $\cos(s) \neq 0$. At the points where $\cos(s) = 0$, the expansion given in Eq. (A10) is not valid, and the integration over $\tau$ in Eq. (A15) is not defined at these points as it stands. Therefore, we define the $\tau$ integration in Eq. (A15), by deforming the line of integration to lie just below the real axis, so that the points in question are avoided. It is not difficult to see that this choice, as well any other that avoids those points (such as, for example, choosing the integration line to lie just above the real axis), amounts to a redefinition of the function $\phi(x, x')$.

**APPENDIX B: EVALUATION OF $J_0$**

In order to evaluate the integral $J_0$ defined in Eq. (3.21), we first observe that, as a function of $\psi$, $J_0$ is analytic in the upper-half plane. Therefore, if we can evaluate it for a particular subregion of that plane, the result is valid for the entire region by analytic continuation. For the subregion $\Re \psi < 0$ (or, equivalently, Euclidean momenta $p_0$) we proceed as follows.
The integrand is an analytic function of $s$ in the lower-half $s$-plane. Therefore we can deform the integration path and integrate over the negative imaginary axis. Then setting

$$s = -it,$$

we have

$$J_0 = -\frac{i}{\varepsilon |B|} \int_0^\infty dt e^{\psi} e^{-u} e^{\frac{2\pi ik}{\varepsilon |B|}},$$

(B2)

where

$$u \equiv e^{-2t},$$

(B3)

and we have used

$$-i \tan(-it) = \frac{2u}{1 + u} - 1.$$

(B4)

Using the generating formula of the Laguerre polynomials

$$e^{-\frac{x}{1-z}} \prod_{n=0}^\infty \frac{L_n(x)z^n}{n!},$$

(B5)

(valid for $0 \leq x < \infty$ and $|z| < 1$), equation (B2) becomes

$$J_0 = -\frac{i}{\varepsilon |B|} \int_0^\infty dt e^{\psi} \sum_{n=0}^\infty D_n(a) e^{2\pi in},$$

(B6)

where $D_n(a)$ is the function defined in Eq. (3.26) in the text. Equation (B6) is easily integrated term by term to yield the final result quoted in Eq. (3.25), which also holds for $\Re \psi > 0$ (or equivalently, Minkowskian values of $\eta$) by analytic continuation.

**APPENDIX C: LOW FREQUENCY LIMIT**

To arrive at Eqs. (5.4) and (5.5) we apply the low frequency conditions to Eq. (4.31) as follows. Let us define

$$D_\pm \equiv \frac{1}{\omega^2 + 2\omega E_n - 2|\varepsilon|B} \pm \frac{1}{\omega^2 - 2\omega E_n - 2|\varepsilon|B}.$$  

(C1)

By combining the denominators,

$$D_+ = \frac{2(\omega^2 - 2|\varepsilon|B)}{(\omega^2 - 2|\varepsilon|B)^2 - 4\omega^2 E_n^2},$$

$$D_- = \frac{-4\omega E_n}{(\omega^2 - 2|\varepsilon|B)^2 - 4\omega^2 E_n^2}.$$  

(C2)

Now define the combinations

$$\Delta_+^{(\pm)} = D_+ \pm (B \rightarrow -B),$$

$$\Delta_-^{(\pm)} = D_- \pm (B \rightarrow -B),$$  

(C3)

in terms of which the formulas for $\pi_{\nu,\nu'}^{(n)}$ given in Eq. (4.31) are expressed as

$$\pi_{\nu,\nu'}^{(n)}(\omega) = \int_{-\infty}^{\infty} \frac{d\eta}{E_n} \left( f_{\nu}(E_n) + f_{\nu'}(E_n) \right) \left\{ \omega E_n \Delta_+^{(\pm)} + 2\eta |\varepsilon|B \Delta_-^{(\pm)} \right\},$$

$$\pi_{\nu}(\omega) = \int_{-\infty}^{\infty} d\eta \left( f_{\nu}(E_n) + f_{\nu}(E_n) \right) \{ \omega D_+ \},$$

$$\pi_{\nu}^{(n)}(\omega) = \int_{-\infty}^{\infty} d\eta \left( f_{\nu}(E_n) - f_{\nu}(E_n) \right) \left\{ \omega E_n \Delta_-^{(\pm)} + 2\eta |\varepsilon|B \Delta_-^{(\pm)} \right\},$$

$$\pi_{\nu}'(\omega) = \int_{-\infty}^{\infty} d\eta \left( f_{\nu}(E_n) - f_{\nu}(E_n) \right) \{ \omega D_- \}.  $$

(C4)
From their definitions, and using Eq. (C1),

\[
\begin{align*}
\Delta_+^{(\pm)} &= \frac{2(\omega^2 - 2|\epsilon|B)}{\omega^2 - 2|\epsilon|B^2 / 4\omega^2 E_n^2} \pm \frac{2(\omega^2 + 2|\epsilon|B)}{(\omega^2 + 2|\epsilon|B^2 / 4\omega^2 E_n^2),\\
\Delta_-^{(\pm)} &= \frac{-4\omega E_n}{\omega^2 - 2|\epsilon|B^2 / 4\omega^2 E_n^2} \pm \frac{-4\omega E_n}{(\omega^2 + 2|\epsilon|B^2 / 4\omega^2 E_n^2).}
\end{align*}
\]  

It is seen that all the \(\Delta\)'s can be approximated by letting \((\omega^2 - 2|\epsilon|B^2) \rightarrow 4|\epsilon|^2 B^2\) in the denominators, except \(\Delta_-^{(\pm)}\), which would give zero. This in particular means that we cannot take the low frequency limit in Eq. (4.31) by just setting \(\omega^2 \rightarrow 0\) in the denominators, or otherwise we would miss some important terms. In order to make a systematic expansion of \(\Delta_-^{(\pm)}\) for \(\omega \ll \sqrt{|\epsilon|B}\), we use the relation

\[
D_- = \frac{1}{\omega E_n} - \frac{\omega^2 - 2|\epsilon|B}{2\omega E_n} D_+, \tag{C6}
\]

which follows from using the trivial identity

\[
\frac{1}{x + y} - \frac{1}{x - y} = \frac{1}{y} \left[ 2 - x \left( \frac{1}{x + y} + \frac{1}{x - y} \right) \right]. \tag{C7}
\]

Then going back to the definition in Eq. (C3), we can re-express \(\Delta_-^{(\pm)}\) as

\[
\Delta_-^{(\pm)} = -\left( \frac{\omega}{2\omega E_n} \right) \Delta_+^{(\pm)} + \left( \frac{|\epsilon|B}{\omega E_n} \right) \Delta_+^{(\pm)}. \tag{C8}
\]

Up to this point the relations are exact. Now we can proceed to make the approximation \(\omega \ll \sqrt{|\epsilon|B}\) systematically, by letting \((\omega^2 \pm 2|\epsilon|B) \rightarrow 2|\epsilon|B\) in Eq. (C5), except for \(\Delta_-^{(\pm)}\), for which we use Eq. (C8). Thus,

\[
\begin{align*}
\Delta_+^{(+)} &= \frac{-\omega^2}{\omega^2 E_n^2 - |\epsilon|^2 B^2}, \\
\Delta_+^{(-)} &= \frac{2|\epsilon|B}{\omega E_n^2 - |\epsilon|^2 B^2}, \\
\Delta_-^{(+)} &= \frac{2\omega E_n}{\omega^2 E_n^2 - |\epsilon|^2 B^2}, \\
\Delta_-^{(-)} &= \left( \frac{2\omega |\epsilon|B}{E_n} \right) \frac{1}{\omega^2 E_n^2 - |\epsilon|^2 B^2}. \tag{C9}
\end{align*}
\]

Similarly, from Eq. (C2)

\[
\begin{align*}
D_+ &= \frac{|\epsilon|B}{\omega^2 E_n^2 - |\epsilon|^2 B^2}, \\
D_- &= \frac{\omega E_n}{\omega^2 E_n^2 - |\epsilon|^2 B^2}. \tag{C10}
\end{align*}
\]

The formulas quoted in Eq. (5.4) follow by substituting these approximate expressions for the \(\Delta\)'s and the \(D\)'s into Eq. (C4).

For \(\pi_L^{(n)}(\omega)\) we assume that \(\omega \ll 2\langle E_e \rangle\) to approximate

\[
\frac{1}{\omega^2 + 2\omega E_n} + \frac{1}{\omega^2 - 2\omega E_n} \approx -\frac{1}{2E_n^2}, \tag{C11}
\]

from which the formula in Eq. (5.5) follows.
APPENDIX D: THE WEAK-FIELD (LINEAR) LIMIT

In order to obtain the weak-field limit formulas quoted in Eq. (5.59), we use the expressions given in Eq. (C4) for \( \pi_{T,p}(\omega) \). From Eq. (C5), as \( B \to 0 \)

\[
\Delta_{+}^{(+)} \to \frac{2\omega E}{\omega^{2}(E^{2} - \omega^{2}/4)},
\]

\[
\Delta_{-}^{(+)} \to -\frac{1}{E^{2} - \omega^{2}/4},
\]

(D1)

Then, remembering that Eq. (5.29) holds,

\[
\pi_{T}(\omega) = 2\epsilon^{2} \int_{-\infty}^{\infty} \frac{d^{3}p}{(2\pi)^{3}2E} \left(f_{\tau}(E) + f_{\bar{\tau}}(E)\right) \left(\frac{1}{E^{2} - \omega^{2}/4}\right) \left[2E^{2} - \frac{p_{z}^{2}}{2}\right],
\]

(D2)

which yields the formula given in Eq. (5.59), after replacing \( \frac{p_{z}^{2}}{2} \to \frac{1}{2}p^{2} \) in the integrand. We proceed similarly for \( \pi_{p} \). From Eq. (C5),

\[
\Delta_{+}^{(-)} \to \frac{-8|\epsilon|B}{\omega^{2} - 4\omega^{2}/E^{2}},
\]

\[
\Delta_{-}^{(-)} \to -\left(\frac{\omega^{2}}{2\omega E}\right) \Delta_{+}^{(-)} + \frac{|\epsilon|B}{\omega E} \Delta_{+}^{(+)}
\]

(D3)

where we have used Eq. (C8) for \( \Delta_{-}^{(-)} \). Then substituting in Eq. (C4),

\[
\pi_{P}(\omega) = 2\epsilon^{2} \int_{-\infty}^{\infty} \frac{d^{3}p}{(2\pi)^{3}2E} \left(f_{\tau}(E) + f_{\bar{\tau}}(E)\right) \left[4m_{c}^{2} + 4p_{z}^{2}\right] \left[E^{2} - \omega^{2}/4\right].
\]

(D4)

Finally, for \( \pi_{L}(\omega) \) we apply Eq. (5.29) directly in Eqs. (4.30) and (4.31) and obtain

\[
\pi_{L}(\omega) = 4\epsilon^{2} \int_{-\infty}^{\infty} \frac{d^{3}p}{(2\pi)^{3}2E} \left(f_{\tau}(E) + f_{\bar{\tau}}(E)\right) [4m_{c}^{2} + 4p_{z}^{2}] \left(E^{2} - \omega^{2}/4\right).\]

(D5)

which can be expressed as quoted in Eq. (5.59).

[18] This result agrees with what can be deduced from the results given in Ref. [11] for the non-relativistic gas. Notice that the quantity denoted by \( d\Phi/dl \) in that reference corresponds to \( \omega \Delta \pi \) in our notation, and is twice the rotation angle.
[19] Our formula for \( U \) differs from the one quoted in the book cited, by a factor of \( \epsilon \) in the term \( \frac{\pi \sigma}{2} F \tau \), which we believe that it is missing there, and in the overall sign, which believe is an error in the determination of the normalization constant \( C \) in the book.