1. Introduction and Summary

We start by formulating the results of [1, 2, 3] in the context of two-dimensional spacetimes. In [1, 2, 3], two-dimensional spacetimes are considered as Lorentzian, and the equations of motion are derived from the action principle. The results in [1, 2, 3] are then applied to the study of entanglement and black hole physics.

In this work, we consider two-dimensional spacetimes as Lorentzian, and the equations of motion are derived from the action principle. The results in [1, 2, 3] are then applied to the study of entanglement and black hole physics.

Over the last few years, there has been considerable interest in the problem of quantum inequalities in two-dimensional curved spacetimes.
In this paper we consider any two dimensional spacetime \((M, g_{ab})\) that is globally conformal to Minkowski spacetime, and again consider a free, massless scalar field \(\Phi\). Our main results are:

(i) For an arbitrary timelike curve \(\gamma\) we have
\[
E_{\gamma, \text{min}}[\gamma, \rho] = -\frac{1}{24\pi} \int_{\gamma} d\tau \left[ \frac{\dot{\gamma}^2}{\rho} + \rho a^2 a_s + \rho R \right],
\]  
where \(a^2 = v^b \nabla_b v^a\) is the acceleration of \(\gamma\) and \(R\) is the Ricci scalar. Here the dot denotes a derivative with respect to proper time; we reinterpret the smearing function \(\rho\) as a function defined on the curve \(\gamma\) rather than on the real line.

(ii) More generally, for any vector field \(v^a\) defined on a timelike curve \(\gamma\), we have
\[
\min_{\omega} \left\langle \left( \int_{\gamma} d\tau T_{ab} v^a v^b \right) \right\rangle_{\omega} = -\frac{1}{24\pi} \int_{\gamma} d\tau \left[ \frac{1}{2} (a^2 + \beta^2) a^a a_s \right. \\
\left. - (a^2 - \beta^2) a + \frac{1}{4} (a + \beta)^2 R + 2\dot{a}^2 + 2\beta^2 \right].
\]  
Here the functions \(a, \beta\) are defined by
\[
a + \beta = -2v^a u_a, \quad a - \beta = 2v^a u_a v_b,
\]  
and dots denote derivatives with respect to \(\tau\). The result (1.8) reduces to (1.7) when \(a = \beta = \sqrt{\rho}\).

(iii) For a null geodesic \(\gamma\), we have
\[
E_{N, \text{min}}[\gamma, \rho] = -\frac{1}{48\pi} \int_{\gamma} d\lambda \frac{\dot{\rho}(\lambda)^2}{\rho(\lambda)}
\]  
which has the same form as the flat spacetime result (1.6). Here the prime denotes a derivative with respect to affine parameter \(\lambda\), where we treat \(\rho\) as a function defined on \(\gamma\).

We derive the results (1.7), (1.8) and (1.11) in Sec. II below, and discuss some implications in Sec. III. We note that Feister has proved that the quantity \(E_{\gamma, \text{min}}[\rho]\) is finite for any spacetime and any curve \(\gamma\) [22]. Thus, the existence of the bound (1.7) follows from the very general result of Feister. Also, with respect to the result (1.8), we note that a quantum inequality for the time average of a null-null component of stress-energy was previously derived by Feister and Roman [37].

We also remark that the average along a null geodesic of any other component of the stress-energy tensor other than \(T_{ab} k^a k^b\) is unbounded below. This can be seen from the flat spacetime analysis of Ref. [35]. Moreover, in two dimensions there is no loss of generality in considering null geodesics instead of more general null curves. Therefore the result (1.11) is the most general quantum inequality that can be derived for null curves.

II. DERIVATION OF THE QUANTUM INEQUALITIES

The basic idea of the proof, following Vollick [28], is to apply conformal transformations to the Minkowski spacetime results (1.5) and (1.6). Our analysis differs from Vollick’s in that we allow accelerated curves.

For any metric \(g_{ab}\), state \(\omega\), timelike curve \(\gamma\) and weighting function \(\rho\) along \(\gamma\) we define
\[
I_{\gamma}[g_{ab}, \rho, \omega] = \int_{\gamma} d\tau \rho(\tau) \left( \bar{g}_{ab} \omega^a \omega^b \right)
\]
\[
= \frac{1}{24\pi} \int_{-\infty}^{\infty} d\tau \rho(\tau) \left[ \frac{\dot{\rho}^2}{\rho^2} + a^2 a_s + R \right].
\]  
To derive the result (1.7) it suffices to show that \(I_{\gamma} \geq 0\) always. Consider now the conformally transformed metric
\[
\bar{g}_{ab} = e^{2\sigma} g_{ab},
\]  
where \(\sigma\) is any smooth function. Let \(\bar{\omega}\) be the state on the spacetime \((M, \bar{g}_{ab})\) that is naturally associated with \(\omega\) (i.e. having the same n-point distributions). We define the transformed smearing function
\[
\bar{\rho} = e^\sigma \rho.
\]
It is now a straightforward computation to show that the functional \(I_{\gamma}\) is a conformal invariant, i.e., that
\[
I_{\gamma}[g_{ab}, \bar{\rho}, \bar{\omega}] = I_{\gamma}[g_{ab}, \rho, \omega].
\]  
We briefly sketch the transformation. The expected stress-energy tensor is given by
\[
\langle T_{ab}\rangle_{\omega} = \langle T_{ab}\rangle_{\omega} + \frac{1}{12\pi} \left[ \nabla_a \nabla_b \sigma - (\nabla_a \sigma)(\nabla_b \sigma) \right. \\
\left. - g_{ab} \nabla_c \nabla^c \sigma + \frac{1}{2} g_{ab}(\nabla^2 \sigma)^2 \right];
\]  
see Eq. (6.133) of Ref. [38]. The transformed velocity is \(\bar{u}^a = e^{-\sigma} u^a\), and the transformed acceleration is
\[
\bar{a}^a = e^{-2\sigma} \left[ a^a + \dot{\sigma} u^a + \nabla^a \sigma \right].
\]  
Finally the transformed Ricci scalar is
\[
\bar{R} = e^{-2\sigma} \left[ R - 2\nabla_c \nabla^c \sigma \right].
\]  

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2 We can ignore here the well known infrared pathologies associated with massless scalar fields in two dimensions, since our smearing functions can be taken to be of compact support, and thus we could impose an infrared cutoff on the theory.

3 The 2-form is normalized according to \(\epsilon_{ab} \delta^c = \delta^c\). It can be seen that the lower bound (1.8) is independent of the choice of sign of \(\epsilon_{ab}\).

4 This result was previously noted in footnote 38 of Ref. [36].
Substituting the quantities (2.3) and (2.5) = (2.7) together with \(d\tau = e^\theta d\tau\) into the definition (2.1) of \(J_\gamma\), and using
\[
v^a u^b \nabla_a \sigma = \vec{\sigma} - d^a \nabla_b \sigma, \tag{2.8}
\]
yields the conformal invariance property (2.4)

Now fix a choice of curve \(\gamma\), a globally-conformally-flat metric \(g_{ab}\), a state \(\omega\) and a smearing function \(\rho\). From the conformal invariance property it suffices to show that \(L_\gamma[g_{ab}, \bar{\rho}, \vec{\omega}] \geq 0\), where \((g_{ab}, \bar{\rho}, \vec{\omega})\) is a conformal transform of \((g_{ab}, \rho, \omega)\). Therefore, without loss of generality we can take \(g_{ab}\) to be the flat, Minkowski metric. We can also without loss of generality take \(a^a = 0\), since any accelerated curve can be made to be geodesic by a conformal transformation which preserves flatness. Thus, it suffices to show that \(L_\gamma \geq 0\) for geodesics in flat spacetime. However, when \(a^a = R = 0\), the condition \(L_\gamma \geq 0\) reduces to the quantum inequality (1.5) previously established.

An exactly analogous argument holds for null geodesics. For any metric \(g_{ab}\), null geodesic \(\gamma\) with tangent \(k^a\), smearing function \(\rho\) and state \(\omega\) we define
\[
J_\gamma[g_{ab}, k^a, \rho, \omega] = \int_\gamma d\lambda \rho(\lambda) \langle T_{ab}\rangle_{\omega} k^a k^b + \frac{1}{48\pi} \int_{-\infty}^\infty d\lambda \frac{(\rho')^2}{\rho}, \tag{2.9}
\]
where \(k^a = (d/d\lambda)^a\). Then one can show that
\[
J_\gamma[g_{ab}, \bar{k}^a, \bar{\rho}, \vec{\omega}] = J_\gamma[g_{ab}, k^a, \rho, \omega], \tag{2.10}
\]
where now
\[
\bar{g}_{ab} = e^{2\sigma} g_{ab}, \quad \bar{k}^a = e^{-\sigma} k^a, \quad \bar{\rho} = e^{2\sigma} \rho. \tag{2.11}
\]
Note that the conformal scaling (2.11) of the smearing function in this null case differs from the scaling (2.3) in the timelike case. As before the conformal invariance property (2.10) allows one to deduce the curved spacetime result (1.11) from the flat spacetime result (1.6).

We now turn to a proof of the result (1.8). Before treating the case of a general vector field \(v^a\) defined on the timelike curve \(\gamma\), it is useful to consider the case of a future-pointing null vector field \(k^a\) normalized according to
\[
k^a u_a = -\frac{1}{2} \tag{2.12}
\]
Note that there are exactly two such null vector fields along \(\gamma\), and they are expressible in terms of a choice of volume form \(\epsilon_{ab}\) by
\[
k^a = \frac{1}{2}(u^a - \epsilon^{ab} u_b). \tag{2.13}
\]
We define
\[
K_\gamma[g_{ab}, k^a, \rho, \omega] = \int_\gamma d\tau \rho(\tau) \langle\bar{T}_{ab}\rangle_{\omega} k^a k^b + \frac{1}{48\pi} \int_{-\infty}^\infty d\tau \rho(\tau) \left(\frac{\rho'}{\rho}^2 + a^a a_a - 2\tilde{a} \right), \tag{2.14}
\]
where
\[
a = \epsilon_{ab} u^a u^b = 2k_a a^a. \tag{2.15}
\]
and dots denote derivatives with respect to proper time \(\tau\). As before, it is straightforward to show that \(K_\gamma\) is a conformal invariant, in the sense that
\[
K_\gamma[e^{\sigma} g_{ab}, e^{-\sigma} k^a, e^{\sigma} \rho, \omega] = K_\gamma[g_{ab}, k^a, \rho, \omega]. \tag{2.16}
\]
Therefore, to show that \(K_\gamma \geq 0\) in general, it suffices to show that \(K_\gamma \geq 0\) for geodesics in flat Minkowski spacetime. In Minkowski spacetime, choose coordinates \((t, x)\) such that the metric is \(d\tau^2 = -dt^2 + dx^2\), and define \(u = \hat{t} + x, v = t - x\), and choose \(k^a = (\partial/\partial u)^a\). Then for the geodesic \(\gamma\) given by \(x = 0\) we have
\[
\int_\gamma d\tau \bar{T}_{ab} k^a k^b \rho(\tau) = \int_{-\infty}^\infty dt \bar{T}_{uu}(t, 0) \rho(t) = \int_{-\infty}^\infty dt \bar{T}_{uu}(t/2, t/2) \rho(t) = \int_{-\infty}^\infty d\lambda T_{ab} k^a k^b \rho(\lambda). \tag{2.17}
\]
Here the second equality follows from the fact that \(\bar{T}_{uu}\) depends only on the \(u\) coordinate and \(\bar{T}_{uv}\) depends only on the \(v\) coordinate, and \(\bar{\gamma}\) is the null geodesic \(\lambda = u\) with affine parameter \(\lambda = u\). It follows from Eq. (2.17) that the quantity \(K_\gamma\) reduces to the integral (2.9) along the null geodesic \(x = t\), which was previously shown to be non-negative. Thus we have proved that \(K_\gamma\) is positive in general.

Consider now a general vector field \(v^a\) defined along \(\gamma\). We fix a volume form \(\epsilon_{ab}\) and define null vectors \(k^a\) and \(l^a\) via
\[
k^a = \frac{1}{2}(u^a - \epsilon^{ab} u_b), \tag{2.18}
\]
\[
l^a = \frac{1}{2}(u^a + \epsilon^{ab} u_b). \tag{2.19}
\]
Then \(k^a\) and \(l^a\) are both future directed null vectors along \(\gamma\) that satisfy the normalization condition (2.13), so the result \(K_\gamma \geq 0\) applies to both \(k^a\) and \(l^a\). We now define the functions \(\alpha\) and \(\beta\) to be the components of the vector \(v^a\) on the basis \((k^a, l^a)\):
\[
v^a = \alpha k^a + \beta l^a. \tag{2.20}
\]
Using the definitions (2.18) and (2.20) one can invert Eq. (2.20) to obtain Eqs. (1.9). We now compute the integrand on the left hand side of Eq. (1.8):

\[ T_{ab} v^a v^b = \alpha^2 T_{ab} k^a k^b + \beta^2 T_{ab} l^a l^b + 2\alpha\beta T_{ab} k^{(q) b}. \]

Using the identity \( k^{(q) b} = \epsilon^{ab}/4 \) together with the trace anomaly

\[ \langle \mathcal{F}^a_a \rangle = \frac{1}{24\pi} R \]

we can rewrite Eq. (2.21) as

\[ T_{ab} v^a v^b = \alpha^2 T_{ab} k^a k^b + \beta^2 T_{ab} l^a l^b - \frac{\alpha\beta R}{48\pi}. \]

We next integrate the quantity (2.23) along the timelike curve \( \gamma \). Since the coefficients \( \alpha^2 \) and \( \beta^2 \) are both nonnegative, we can apply the result (2.14) to bound the first two terms in Eq. (2.23). The result is the expression (1.8). Note that the bound is optimal or sharp, since taking the minimum over states for the first two terms in Eq. (2.23) involves the two, independent, right-moving and left-moving sectors of the theory.

### III. IMPLICATIONS

In this section we discuss some implications of the quantum inequalities (1.7), (1.8) and (1.11) in some specific spacetimes, and their physical interpretation.

Consider first a uniformly accelerated or Rindler observer with acceleration \( a \) in Minkowski spacetime. From Eq. (1.7) it follows that the energy density measured by such an observer can be as negative as \( -\alpha^2/(24\pi) \) over arbitrarily long timescales. This behavior is in marked contrast to that of inertial observers, who can only measure negative energy densities over limited timescales. The reason for the difference can be understood by considering a state containing a burst of negative energy radiation \( \langle T_{ab}(u) \rangle \) followed by a compensating burst of positive energy radiation. An accelerated observer can intercept the negative energy burst while avoiding the positive energy burst if the Rindler horizon lies between the two bursts.

A similar argument was given by Borde, Ford and Roman to explain the behavior of the total energy on asymptotically null spacelike surfaces in four dimensions; see Fig. 2 of Ref. [16].

Consider next a static observer near the horizon of the static, two dimensional black hole spacetime

\[ ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2. \]

As shown by Vollick [28], the lower bound on the time averaged energy density measured by such an observer becomes arbitrarily negative near the horizon. From Eq. (1.7), we can see that this effect is due to the acceleration of the static observer, since the Ricci scalar term in Eq. (1.7) is finite at the horizon.

Next, one can derive from the general result (1.8) a constraint on the time averaged pressure measured by an observer. By taking \( \alpha = \sqrt{\beta}, \beta = -\sqrt{\beta} \) in Eq. (1.8), one obtains

\[ \min \left\langle \int d\tau \rho T_{ab} \epsilon^a \epsilon^b \right\rangle_\omega = -\frac{1}{24\pi} \int d\tau \left[ \frac{\beta^2}{\rho} + \rho a^2 a^a \right] \]

where \( \epsilon^a \) is any unit spacelike vector field along \( \gamma \) orthogonal to \( v^a \). This is identical in form to the result (1.7) except that there is no Ricci scalar term.

Finally, consider averages of energy densities over a spacelike curve rather than over a timelike curve. One can derive a constraint on such averages from Eq. (1.8) as follows. Consider the transformation \( g_{ab} \rightarrow -g_{ab} \). Under this transformation, the curve \( \gamma \) becomes a spacelike curve, \( \tau \) becomes proper length rather than proper time, while the set of allowed stress energy tensors \( \langle T_{ab} \rangle \) is invariant. Also, the quantities \( u^a, v^a, a^a, a = \epsilon_{ab} u^b \), and \( \hat{a} \) are even under the transformation, while the quantities \( u_a, v_a, \) and \( R \) flip sign. We therefore obtain that

\[ \min \left\langle \int \gamma ds T_{ab} u^a v^b \right\rangle = -\frac{1}{24\pi} \int \gamma ds \left[ -\frac{1}{2}(a^2 + \beta^2) a^a a_a - (\alpha^2 + \beta^2) \hat{a} - \frac{1}{4}(\alpha + \beta)^2 R + 2\alpha^2 + 2\beta^2 \right], \]

where \( \gamma \) is a spacelike curve with proper length parameter \( s \), tangent \( t^a = (d/ds)^a \), and acceleration \( a^a = t^b t^c \nabla_b t_c \). Also dots denote derivatives with respect to \( s \), and \( \alpha, \beta \) and \( a \) are defined by [cf. Eqs. (1.9) above]

\[ \alpha + \beta = 2v^a t_a, \quad \alpha - \beta = -2v^a t_a v_b, \quad a = \epsilon_{ab} t_b. \]

By taking \( \alpha = \sqrt{\beta}, \beta = -\sqrt{\beta} \) we obtain

\[ \min \left\langle \int \gamma ds T_{ab} u^a n^b \right\rangle = -\frac{1}{24\pi} \int \gamma ds \left[ \frac{\beta^2}{\rho} - \rho a^2 a^a \right], \]

where \( n^a \) is the unit, future directed normal to the spacelike curve \( \gamma \). As an example, consider the static spacetime

\[ ds^2 = e^{2\sigma(x)} (-dt^2 + dx^2). \]

Then the Killing energy of a state on the hypersurface \( t = 0 \) is

\[ \int ds \langle T_{ab} \rangle n^a \xi^b, \]

where \( \xi^b \) is a Killing vector.
where \( ds = e^\sigma dx \), \( \xi^a = (\partial / \partial t)^a \) is the Killing vector field and \( n^a = e^{-\sigma} \xi^a \) is the unit normal. From Eq. (3.5) with \( \rho = e^\sigma \) it follows that the lower bound on this Killing energy is

\[
- \frac{1}{24\pi} \int dx \left( \frac{d\sigma}{dx} \right)^2. \tag{3.8}
\]

**IV. CONCLUSION**

We have derived very general constraints on averages of components of the stress energy tensor along timelike, null and spacelike curves in two dimensional spacetimes. The bounds are all optimal and are expressed in terms of covariant quantities. Unfortunately the methods used here do not generalize to the more interesting case of four dimensions.

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