A Test of The Source Galerkin Method

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Some results of the ongoing development of our Source Galerkin (SG) nonperturbative approach to numerically solving Quantum Field theories are presented. This technique has the potential to be much faster than Monte Carlo methods. SG uses known symmetries and theoretical properties of a theory. In order to test this approach, we applied it to \( \phi^4 \) theory in zero dimensions. This model has been extensively studied and has a known set of exact solutions. This allows us to broaden the understanding of various properties of the SG method and to develop techniques necessary for the successful application of this method to more sophisticated theories.

1. Introduction.

The Source Galerkin method is being developed as a flexible alternative to Monte Carlo approaches to solving Quantum Field Theories.

1.1. Overview of the Method

To illustrate the application of the Source Galerkin approach to solving Quantum Field Theories we consider a theory defined by a Lagrangian \( \mathcal{L} \). All information of the theory is contained in the generating functional

\[
Z(j) = \int \mathcal{D}\phi \exp \left[ \int (-\mathcal{L} + j\phi)d^4x \right].
\]  (1)

This satisfies a Schwinger-Dyson set of differential equations. A procedure to determine an approximate solution of these equations is specified by our SG technique. To start, a trial solution or an ansatz must be introduced. In general, it is constructed from a predefined set of trial functions. The proper choice of trial functions allows us to take advantage of symmetries of the theory as well as of other known analytical properties of the generating functional. A simple example of an approximate solution is:

\[
Z_a = \exp \left[ \int j_x G_{xy} j_y + \int j_w j_x H_{wxyz} j_y j_z + \ldots \right].
\]  (2)

A set of residues is obtained by substituting this ansatz into the Schwinger-Dyson equations. The undefined coefficients of the trial solution are determined by solving a system of nonlinear equations obtained by projecting the residues on the trial functions and requiring these projections to be equal to zero. The procedure outlined above guarantees that the error associated with the approximate solution converges to zero in the mean as the number of members in the set of trial functions goes to infinity.

1.2. Motivation

In order to demonstrate the validity of the method, it was applied to the \( O(3) \) nonlinear \( \sigma \) model [1]. This model is asymptotically free and is a useful toy model for approaching Non-Abelian gauge theories. First order solutions were obtained, but during our attempts to extend this work to higher orders it became evident that a better understanding of the general properties of the SG method is necessary. Theorems guarantee that Galerkin methods produce an approximate solution, which converges to the exact solution as the number of terms goes to infinity. In practice, the approximate solution can only have a fairly small number of parameters. Therefore, we must ensure that high accuracy and rapid convergence can reasonably be achieved with only a few terms in the ansatz. The fact that the error approaches zero in the mean implies that the accuracy of the result is heavily affected by the choice of inner product. We attempt to investigate the effectiveness of the Source Galerkin approach by using it to solve \( \phi^4 \) theory in zero dimensions. We study its performance for several choices of trial func-
tions and scalar products.

2. Ultralocal Model

After introducing an external source \( j \), we define the ultralocal model by the following lagrangian

\[
\mathcal{L} = \frac{g}{4} \phi^4 + \frac{\mu}{2} \phi^2 - j \phi. \tag{3}
\]

Theoretical solutions for this theory can be obtained by expressing generating functional as a power series

\[
Z_a(j) = \sum_{k=0}^{\infty} \frac{1}{k!} a_k j^k. \tag{4}
\]

The coefficients of this series are given by \[2\]

\[
a_{2n} = (2n-1)!! \frac{U(n, t) + (-1)^n \rho U(n, -t)}{U(0, t) + \rho U(0, -t)},
\]

\[
a_{2n+1} = \frac{2n!!}{n!} (-1)^n \frac{V(n + \frac{1}{2}, t)}{V(\frac{1}{2}, t)} \frac{\epsilon^2 e^{\frac{\epsilon^2}{2}}}{U(0, t) + \rho U(0, -t)}.
\]

Here \( U \) and \( V \) are parabolic cylinder functions, coefficients \( \rho \) and \( \alpha \) fix boundary conditions.

2.1. Numerical Solution

In analogy with the theoretical solution, a truncated power series of order \( N \) can be chosen as a trial solution. After obtaining the residues by substituting this expression into the Schwinger-Dyson equations, we need to eliminate source dependence by projecting them on the trial functions. Two possible ways to define scalar product are presented below.

\[
\int_{-c}^{c} j^n R(j) dj, \quad n = 0..N - 3 \tag{5}
\]

and

\[
\int_{-\infty}^{\infty} j^n R(j) \exp \left[ -\frac{j^2}{\epsilon^2} \right] dj, \quad n = 0..N - 3. \tag{6}
\]

Both definitions are localized in the region close to \( j = 0 \) since eventually we want to set the source to zero in order to compute physical values. Using these definitions produces similar results as shown in figure 1. However, the second expression is more general and it is easier to extend to calculations with space-time dimensions. Inspection of these results shows that the error in determination of coefficients \( a_i \) increases rapidly at high orders. In spite of this, the generating functional can be determined with accuracy as high as several parts in \( 10^7 \) for a significant range of \( j \). However, it must be noted that correct determination of the proper range of integration or a value of the parameter \( \epsilon \) is crucial for achieving high accuracy. The SG approach by itself does not provide an algorithm for setting these parameters. In a real problem when the exact result is not available it is necessary to have some external source of information which allows us to determine the correct values of the parameters.

2.2. Solution in Terms of Hermite Polynomials

A set of Hermite polynomials can be defined by

\[
H_n(\xi) = (-1)^n \exp \left\{ \frac{\xi^2}{\epsilon^2} \right\} \frac{d^n}{d \xi^n} \exp \left\{ -\frac{\xi^2}{\epsilon^2} \right\}. \tag{7}
\]
Figure 2. Order by order relative error is plotted for $N = 7$ and $N = 5$. Residual equations derived from $\int_{-\infty}^{\infty} H_n(j)R(j) \exp \left[ -\frac{j^2}{\epsilon^2} \right] dj$ with respect to the parameter $\epsilon$.

These polynomials are orthonormal under the following inner product

$$\int_{-\infty}^{\infty} H_k(x)H_l(x) \exp \left( -\frac{x^2}{\epsilon^2} \right) dx = \delta^{kl}. \quad (8)$$

This suggests a way to improve our implementation of the Source Galerkin procedure for the ultralocal theory. Trial functions can be constructed as a linear combination of Hermite polynomials of different orders, while the expression shown above is used to define the scalar product. This choice leads to significant simplification of the system of equations which give the coefficients in the ansatz. Figure 2 shows the dependence of relative error in determination of values $a_i$ on the parameter $\epsilon$. From this graph we observe that there is a significant range of values of the parameters in which numerical results remain stable. We can use this fact to resolve the problem outlined in the end of previous section. In a real problem, it would be possible to set all the parameters without any knowledge of theoretical solution just by finding solutions for all possible values of parameters and then choosing a solution which is stable within the required accuracy for some range of the parameters. The accuracy of determination of the generating functional using this method is consistent with the power series ansatz approach. This implies that it is limited only by error introduced by the method used to solve the system of equations produced by the Source Galerkin procedure and by the numerical precision of the hardware used for computation.

3. Conclusion

This investigation demonstrates that very high precision computations can be performed using the Source Galerkin procedure. On the other hand, the method is very sensitive to the choice of parameters. Application of this method to a more general set of problems will require development of some consistent method of determination of proper values of parameters. Current results suggest that use of orthogonal trial functions may simplify this task. Source Galerkin technique is still under development. Additional work is needed to fully investigate it’s properties and to apply it to various field theories.

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