Quantum computation with unknown parameters

We show how it is possible to achieve quantum computation on a system in which most of the parameters are unknown. We develop quantum algorithms which can be executed in polynomial time in the number of parameters, and which depend on the quantum computation of the problem in question.

The use of quantum computation is a powerful tool in a variety of applications. Quantum algorithms have been used to solve problems in areas such as cryptography, optimization, and machine learning. In this paper, we focus on the problem of quantum computation with unknown parameters.

We consider a system of qubits, each of which is initialized in a known state. The goal is to perform a quantum computation on these qubits, where the parameters of the computation are unknown. The parameters can be any real numbers, and the computation is assumed to be a unitary transformation.

The key insight is that it is possible to perform a quantum computation in polynomial time in the number of parameters, even if these parameters are unknown.

We present two main results in this paper. First, we show how to perform quantum computation with unknown parameters in polynomial time. Second, we show how to perform quantum computation with unknown parameters in sub-exponential time.

These results have important implications for the practical implementation of quantum computing. They suggest that it may be possible to perform quantum computations in scenarios where the parameters are not fully known, which could be useful in a variety of applications.
such that the eigenvectors evolve according to the desired gate, and then to repeat the procedure changing the parameters appropriately in order to cancel the geometric and dynamical phases.

Let us first show how to perform the phase gate $U = e^{i\theta \sigma_z/2}$. We set $\Delta = 0$ for all times. The parameters $(\Omega, \varphi)$ have to be changed as follows [see Fig. 1(a)]:

$$
\begin{align*}
\left(0, 0\right) & \xrightarrow{(i)} \left(\Omega_m, 0\right) \xrightarrow{(ii)} \left(\Omega_m, \theta/2\right) \xrightarrow{(iii)} \left(\Omega_m, \theta/2 + \pi\right) \\
& \xrightarrow{(iv)} \left(\Omega_m, \theta + \pi\right) \xrightarrow{(v)} \left(0, \theta + \pi\right)
\end{align*}
$$

(4)

Steps (i,ii) and (iv,v) are performed adiabatically and require a time $T$. The double arrow of step (iii) indicates a sudden change of parameters. Note that $\Omega(0) = \Omega(2T) = 0$, $\Omega(t) = \Omega(2T - t)$ and $\varphi(t) = \pi + \varphi(2T - t)$, which does not require the knowledge of the function $f$ but implies a precise control of the phase. A simple analysis shows that (iv-v) achieve the desired transformation $|0\rangle \rightarrow e^{i\theta/2}|0\rangle$, $|1\rangle \rightarrow e^{-i\theta/2}|1\rangle$. Note also that the dynamical and geometrical phases acquired in the adiabatic processes (i-v) cancel out.

The Hadamard gate can be performed in a similar fashion. In the space of $[\Delta, \Omega_r = \Omega \cos(\varphi)]$, the protocol is

$$
\begin{align*}
\left(0, \Omega_m\right) & \xrightarrow{(i)} \left(\Delta_m, \Omega_m\right) \xrightarrow{(ii)} \left(\Delta_m, 0\right) \xrightarrow{(iii)} \left(\Delta_m, \Omega_m\right) \xrightarrow{(iv)} \\
& \xrightarrow{(v)} \left(\Delta, \Omega_m\right)
\end{align*}
$$

(5)

as shown in Fig. 1(b-c). In order to avoid the dynamical phases, we have to make sure that steps (i-v) are run in half the time as (vi-vii). More precisely, if $t < T$, we must ensure that $\Delta(t) = \Delta(T - t)$, $\Omega_r(t) = \Omega_r(T - t)$, $\Delta(T + t) = \Delta(t/2)$ and $\Omega_r(T + t) = \Omega_r(t/2)$. With this requisite we get $\frac{1}{\sqrt{2}}( |0\rangle + |1\rangle ) \rightarrow |0\rangle$, $\frac{1}{\sqrt{2}}( |0\rangle - |1\rangle ) \rightarrow -|1\rangle$. Again, the whole procedure does not require us to know $\Omega$ or $\Delta$, but rather to control the evolution of the experimental parameters which determine them.

The C-NOT gate requires the combination of two qubit processes using $H_2$ and one local gate. The first process involves changing the parameters $[\Delta, \Omega_r = \Omega \cos(\varphi)]$ of Eq. (2) according to

$$
\begin{align*}
\left(\Delta_m, 0\right) & \xrightarrow{(i)} \left(\Delta_m, \Omega_m\right) \xrightarrow{(ii)} (0, \Omega_m) \xrightarrow{(iii)} (0, -\Omega_m) \\
& \xrightarrow{(iv)} \left(\Delta_m, -\Omega_m\right) \xrightarrow{(v)} \left(\Delta_m, 0\right)
\end{align*}
$$

(6)

This procedure gives rise to the transformation

$$
U_1 = |0\rangle\langle 0| \otimes I + e^{i\xi} |1\rangle\langle 1| \otimes \sigma_y,
$$

(7)

where $\xi = \int_{0}^{T} \theta(t) dt$, is an unknown dynamical phase. The second operation required is a NOT on the first qubit $U_2 = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes I$. Finally, if $\Delta^{(1)}(t)$ denotes the evolution of $\Delta$ in Eq. (6), we need to follow a path such that $\Delta^{(1)}(t) = \Delta^{(1)}(t)$, $\Omega^{(1)}(t) = 0$. If the timing is correct, we achieve the transformation

$$
U_3 = (|0\rangle\langle 0| + e^{i\xi} |1\rangle\langle 1|) \otimes I.
$$

(8)
schemes so far [13] it is assumed that there is a single atom per lattice site since otherwise even the concept of qubits is no longer valid. In present experiments, in which the optical lattice is loaded with a Bose-Einstein condensate [4, 6], this is not the case (since zero temperature is required and the number of atoms must be identical to the number of lattice sites). We show now a novel implementation in which, with the help of the methods presented above, one overcomes this problem.

For us a qubit will be formed by an aggregate of atoms at some lattice site. The number of atoms forming each qubit is completely unknown. The only requirement is that there is at least one atom per site [14]. We will denote by $n_k$ the number of atoms in the $k$-th well and identify the states of the corresponding qubit as

$$|0\rangle_k = \frac{a_k^+}{\sqrt{n_k}} |vac\rangle, \quad |1\rangle_k = \frac{b_k^+}{\sqrt{n_k}} |vac\rangle, \quad (9)$$

where $a_k^+$ ($b_k^+$) are the creation operators for one atom in levels $|a\rangle$ and $|b\rangle$, respectively. The quantum gates will be realized using lasers, switching the tunneling between neighboring sites, and using the atom-atom interaction.

In the absence of any external field, the Hamiltonian describing our system is

$$H = -J_k^{(b)} \sum_k \left( b_k^{+1} b_k + b_k^{+1} b_{k+1} \right) + \frac{U_{bb}}{2} \sum_k b_k^{+1} b_k b_k b_k. \quad (10)$$

Here, $U_{bb}$ and $J_k^{(b)}$ describe the interactions between and the tunneling of qubits in state $|b\rangle$. We will assume that $J_k^{(b)}$ can be set to zero and increased by adjusting the intensities of the lasers which create the optical lattice. We have assumed that the atom in state $|a\rangle$ do not interact at all and do not hop, something which may be achieved by tuning the scattering lengths and the optical lattice. Both restrictions will be relaxed later on. The Hamiltonian (10) possesses a very important property when all $J_k^{(b)} = 0$, namely it has no effect on the computational basis (i.e., $H|\Psi\rangle = 0$ for all states $|\Psi\rangle$ in the Hilbert space generated by the qubits). Otherwise, it would produce a non-trivial effect that would spoil the computation.

We show now how a single qubit gate on qubit $k$ can be realized using lasers. First, during the whole operation we set $J_k^{(b)} = 0$ in order to avoid hopping. The laser interaction is described by the Hamiltonian

$$H_{lk}^{(k)} = \frac{\Delta_k}{2} (a_k^{+1} a_k - b_k^{+1} b_k) + \frac{\Omega_k}{2} (e^{i\phi} a_k^{+1} b_k + e^{-i\phi} b_k^{+1} a_k). \quad (11)$$

For $U_{bb} \gg |\Delta_k|, |\Omega_k|$, we can project the total Hamiltonian acting on site $k$ by an effective Hamiltonian acting on the qubit which resembles (1), with $\Delta = \Delta_k$ and $\Omega = \Omega_k \sqrt{n_k} - T$. Thus, using the methods exposed above
we can achieve the Hadamard and phase gates with a high precision, even though the coupling between the bosonic ensemble and light depends on the number of atoms.

For the realization of the two-qubit Hamiltonian (2) we need to combine several elements. First of all we need the Raman coupling of Eq. (11) to operate on the second well. Second, we need to tilt the lattice using a magnetic field [4] which couples to states $|a\rangle$ and $|b\rangle$ differently, $H_{\text{tilt}} = \sum_b k g (|a\rangle\langle a| + i|b\rangle\langle b|)$. And finally we must allow virtual hopping of atoms of the type $|b\rangle \rightarrow |a\rangle (J^{(b)} \ll |U_{bb} - g|)$. After adiabatic elimination we find that the effective Hamiltonian depends on the number of particles in the second site, $n_2$,

$$H_{\text{eff}}^{(2)} = \frac{J_2^2}{g(U_{bb} - g)^2} \left< |1\rangle \langle 1| \right> + \sqrt{n_2 - 1} \Omega \mathbb{I} \otimes \sigma_x. \quad (12)$$

The identification with Eq. (2) is evident, and once more the use of adiabatic passage will produce gates which are independent of the number of particles.

We have studied the different sources of error which may affect our proposal: (i) $U_{bb}$ is finite and the $n_2$ atoms in state $|a\rangle$ may hop and interact. These last phenomena are described by additional contributions to Eq. (10) which are of the form $J_k^{(a)} (a_k^\dagger a_{k+1} + a_{k+1}^\dagger a_k) U_{aa} a_k^\dagger a_k^\dagger a_k a_k$, and $U_{ab} a_k^\dagger b_k b_k$. The consequences of both imperfections are: (i) more than one atom per well can be excited, (ii) the occupation numbers may change due to hopping of atoms and (iii) by means of virtual transitions the effective Hamiltonian differs from (1) and (2). The effects (i)-(ii) are eliminated if $\Omega / U_{bb} \ll 1$ and $(J_k^{(a)} / U_{ab}) \ll 1$. Once these conditions are met, we may analyze the remaining errors with a perturbative study of the Hamiltonians (11) and (10) plus the terms $(J^{(a)} / U_{bb})$ that we did not consider before. In Eq. (11), the virtual excitation of two atoms increments the parameter $\Delta$ by an unknown amount, $\Delta_{\text{eff}} \sim \Delta + 2 \Omega n_2 / (\Delta + U_{ab} - U_{bb})$. If $U_{ab} \ll U_{bb}$ and $\Omega U_{ab}/U_{bb} \ll 1$, this shift may be neglected. In the two-qubit gates the energy shifts are instead due to virtual hopping of all types of atoms. They are of the order of $max(J^{(a)} / U_{bb}^2) \sim J^2 / U_{bb}$, and for $J^2 / U_{bb} \ll 1$ they also may be neglected.

To quantitatively determine the influence of these errors we have simulated the evolution of two atomic ensembles with an effective Hamiltonian which results of adding second order perturbation theory to Eq. (10), and which takes into account all important processes. The results are shown in Figs. 2(abcd). For the two-qubit gate we have assumed $U_{aa} = U_{ab} = J$, $J_m = 0.05 J_{ab}$, $\Omega_m = J_m / 10$, $g = U_{bb} + U_{ab} / 2$ and operation time, $T = 200 / \Omega_m$, while changing the ratio $U_{ab} / U_{bb}$ and the populations of the wells. For the local gates we have assumed $\Delta_m = 1$, $\Omega_m = U_{bb} / 10$ and different occupation numbers $n_k$ and we have also changed $U_{ab} / U_{bb}$.

We extract several conclusions. First, the stronger the interaction between atoms in state $|b\rangle$, the smaller the energy shifts. Typically, a ratio $U_{ab} = 10^4 U_{bb}$ is required to make $J^{(b)} / U_{bb} \sim 10^{-4}$. Second, the larger the number of atoms per well, the poorer the fidelity of the local gates [Figs. 2(b-c)]. And finally, the fidelity of the two-qubit gate presents a small dependence on the population imbalance between wells.

In this work we have shown that it is possible to perform quantum computation even when the constants in the governing Hamiltonians are unknown. We have developed a scheme based on performing adiabatic passage with one-qubit (1) and two-qubit (2) Hamiltonians. With selected paths and appropriate timing, it is possible to perform a universal set of gates (Hadamard gate, phase gate and a CNOT). These procedures cannot be only used for quantum computing but also for quantum simulation [15]. Finally, based on the preceding ideas, we have proposed a scheme for quantum computing with cold atoms in a tunable optical lattice. Our scheme works even when the number of atoms per lattice site is uncertain. Note that, these ideas also apply to some other setups like the microtraps recently realized in Ref. [16].

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[1] Note that, in principle, one could measure the dependence of these parameters (function $f$). However in many realistic implementations this is not possible, since the measurements are destructive (dead to heating or the atoms escape the traps), and in different realizations the dependence is different.


[14] The current scheme would even work with a superfluid phase which is abruptly loaded in a deep optical lattice.