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Abstract

We apply supersymmetric discrete light-cone quantization (SDLCQ) to the study of supersymmetric Yang–Mills-Chern–Simons (SYM-CS) theory on $\mathbb{R} \times S^1 \times S^1$. One of the compact directions is chosen to be light-like and the other to be space-like. Since the SDLCQ regularization explicitly preserves supersymmetry, this theory is totally finite, and thus we can solve for bound-state wave functions and masses numerically without renormalizing. The Chern–Simons term is introduced here to provide masses for the particles while remaining totally within a supersymmetric context. We examine the free, weak and strong-coupling spectrum. The transverse direction is discussed as a model for universal extra dimensions in the gauge sector. The wave functions are used to calculate the structure functions of the lowest mass states. We discuss the properties of Kaluza–Klein states and focus on how they appear at strong coupling. We also discuss a set of anomalously light states which are reflections of the exact Bogomol’nyi–Prasad–Sommerfield states of the underlying SYM theory.
1 Introduction

In 2+1 dimensions Chern–Simons (CS) theories are some of the most interesting field theories. For a review of these theories see [1]. There is also considerable work on SYM-CS theories. These theories have their own remarkable properties. It has been shown that there is a finite anomaly that shifts the CS coupling [2], and it has been conjectured by Witten [3] that this theory spontaneously breaks supersymmetry for some values of the CS coupling.

It is certainly useful to numerically simulate these theories, since they are interesting from so many different points of view. The method we will use is SDLCQ (Supersymmetric Discrete Light-Cone Quantization). This is a numerical method that can be used to solve any theory with enough supersymmetry to be finite. The central point of this method is that using DLCQ [4, 5] we can construct a finite dimensional representation of the superalgebra [6]. From this representation of the superalgebra, we construct a finite-dimensional Hamiltonian which we diagonalize numerically. We repeat the process for larger and larger representations and extrapolate the solution to the continuum.

We have already solved (2+1)-dimensional supersymmetric Yang–Mills (SYM) theories by this method [7, 8] and also the dimensionally reduced SYM-CS theory [9, 10]. Two-dimensional CS theory is very QCD-like. The states with more partons tend to be more massive, and many of the low-mass states have a valence-like structure. The states have a dominant component of the wave function with a particular number and type of parton. We found that (1+1) and (2+1)-dimensional \( \mathcal{N} = 1 \) SYM theories have an interesting set of massless Bogomol’nyi–Prasad–Sommerfield (BPS) bound states [7, 8]. These states are reflected in the (1+1)-dimensional CS theory as a set of states whose masses are approximately independent of the YM coupling \( g \) and equal to the square of the sum of the CS masses of the partons in the bound state [9, 10]. Massless BPS states are also present in the (2+1)-dimensional SYM theory, and we again see their reflection in the (2+1)-dimensional SYM-CS theory [11]. They have anomalously light masses, and we have already published some results for one of these states [11]. We will report more results here.

String theory suggests that a fundamental theory of the world has more than 3+1 dimensions. It has therefore become interesting to consider how higher dimensional theories behave when viewed from a universe with a smaller number of dimensions. In particular the momentum of a particle in the extra dimension plays the role of an additional mass in lower dimensions. Excitations of this degree of freedom are called Kaluza–Klein (KK) particles. Finding such KK particles could be considered a sign of extra dimensions. Since we are focusing here on (2+1) and (1+1)-dimensional theories, it is natural to think of the transverse dimension as the extra dimension and look at its effect in the two-dimensional world. We can view the spectrum of this model as a model of a (3+1)-dimensional world that is part of a (4+1)-dimensional “fundamental” theory. From this point of view, the states with nonzero transverse momentum are KK states, i.e. states that have contributions to their mass from the transverse momentum in the “extra” dimension. This is a useful model to have because it allows us to identify KK particles that are a part of bound states and ask how they will look both in the higher dimensional space and in the reduced dimensional space. This will be one of the
main themes when we analyze the bound states of this theory.

We investigate the various physical effects that we just discussed in terms of the structure functions of the bound states. We find that the anomalously light states, which are a reflection of the BPS states of the SYM theory, have a unique structure function. This gives hope that such states might be disentangled from the rest of the spectrum. We discuss the structure functions of KK states both at weak and strong coupling. We also present the structure functions of the states that are mixtures of KK states and low-energy states.

In constructing the discrete approximation we drop the longitudinal zero-momentum mode. For some discussion of dynamical and constrained zero modes, see the review [5] and previous work [12]. Inclusion of these modes would be ideal, but the techniques required to include them in a numerical calculation have proved to be difficult to develop, particularly because of nonlinearity. For DLCQ calculations that can be compared with exact solutions, the exclusion of zero modes does not affect the massive spectrum [5]. In scalar theories it has been known for some time that constrained zero modes can give rise to dynamical symmetry breaking [5], and work continues on the role of zero modes and near zero modes in these theories [13].

Dropping the zero modes in CS theory does result in the loss of many of the interesting aspects of CS theory, most notably the quantization of the CS coupling. However, one particularly interesting property that will be preserved is the fact that the CS term simulates a mass for the theory. It is well known that supersymmetric abelian CS theory is simply the theory of a free massive fermion and a free massive boson. In the non-abelian theory additional interactions are introduced, but we expect to also see this mass effect. This is particularly interesting here because reduced $\mathcal{N} = 1$ SYM theories are very stringy. The low-mass states are dominated by Fock states with many constituents, and as the size of the superalgebraic representation is increased, states with lower masses and more constituents appear [7, 8, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The connection between string theory and supersymmetric gauge theory leads one to expect this type of behavior; however, these gauge theories are not very QCD-like. Ultimately one might like to make contact with the low-mass spectrum observed in nature.

The remainder of the paper is structured as follows. In Sec. 2 we provide a summary of SYM-CS theory and our numerical approximation; much of this is given in [9] but is repeated here for completeness. In Sec. 3, we set the stage by reviewing what we already know about the pieces of this theory, including the free discrete spectrum of pure CS theory. In Sec. 4 we present our main results. A discussion of weak coupling, which includes a nice picture of the KK modes, is given in Sec. 4.1. The coupling dependence of the theory and our results at strong coupling are discussed in Sec. 4.2. Then in Sec. 4.3 we define structure functions for the bound states and discuss in terms of them the various physical phenomena. Section 5 contains a summary of our conclusions and some discussion of what could be done next.
We consider $N = 1$ supersymmetric CS theory in 2+1 dimensions. The Lagrangian of this theory is

$$\mathcal{L} = \text{Tr}(-\frac{1}{4} \mathcal{L}_{YM} + i \mathcal{L}_F + \frac{\kappa}{2} \mathcal{L}_{CS}), \quad (1)$$

where $\kappa$ is the CS coupling and

$$\mathcal{L}_{YM} = F_{\mu\nu} F^{\mu\nu}, \quad (2)$$

$$\mathcal{L}_F = \bar{\Psi} \gamma_\mu D^\mu \Psi, \quad (3)$$

$$\mathcal{L}_{CS} = \epsilon^{\mu\nu\lambda} \left( A_\mu \partial_\nu A_\lambda + \frac{2i}{3} g A_\mu A_\nu A_\lambda \right) + 2 \bar{\Psi} \Psi. \quad (4)$$

The two components of the spinor $\Psi = 2^{-1/4}(\psi \chi)$ are in the adjoint representation of $U(N_c)$ or $SU(N_c)$ and are the chiral projections of the spinor $\Psi$, also defined by

$$\psi = \frac{1 + \gamma^5}{2^{1/4}} \Psi, \quad \chi = \frac{1 - \gamma^5}{2^{1/4}} \Psi. \quad (5)$$

We will work in the large-$N_c$ limit. The field strength and the covariant derivative are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \quad D_\mu = \partial_\mu + ig[A_\mu, \ ] \quad (6)$$

The supersymmetric variations of the fields are

$$\delta A_\mu = i \bar{\epsilon} \gamma_\mu \Psi, \quad (7)$$

$$\delta \Psi = \frac{1}{4} i \epsilon^{\mu\nu\lambda} \gamma_\lambda F_{\mu\nu} = \frac{1}{4} \Gamma^{\mu\nu} \epsilon F_{\mu\nu}, \quad (8)$$

where

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i \sigma_1, \quad \gamma^2 = i \sigma_3, \quad \Gamma^{\mu\nu} \equiv \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} = i \epsilon^{\mu\nu\lambda} \gamma_\lambda. \quad (9)$$

This leads to the supercurrent $Q^{(\mu)}$ in the usual manner via

$$\delta \mathcal{L} = \bar{\epsilon} \partial_\mu Q^{(\mu)}. \quad (10)$$

Light-cone coordinates in 2+1 dimensions are $(x^+, x^-, x^\perp)$ where $x^+ = x_-$ is the light-cone time and $x^\perp = -x_\perp$. The totally anti-symmetric tensor is defined by $\epsilon^{+\perp\perp} = -1$. The variations of the three parts of the Lagrangian in Eq. (1) determine the (‘chiral’) components $Q^\pm$ of the supercurrent via Eq. (10) to be

$$\int d^2 x Q^{(+)} = \begin{pmatrix} Q^+ \\ Q^- \end{pmatrix} = \frac{i}{2} \int d^2 x \Gamma^{\alpha\beta} \gamma^+ \Psi F_{\alpha\beta}. \quad (11)$$

Explicitly they are

$$Q^- = -i 2^{3/4} \int d^2 x \psi \left( \partial^+ A^- - \partial^- A^+ + ig[A^+, A^-] \right), \quad \text{and} \quad (12)$$

$$Q^+ = -i 2^{5/4} \int d^2 x \psi \left( \partial^+ A^2 - \partial^2 A^+ + ig[A^+, A^+] \right). \quad (12)$$
One can convince oneself by calculating the energy-momentum tensor $T^{\mu\nu}$ that the supercharge fulfills the supersymmetry algebra
\begin{equation}
\{Q^+, Q^-\} = 2\sqrt{2}P^\perp, \quad \{Q^+, Q^-\} = -4P^\perp. \tag{13}
\end{equation}

In order to express the supercharge in terms of the physical degrees of freedom, we have to use equations of motion, some of which are constraint equations. The equations of motion for the gauge fields are
\begin{equation}
D_\nu F^{\alpha\nu} = -J^\alpha, \tag{14}
\end{equation}
where
\begin{equation}
J^\alpha = \frac{\kappa}{2} \epsilon^{\alpha\lambda\mu} F_{\lambda\mu} + 2g \bar{\Psi} \gamma^\alpha \Psi. \tag{15}
\end{equation}
For $\alpha = +$ this is a constraint for $A^-$,
\begin{equation}
D_- A^- = -(D_2 - \kappa) A^\perp - \frac{1}{D_-} (D_2 - \kappa) \partial_2 A^+ + 2g \frac{1}{D_-} \bar{\Psi} \gamma^+ \Psi. \tag{16}
\end{equation}
In light-cone gauge, $A^+ = 0$, this reduces to
\begin{equation}
D_- A^- = \frac{1}{D_-} [(\kappa - D_2) D_- A^\perp + 2g \bar{\Psi} \gamma^+ \Psi]. \tag{17}
\end{equation}
The equation of motion for the fermion is
\begin{equation}
\gamma^\mu D_\mu \Psi = -i\kappa \Psi. \tag{18}
\end{equation}
Expressing everything in terms of $\psi$ and $\chi$ leads to the equations of motion
\begin{align}
\sqrt{2} D_+ \psi &= (D_2 + \kappa) \chi, \\
\sqrt{2} D_- \chi &= (D_2 - \kappa) \psi,
\end{align}
the second of which is a constraint equation. The constraint equations are used to eliminate $\chi$ and $A^-$. At large $N_c$ the canonical (anti-)commutators for the propagating fields $\phi \equiv A_\perp$ and $\psi$ are, at equal light-cone time $x^+$,
\begin{equation}
[\phi_{ij}(x^-, x_\perp), \partial_- \phi_{kl}(y^-, y_\perp)] = \{\psi_{ij}(x^-, x_\perp), \psi_{kl}(y^-, y_\perp)\} = \frac{1}{2} \delta(x^- - y^-) \delta(x_\perp - y_\perp) \delta_{ij} \delta_{jk}. \tag{21}
\end{equation}
The expansions of the field operators in terms of creation and annihilation operators for the Fock basis are
\begin{align}
\phi_{ij}(0, x^-, x_\perp) &= \\
\psi_{ij}(0, x^-, x_\perp) &= \frac{1}{\sqrt{2\pi L}} \sum_{n_{\pm} = -\infty}^{\infty} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left[ a_{ij}(k^+, n^\perp) e^{-ik^+ x^- + i\frac{2\pi n^\perp}{L} x_\perp} + a_{ji}^\dagger(k^+, n^\perp) e^{ik^+ x^- - i\frac{2\pi n^\perp}{L} x_\perp} \right], \\
\psi_{ij}(0, x^-, x_\perp) &= \\
\phi_{ij}(0, x^-, x_\perp) &= \frac{1}{\sqrt{2\pi L}} \sum_{n_{\pm} = -\infty}^{\infty} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left[ b_{ij}(k^+, n^\perp) e^{-ik^+ x^- + i\frac{2\pi n^\perp}{L} x_\perp} + b_{ji}^\dagger(k^+, n^\perp) e^{ik^+ x^- - i\frac{2\pi n^\perp}{L} x_\perp} \right].
\end{align}
From the field (anti-)commutators one finds
\[
[a_{ij}(p^+, n_\perp), a^\dagger_{lk}(q^+, m_\perp)] = \{ b_{ij}(p^+, n_\perp), b^\dagger_{lk}(q^+, m_\perp) \} = \delta(p^+ - q^+)\delta_{n_\perp m_\perp} \delta_{il} \delta_{jk}. \tag{22}
\]

Notice that the compactification in \( x^\perp \) means that the transverse momentum modes are summed over a discrete set of values \( 2\pi n_\perp/L \). In order to have a finite matrix representation for the eigenvalue problem, we must truncate these sums at some fixed integers \( \pm T \). The value of \( T \) defines a physical transverse cutoff \( \Lambda_\perp = 2\pi T/L \); however, given this definition, \( T \) can be viewed as a measure of transverse resolution at fixed \( \Lambda_\perp \).

The supercharge \( Q^+ \) takes the following form:
\[
Q^+ = i2^{1/4} \sum_{|n_\perp| \leq T} dk \sqrt{k} \left[ b^\dagger_{ij}(k, n_\perp) a_{ij}(k, n_\perp) - a^\dagger_{ij}(k, n_\perp) b_{ij}(k, n_\perp) \right]. \tag{23}
\]

The supercharge \( Q^- \) can be written as
\[
Q^- = gQ_{\text{SYM}}(T) + Q_\perp(T) + i\kappa Q_{\text{CS}}(T), \tag{24}
\]
where
\[
Q_\perp(T) = \frac{2^{3/4} \pi i}{L} \sum_{|n_\perp| \leq T} dk \frac{n_\perp \sqrt{k}}{k} \left[ a^\dagger_{ij}(k, n_\perp) b_{ij}(k, n_\perp) - b^\dagger_{ij}(k, n_\perp) a_{ij}(k, n_\perp) \right], \tag{25}
\]
\[
Q_{\text{CS}}(T) = 2^{-1/4} \sum_{|n_\perp| \leq T} dk \frac{1}{\sqrt{k}} \left[ a^\dagger_{ij}(k, n_\perp) b_{ij}(k, n_\perp) + b^\dagger_{ij}(k, n_\perp) a_{ij}(k, n_\perp) \right], \tag{26}
\]
and
\[
Q_{\text{SYM}}^- = \frac{i2^{-5/4}}{\sqrt{L}\pi} \sum_{|n_\perp| \leq T} \int_0^\infty dk_1 dk_2 dk_3 \delta(k_1 + k_2 - k_3) \delta_{n_\perp} \left\{ 2 \left( \frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} \right) \right. \\
\left. + \frac{k_2 - k_1}{k_3 \sqrt{k_1 k_2}} \left[ a^\dagger_{ik}(k_1, n_\perp^1) b^\dagger_{kj}(k_2, n_\perp^2) b_{ij}(k_3, n_\perp^3) + b^\dagger_{ij}(k_3, n_\perp^3) b_{ik}(k_1, n_\perp^1) b_{kj}(k_2, n_\perp^2) \right] \right. \\
\left. + \frac{k_1 + k_3}{k_2 \sqrt{k_1 k_3}} \left[ a^\dagger_{ik}(k_3, n_\perp^3) a_{kj}(k_1, n_\perp^1) b_{ij}(k_2, n_\perp^2) - b_{ij}(k_2, n_\perp^2) a_{ik}(k_1, n_\perp^1) a_{kj}(k_3, n_\perp^3) \right] \right. \\
\left. + \frac{k_2 + k_3}{k_1 \sqrt{k_2 k_3}} \left[ b^\dagger_{ik}(k_1, n_\perp^1) a^\dagger_{kj}(k_2, n_\perp^2) a_{ij}(k_3, n_\perp^3) - a_{ij}(k_3, n_\perp^3) b^\dagger_{ik}(k_1, n_\perp^1) a_{kj}(k_2, n_\perp^2) \right] \right\}. \tag{27}
\]

For a massive particle the light-cone energy is \( (k^2_\perp + m^2)/k^+ \), so \( k_\perp \) behaves like a mass. In comparing \( Q^-_{\perp} \) and \( \kappa Q_{\text{CS}} \) we see that \( \kappa \) appears in a very similar way to \( k_\perp \). Therefore \( \kappa \) behaves in many ways like a mass.

To fully discretize the theory we impose periodic boundary conditions on the boson and fermion fields alike, and obtain an expansion of the fields with discrete momentum modes. The discrete longitudinal momenta \( k^+ \) are written as fractions \( n P^+/K \) of the total longitudinal momentum \( P^+ \), with \( n \) and \( K \) positive integers. The longitudinal resolution is set by \( K \), which is known in DLCQ as the harmonic resolution \([4]\). This
converts the mass eigenvalue problem \(2P^+P^-|M\rangle = M^2|M\rangle\) to a matrix eigenvalue problem. As discussed in the introduction, this is done in SDLCQ by first discretizing the supercharge \(Q^-\) and then constructing \(P^-\) from the square of the supercharge: \(P^- = (Q^-)^2/\sqrt{2}\). When comparing the \(\perp\) and CS contributions to the supercharge, we find a relative \(i\) between them. Thus the usual eigenvalue problem

\[
2P^+P^-|M\rangle = \sqrt{2}P^+ \left( gQ_{\text{SYM}}^-(T) + Q_{\perp}(T) + i\kappa Q_{\text{CS}}^-(T) \right)^2 |M\rangle = M^2|M\rangle
\]

has to be solved by using fully complex methods. The continuum limit is obtained by taking \(T \to \infty\) and \(K \to \infty\). The largest matrices are diagonalized by the Lanczos technique [24] which easily yields several of the lowest eigenvalues and their eigenvectors. For a more complete discussion of our numerical methods, see [11] and [7].

We retain\(^1\) the \(S\)-symmetry, which is associated with the orientation of the large-\(N_c\) string of partons in a state [25]. In a \((1+1)\)-dimensional model this orientation parity is usually referred as a \(\mathbb{Z}_2\) symmetry, and we will follow that here. It gives a sign when the color indices are permuted

\[
Z_2 : a_{ij}(k) \to -a_{ji}(k), \quad b_{ij}(k) \to -b_{ji}(k).
\]

We will use this symmetry to reduce the Hamiltonian matrix size and hence the numerical effort. All of our states will be labeled by the \(\mathbb{Z}_2\) sector in which they appear. We will not attempt to label the states by their normal parity; on the light cone this is only an approximate symmetry. Such a labeling could be done in an approximate way, as was shown by Hornbostel [26], and might be useful for comparison purposes if at some point there are results from lattice simulations of the present theory.

\section{Limiting Cases}

Before considering results for the full theory we discuss what we have learned about various limiting cases. These are dimensionally reduced pure SYM theory \((\kappa = 0, T = 0)\) [22, 23], \((2+1)\)-dimensional pure SYM theory \((\kappa = 0)\) [7], dimensionally reduced SYM-CS theory \((T = 0)\) [9, 10], and pure CS theory \((g = 0)\). These will provide convenient points of reference when we discuss the full theory.

Let us start with dimensionally reduced SYM theory [22, 23], for which

\[
Q^- = gQ_{\text{SYM}}^-(0).
\]

There are two properties that characterize this theory. As we increase the resolution we find that there are new lower mass states that appear, and this sequence of states appears to accumulate at \(M^2 = 0\). In addition, there are massless BPS states, and the dominant component of the wave function of this sequence of states can be arranged to have 2 particles, 3 particles, etc., up to the maximum number of particles allowed by the resolution, \(i.e., K\) particles. Therefore at resolution \(K\) there are \(K-1\) boson BPS states and \(K-1\) fermion BPS states. In Fig. 1(a) we see the spectrum of two-dimensional SYM as a function of the inverse of the resolution \(1/K\) from earlier work [22, 23].

\(^1\)We note that the CS term breaks transverse parity.
When we extend the SYM theory to 2 + 1 dimensions [7], the supercharge $Q^-$ becomes

$$Q^- = gQ_{\text{SYM}}(T) + Q_{\perp}(T).$$

Again there are a number of unique properties that characterize this theory. At small coupling $g$ and low energy we recover completely the (1+1)-dimensional theory at $g = 0$. At higher energies we find a series of KK states. This is an expected result and was fully verified in earlier work. At large $g$ one might expect that the $n_{\perp}/g$ term would freeze out and one would again see a reflection of the 1+1 theory. The fact is that while we see these states, they are just a small part of the spectrum. $Q_{\text{SYM}}^-$ by itself wants to have a large number of particles, as we discussed above. The contributions from $gQ_{\text{SYM}}^-$ and $Q_{\perp}$ are therefore minimized by a larger number of particles, each with a small transverse momentum. We therefore find that the average number of particles in the bound states grows with the YM coupling. The massless BPS states of the 1+1 SYM persist here in 2+1 dimensions, but now the number of particles in all of these states increases rapidly with YM coupling.

When we add a CS term to the (1+1)-dimensional theory [9], we find that the most important role of the CS term is to provide a mass for the constituents. This freezes out the long, lower mass states that characterized (1+1)-dimensional SYM theory. Interestingly, however, the massless BPS states become massive approximate BPS states and have masses that are nearly independent of the YM coupling [10], as seen in Fig. 1(b).

If we set $g = 0$ in Eq. (24) we are left with the CS term and the transverse momentum term. It is not surprising that the square of this supercharge gives the light-cone energy of a set of noninteracting massive adjoint fermions and adjoint bosons. Understanding the discrete version of the continuum of free-particle states in the DLCQ approximation is...
is helpful, because once the coupling is turned on SYM-CS theory becomes a confining theory and each of the continuum states becomes a bound state. Therefore at small coupling we expect to see bound-state masses close to the masses seen in the discrete version of the continuum.

The discrete version of the two particle continuum spectrum is described by

$$\frac{M^2(K)}{K} = \frac{\kappa^2 + k_1^2}{n_1} + \frac{\kappa^2 + k_2^2}{n_2},$$

(32)

where \( n_1 + n_2 = K \) and \( k_{1\perp} + k_{2\perp} = 0 \). The lowest states in the \( Z_2 = +1 \) sector are part of the two-particle continuum spectrum. The discrete version of the three-particle continuum spectrum is given by

$$\frac{M^2(K)}{K} = \frac{\kappa^2 + k_{1\perp}^2}{n_1} + \frac{\kappa^2 + k_{2\perp}^2}{n_2} + \frac{\kappa^2 + k_{3\perp}^2}{n_3},$$

(33)

where \( n_1 + n_2 + n_3 = K \) and \( k_{1\perp} + k_{2\perp} + k_{3\perp} = 0 \). The lowest states in the \( Z_2 = -1 \) sector are part of the three-particle continuum spectrum. The lowest mass states in the \( Z_2 = +1 \) and \( Z_2 = -1 \) continuum spectrum, and their Fock-state composition, are shown in Table 1. Higher contributions are seen at the appropriate places in the free particle spectrum as well. For example, among the higher mass states in the \( Z_2 = +1 \) sector is the state \( a(3,0)b(3,0)b(3,0) \) which is degenerate with the lowest mass state in the \( Z_2 = -1 \) sector.

In SYM theory in two and three dimensions the bound-state spectrum has a fourfold degeneracy with two boson states and two fermion states at each mass. When we add a CS term the theory becomes complex, and the degeneracy is reduced to a two-fold degeneracy, that is one boson and one fermion bound state at each mass. At the special point \( g = 0 \) we see in the table a number of accidental degeneracies, at least from the supersymmetry point of view. In particular, the two-parton states in the bosonic sector will consist of either two bosons or two fermions. We thus expect a double degeneracy of the multi-particle states, unless the fermion-fermion state is forbidden by the Pauli principle, as happens when the momenta are equal.

Table 1: The Fock-state composition and mass squared \( M^2 \) in units of \( 4\pi^2/L^2 \) for the continuum states in the two-particle \( (Z_2 = 1) \) and three-particle \( (Z_2 = -1) \) sectors for resolutions \( K = 9, 8 \). The Chern–Simons coupling is \( \kappa = 2\pi/L \); the Yang–Mills coupling is zero.
Figure 2: The spectrum as a function of (a) $1/T$, at $K = 9$, and (b) the longitudinal resolution $K$, for $g = 0.1 \sqrt{4\pi^3/N_c L}$. In (b) the horizontal lines indicate the discrete continuum spectrum.

4 Results for the Full Theory

4.1 Weak coupling

It is interesting to first consider weak coupling because it provides a clear connection between a variety of interesting phenomena that we want to consider in this paper. At low energy and small coupling the spectrum is essentially a $(1+1)$-dimensional spectrum. The lowest energy bound states are the two-parton states that we saw in the $g = 0$ spectrum, with a binding energy proportional to the YM coupling that breaks the accidental degeneracy at $g = 0$. Since none of the partons have even one unit of transverse momentum at very low energy, the transverse momentum term in the supercharge does not contribute, and the supercharge is the $(1+1)$-dimensional supercharge scaled by $L$. It is therefore obvious that at low energy and small coupling the $(2+1)$-dimensional spectrum is a reflection of the two-dimensional spectrum.

As we move up in energy at weak coupling we find the KK states. The masses are closely related to the masses of the KK states that we found at $g = 0$. The lowest energy states are two-parton states with the partons having $+1$ and $-1$ units of transverse momentum. This essentially doubles the free energy of the bound state from 4 to 8. In addition, the transverse momentum gives an interaction energy that breaks the accidental degeneracies we saw at $g = 0$.

We have done a detailed calculation for weak coupling with\(^2 g = 0.1\). We gain an additional numerical advantage at small coupling in that the average number of particles is well behaved and well understood because each of the states is close to a continuum state. We are therefore able to put an additional cutoff on the number of

\(^2\)For the remainder of the paper, values of $g$ will be quoted in units of $\sqrt{4\pi^3/N_c L}$.
partons in each state. For the ten lowest mass states in each sector we are able to limit ourselves to 4 partons. This allows us to carry the calculation to 9 units of transverse resolution and 9 units of longitudinal resolution. We have checked that calculating with and without this cutoff produces the same results at the highest values of the resolution where we can perform the complete calculation.

We carry out a standard DLCQ analysis of the bound states. At each value of the resolution $K$ and in each $Z_2$ sector we look at the ten lowest energy bound states at each transverse resolution $T$. For $Z_2 = +1$ we fit the curves of mass squared vs. $1/T$, as shown in Fig. 2(a), and extrapolate to infinite transverse momentum cutoff. The intercepts of these curves are the infinite momentum extrapolations, and we plot these as a function of $1/K$ in Fig. 2(b). The $Z_2 = -1$ sector behaves very similarly to the $Z_2 = +1$ sector except that the lowest energy bound states consist primarily of three partons rather than two.

We plot the discrete version of the continuum spectrum as horizontal lines at each resolution $K$ and the bound states as dots. We see that the accidental degeneracy we found at $g = 0$ is completely broken, as expected. If we connect the lowest mass states at each $K$ we will get a saw tooth pattern. This saw tooth pattern is a reflection of the close connection of these states with the discrete continuum spectrum. We can nevertheless fit these points using $M^2 = M_\infty^2 + b K$. The intercept of this curve yields the continuum mass $M_\infty$ of the particular state. We find for example that the lowest state is at approximately $3 M^2 = 4.5$.

We note that at about 9.0 we see the first KK excitation of these states. In the section below on structure functions we will show the structure functions of these states and see the characteristic behavior of each of these types of states.

### 4.2 Strong coupling

In Fig. 3 we show the coupling dependence of SYM-CS theory in 2+1 dimensions. For numerical convenience the results presented here are at specific and low values of the longitudinal and transverse cutoffs $K$ and $T$ but at many values of the coupling. We can distinguish two regions for the low-lying spectrum. For very small $g$ we have a spectrum of almost free particles as we discussed in the previous section. There is a transition that occurs around $g = 0.3$ where we see a number of level crossings, and beyond this point the character of the spectrum changes. For larger $g > 0.3$ we will have a *bona fide* strong-coupling bound-state spectrum.

We will now perform a detailed investigation of the strong-coupling spectrum at $g = 0.5$. At this stronger coupling, however, we are unable to make the additional approximation of cutting off the number of partons at 4 particles. The reason for this is that as we increase $K$ the average number of partons in a state also increases as we described earlier. We show this behavior in Fig. 4, and we see that at the largest values of $K$ the average number of particles is approaching 4. This tells us that, if we cut off the number of particles at 4, we are missing a significant part of the wave function, and the approximation is starting to break down. We have checked this by looking at the spectrum with and without a 4-parton cutoff at $K = 6$, and we find that we

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3Masses will be given in units of $4\pi^2/L^2$. 

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Figure 3: The mass spectrum as a function of $g$, in units of $\sqrt{4\pi^3/N_c L}$, at $K = 6$, $T = 1$, and $Z_2 = +1$.

<table>
<thead>
<tr>
<th>$g$</th>
<th>$M^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4.36</td>
</tr>
<tr>
<td>0.5</td>
<td>8.93</td>
</tr>
</tbody>
</table>

Table 2: Mass squared in units of $4\pi^2/L^2$ of the lowest mass states at $g = 0.1$ and $g = 0.5$, in units of $\sqrt{4\pi^3/N_c L}$ in the $Z_2 = +1$ sector, showing the development of a gap above the lowest state. Results in both cases are for resolutions $K = 6$ and $T = 4$ and for Chern–Simons coupling $\kappa = 2\pi/L$.

We can use these to extrapolate to the continuum. While we have always been able to make this linear fit in SDLCQ, it remains an assumption that linear fits remain valid outside of any region where we cannot calculate. Keeping 6 particles at resolution $K = 6$ means that we are only able to go to a transverse resolution of $T = 4$.

At $g = 0.5$ we repeat the standard analysis of the 10 lowest mass states in each of the sections. The spectrum as a function of the transverse cutoff $T$ is shown in Fig. 5, and we again see that the curves are well described by a linear fit in $1/T$. The infinite transverse cutoff values of the masses are plotted as a function of $1/K$; we find good linear fits for the lowest state indicating that we again have good convergence. A linear fit for the higher states appears less convincing and clearly would benefit from additional data. We have done the same analysis in the $Z_2 = -1$ sector, and we will present some of the results later in this section.

It is important at this point to comment on a feature that appears in Fig. 3. We see a decoupling of the lowest state from the other states in the spectrum, whose masses grow more rapidly with coupling. In addition there is one low-mass state that appears to fall with increased coupling and become nearly degenerate with the lowest mass.
Figure 4: The average number of particles in the 10 lowest bound states vs. $1/K$ for (a) $g = 0.1$ and (b) $g = 0.5$ in units of $\sqrt{4\pi^3/N_cL}$. The Chern–Simons coupling is $\kappa = 2\pi/L$.

Figure 5: The spectrum as a function of (a) $1/T$, with a linear fit starting at $T = 3$, and (b) the longitudinal resolution $K$, with a linear fit for the lowest state. The coupling values are $g = 0.5\sqrt{4\pi^3/N_cL}$ and $\kappa = 2\pi/L$. 
state. Calculations at larger values\(^4\) of \(T\) indicate that this behavior is a numerical artifact and disappears when we increase \(T\). With this state removed the gap between the lowest state and the remaining states is even more apparent. In Table 2 we present the masses of the 10 lowest mass bound state that we obtained at \(K = 6, T = 4\) for \(g = 0.1\) and \(g = 0.5\). We see that at weak coupling the states are all closely spaced while at strong coupling a gap appears. From Fig. 5(b) it appears that gap is in fact increasing with \(K\) so that in the continuum limit the effect is even larger.

The low-mass state separating from the rest of the spectrum is related to an effect that we saw in SYM-CS theory in 1+1 dimensions [9, 10]. There most of the spectrum behaves like \(g^2\). There was, however, a special set of states, which we called approximate BPS states, that were nearly constant in mass as a function of the coupling. They are a reflection of the exactly massless BPS states in SYM theory in 1+1 dimensions. In the limit of very large coupling we found that the spectrum of these approximate BPS states was proportional to the average number of particles in the bound state. In 1+1 dimensions the average number of particles was itself independent of the coupling.

In the (2+1)-dimensional SYM theory without a CS term there are again exactly massless BPS states [7, 8]. The average number of particles in these and other states grows with the coupling. This increase in the average number of particles causes the approximate BPS states to increase in mass as we increase the coupling. This increase is slower, however, than the \(g^2\) behavior we see in the other states.

In the \(Z_2 = -1\) sector the lowest energy state is a three-parton state. Here again we see the same approximate BPS phenomena that we saw in the \(Z_2 = 1\) sector. In Table 3 we see that at small coupling all the masses of the states are closely spaced but at strong Yang–Mills coupling a gap develops between the lowest and the remaining states. There is again a state with anomalously light mass at strong coupling. The effect here is not as dramatic because the states are more massive and the gap is a smaller relative to the masses.

Presumably there are higher mass approximate BPS states in this theory, but at coupling of the order \(g = 0.5\) they are intermixed with normal states, and it is difficult to study them. At large enough coupling of course we expect them to separate, but as discussed we have problems going to higher couplings. A more complete discussion of the approximate BPS states in (2+1)-dimensional SYM-CS theory can be found in [11].

<table>
<thead>
<tr>
<th>(g)</th>
<th>(M^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>9.61</td>
</tr>
<tr>
<td>0.5</td>
<td>15.71</td>
</tr>
</tbody>
</table>

Table 3: Same as Table 2, but for \(Z_2 = -1\).

\(^4\)For these increased resolutions the time required for diagonalizations at many values of \(g\) is prohibitive; therefore, we studied the \(T\) dependence at only two values of \(g\).
structure function is basically the probability that a constituent of type A has a longitudinal momentum fraction \( x = k^+/P^+ \) and transverse momentum \( k^\perp \). It is given as follows in terms of the light-cone wave function \( \psi \):

\[
g_A(x, k^\perp) = \sum_q \int_0^1 dx_1 \cdots dx_q \int_{-\infty}^{\infty} dk_1^\perp \cdots dk_q^\perp \delta \left( \sum_{i=1}^q x_i - 1 \right) \delta \left( \sum_{j=1}^q k_j^\perp \right) \times \prod_{i=1}^q \delta (x_i - x) \delta (k_i^\perp - k^\perp) \delta_{A_l} |\psi(x_1, k_1^\perp; \ldots, x_q, k_q^\perp)|^2,
\]

(34)

where \( A_l \) is the type for the \( l \)-th constituent.

It is instructive to consider first the structure function for a few states at weak coupling. In Fig. 6 we see the structure function of two two-particle bound states and a bound state composed of KK particles. Fig. 6(a) is the lowest mass state in the \( Z_2 = 1 \) sector has primarily two particles, one with longitudinal momentum \( n^\| = 4 \) and one with longitudinal momentum \( n^\| = 5 \), and no units of transverse momentum. Fig. 6(b) is another two-parton bound state with clearly separated peaks. The partons have \( n^\| = 3 \) and \( n^\| = 6 \) units of longitudinal momentum and no units of transverse momentum. The state in Fig. 6(c) is very similar to the bound state shown in Fig. 6(a) except that it is a bound state of particles that have \( n^\perp = 1 \) and \( n^\perp = -1 \) units of transverse momentum and is therefore an example of a state that is viewed in two dimensions as a bound state of KK particles. These structure functions are very simple because at weak coupling these bound states are composed of nearly free partons.

As we move to strong coupling, the structure functions change significantly. An overall effect which we see in all the strong-coupling structure functions is the stringiness of the states, i.e. states have many partons. Therefore the average momentum per parton decreases, and we see the peaks in the structure function move to lower \( n^\| \). Within the range of parameters we have selected here, the interaction energy at \( g = 0.5 \) is as large as the KK excitation energy. Therefore we will see mixing between the KK modes and the low-energy modes.

In Fig. 7(a) we see the structure function of the lowest energy two-parton state at \( g = 0.5 \). As we discussed earlier this is an anomalously light state and is a reflection of the BPS states in the pure SYM theory. Since it is so light it does not mix with the KK modes, and it is very flat in \( n^\| \). This flatness seems to be unique to this state. The state in Fig. 7(b) appears to be at least partially a three-parton state. The shape of its structure function is consistent with having a component with \( n^\| = 4 \) and \( n^\perp \pm 1 \), \( n^\| = 1 \) and \( n^\perp \pm 1 \), and \( n^\| = 1 \) and \( n^\perp = 0 \). There also appears to be a two-parton component with \( n^\perp = 0 \) for both partons and \( n^\| = 5 \) and \( n^\| = 1 \). This is therefore an example of mixing between the KK sector and the low-energy sector as well as a mixing between the two-parton and three-parton sectors. Finally, in Fig. 7(c) we have a state that is dominantly a KK state. It has the lowest energy for such a state and is therefore most likely the evolution of the pure KK state we saw at \( g = 0.1 \). We note again that there is mixing with the low-energy sector, since there is a significant probability of finding a parton with one unit of \( n^\| \) and no transverse momentum.

It is clear from this study that a search for a signal of extra dimensions in the form of KK bound states may be frustrated by strong mixing with ordinary states, a situation reminiscent of the long-standing hunt for glue balls. One hopes that the energy scales
Figure 6: Bosonic structure functions of the lowest energy state with (a) and (b) primarily two partons, and (c) two KK partons. The symmetry sector is $Z_2 = +1$, and the couplings are $g = 0.1\sqrt{4\pi^3/N_cL}$ and $\kappa = 2\pi/L$. The resolutions are $K = 9$ and $T = 9$.

Figure 7: Same as Fig. 6, but for $g = 0.5\sqrt{4\pi^3/N_cL}$ and resolutions $K = 6$ and $T = 4$.

are well separated, as in the weak-coupling example, so that these bound states of KK particles can be identified from the mass spectrum alone.

In the $Z_2 = -1$ bosonic sector the lowest states are three-particle states at weak coupling which broaden to stringy states as the coupling is increased, just as in the $Z_2 = +1$ sector. There are several interesting structure functions in this sector, both at weak coupling and at strong coupling. They are shown in Figs. 8 and 9. The lowest energy state at $g = 0.1$ is a classic pure three-parton state. The state at $M^2 = 10.544$ is a three-parton state with bosons at $n_\parallel = 2$ and 3 as well as a fermionic component that does not show up in this structure function. The state at $M^2 = 11.606$ is interesting because it is clearly a pure state with two partons with 2 units of $n_\parallel$ and one with $n_\parallel = 5$ units.

The three-parton state in Fig. 9(a) at $g = 0.1$ is an approximate BPS state and therefore has an anomalously light mass relative to the other states at $g = 0.5$. The structure function has the characteristic long ridge in $n_\parallel$ that we saw in the $Z_2 = 1$
Figure 8: Structure functions for the lowest energy states with primarily three partons. The masses squared are (a) 9.8, (b) 10.54, and (c) 11.61, in units of $4\pi^2/L^2$. The symmetry sector is $Z_2 = -1$, and the couplings are $g = 0.1\sqrt{4\pi^3/N_cL}$ and $\kappa = 2\pi/L$. The resolutions are $K = 9$ and $T = 9$.

![Figure 8](attachment:image8.png)

(a) (b) (c)

Figure 9: Same as Fig. 8, but for the corresponding states at Yang–Mills coupling $g = 0.5\sqrt{4\pi^3/N_cL}$ and resolutions $K = 6$ and $T = 4$. The mass values are (a) 15.71, (b) 19.25, and (c) 22.60. The peak of structure function in Fig. 9(b) has moved to smaller $n_{||}$ which is a characteristic of these supersymmetric bound states at strong coupling. We have included Fig. 9(c) because it has a high enough mass that it mixes with the KK modes in this sector.

![Figure 9](attachment:image9.png)

(a) (b) (c)

5 Summary

We have considered SYM-CS theory in 2+1 dimensions and presented numerical solutions for the spectrum and structure functions, as well other properties of the bound states. There are several motivations for this calculation. Previously we found two very interesting features in the solutions of SYM in 2+1 dimensions [11]. The states become very stringy as we increase the Yang–Mills coupling, in the sense that the average number of partons in all the bound states grows rapidly. The question is whether...
this property will change significantly when we add a mass for the partons. Therefore
the technical question that we are attempting to answer here is how these properties of
the solutions of SYM theory in 2+1 dimensions change when we add a CS term, which
effectively gives a mass to partons of the theory without breaking the supersymmetry.
From a physics point of view we are looking for new phenomena that might appear,
particularly phenomena that are too complicated to be investigated analytically.

At weak Yang–Mills coupling we find a spectrum that is in many ways very similar
to the free spectrum. Namely, we find bound states at weak coupling which are very
close in mass to the discrete approximation to the multi-particle continuum but with
the squares of the masses shifted by approximately \( \Delta M^2 = 2\kappa g \sqrt{N_c} \). We find a full
spectrum of KK states above the low-energy states. Since KK states are of considerable
interest within the context of extra dimensions, we investigated the relation between
the low-energy states and the KK states. This could be considered as a toy model of
universal extra dimensions in the gauge sector, with the extra twist that the toy model
is fully supersymmetric. At weak coupling the KK states are well separated from the
low-energy states, and their spectrum is easily understood in terms of the low-energy
spectrum. As we go to larger coupling we find extensive level crossing and beyond this
what we call the strongly coupled region. In the numerical calculation we have taken
the interaction energy to be of the same scale as a unit of KK mass. We find that the
KK states and the low-energy states mix. We looked at the structure functions of these
states and found that the structure functions give a clear picture of the mixing. When
viewed from a lower number of dimensions, there does not appear to be a simple way
to disentangle the KK states from the low-energy states.

We studied the strong-coupling region in considerable detail and find another interesting feature. The BPS states of the underlying SYM theory become states with anomalous low mass. These states are the (2+1)-dimensional versions of the approximate BPS states that we saw in the dimensionally reduced theory \([9, 10]\). We looked at the lowest such states in both \(Z_2\) sectors. It seems entirely possible that this could be a novel mechanism for generating states at a new scale significantly below the fundamental scale of a theory. The structure functions of these states appear to be unique in the sense that they are much flatter in \(n_{||}\) than those of the other bound states of the theory.

There are a number of interesting research directions from here. The CS term is
clearly an efficient way to introduce a mass without breaking supersymmetry, and it
would be interesting to investigate coupling this gauge sector to fundamental matter.

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References


