$M_\pi^2$ versus $m_q$: comparing CP-PACS and UKQCD data to Chiral Perturbation Theory

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Abstract

I present a selection of CP-PACS and UKQCD data for the pseudo-Goldstone masses in $N_f = 2$ QCD with valence quarks exactly degenerate with the sea quarks. At least the more chiral points should be consistent with Chiral Perturbation Theory for the latter to be useful in extrapolating to physical masses. I discuss the chiral convergence behaviour and possible reasons why the expected chiral logs are barely visible in the data.

Introduction

Ever since numerical lattice QCD computations have been done, the light meson spectrum has served as a benchmark problem. The aim has been, in the pioneering days, to demonstrate the viability of the approach itself and to test, thereafter, the efficiency of new actions and algorithms. It still serves today, as actions respecting the chiral symmetry among the light flavours get tested and several collaborations have embarked on ambitious simulations of full QCD. These developments address one of the key issues in low-energy QCD. That chiral symmetry is both spontaneously and explicitly broken gives rise to pseudo-Goldstone bosons, i.e. particles that dominate (for small enough quark masses) the long-range behaviour of the Green’s functions between external currents and which are collectively called “pions”.

Lattice QCD is, however, not the only framework to address the low-energy structure of the theory. In the old days PCAC relations were exploited to predict the dependence of low-energy observables on shifts in the quark masses and external momenta. The one best known is

$$F_\pi^2 M_\pi^2 = (m_u + m_d) \left| \langle 0 | \bar{q} q | 0 \rangle \right| + O(m^2)$$

which connects the pion observables to the product of the explicit and spontaneous symmetry breaking parameters. The limitations of the Gell-Mann–Oakes–Renner relation (1) are twofold: It does not give separate predictions how $F_\pi$ and $M_\pi$ depend on the quark masses. And the latter may be shifted, in the real world, only by a discrete amount; replacing $d \rightarrow s, \pi \rightarrow K$ one gets a LO prediction for the quark mass ratios, but not more.

Today, the first limitation is overcome, since the successor of PCAC, Chiral Perturbation Theory, gives detailed predictions how either observable $M_\pi, F_\pi$ individually depends on the quark mass $m$. And the second problem is gone, in principle, since one may vary $m$ continuously in a lattice computation. Hence, it seems natural to combine the two approaches to benefit from their respective advantages. However, for that aim quarks need to be taken sufficiently light, and this is a numerical challenge on the lattice. Below, an elementary test whether this is already achieved in present simulations is attempted by considering the way the pion mass depends on the quark mass. We will restrict ourselves to the doubly degenerate case (where the “sea”- and “valence”-quarks are taken identical) with $N_f = 2$ degenerate Wilson-type quarks. This means that we shall neglect isospin breaking and electromagnetic effects and use the average mass $m \equiv (m_u + m_d)/2$ for the first generation quarks throughout.
Prediction by Chiral Perturbation Theory

Gasser and Leutwyler have calculated the substitute for the GOR relation (1) to NLO in Chiral Perturbation Theory (XPT) with $N_f = 2$ quarks [1]. To that order, two low-energy “constants” 1 from the LO Lagrangian appear ($F = \lim_{\mu \to 0} F_\pi$ and $B = -\lim_{\mu \to 0} (\bar{q}q|0\rangle/F^2$) together with some of the Gasser-Leutwyler (GL) coefficients $l_i$ (or $L_i$ in the $N_f = 3$ case) from the NLO Lagrangian. After specifying to the degenerate case their result [1] reads

\begin{align}
M_{\pi}^2 &= M^2 + \frac{1}{F^2} \left( \frac{M^4}{32\pi^2} \log\left(\frac{M^2}{\mu^2}\right) - 2M^4 l_3^* \right) \\
F_\pi &= F + \frac{1}{F} \left( -\frac{M^2}{16\pi^2} \log\left(\frac{M^2}{\mu^2}\right) + M^2 l_4^* \right) \\
M^2 &\equiv 2mB.
\end{align}

Note that (3, 4) has the typical structure of a NLO prediction: A chiral logarithm summarizing the contribution from the pion loops appears together with a counterterm. The $l_i^*$ are the renormalized GL coefficients, i.e. they are the descendents of the $l_i$ (or $L_i$ for $N_f = 3$) which appear in the NLO Lagrangian and which are divergent quantities. As a result, the $l_i^*$ depend on the (chiral) renormalization scale $\mu$. In the dimensionally regularized theory one has

$$l_i = l_i^*(\mu) + \gamma_i\lambda(\mu)$$

$$\lambda(\mu) = \frac{-1}{16\pi^2\mu^{4-d}} \left( \frac{1}{4-d} + \frac{\log(4\pi) + \Gamma'(1) + 1}{2} \right)$$

$$l_i^*(\mu) = l_i^*(\mu^*) - \frac{\gamma_i}{16\pi^2} \log\left(\frac{\mu}{\mu^*}\right)$$

and the $\beta$-function coefficients $\gamma_i$ are known [1]. In the present context only $\gamma_3 = -\frac{1}{2}$, $\gamma_4 = 2$ are relevant, and this suggests that one rewrites (3, 4) with the help of

$$l_i^* = \frac{\gamma_i}{32\pi^2} \left( l_i + \log\left(\frac{M^2}{\mu^2}\right) \right)$$

where the $\mu$ dependence in $l_i^* = l_i^*(\mu)$ is traded for an $M$ dependence in $\tilde{l}_i = \tilde{l}_i(M)$. The result is an expression in terms of four low-energy “constants” ($F, B, \tilde{l}_3, \tilde{l}_4$) and the quark mass $m$ [1]

\begin{align}
M_{\pi}^2 &= M^2 \left( 1 - \frac{M^2}{32\pi^2 F^2} \tilde{l}_3 + O(M^4) \right) \\
F_\pi &= F \left( 1 + \frac{M^2}{16\pi^2 F^2} \tilde{l}_4 + O(M^4) \right).
\end{align}

In view of the analysis below, it is worth emphasizing that the NLO part is given in terms of the LO parameters $F, B$ and the NLO coefficients $\tilde{l}_3, \tilde{l}_4$. While the former two are known quite accurately, for the $\tilde{l}_i$ only their running in $M^2$ is known exactly and the phenomenological estimate of the integration constants has comparatively large error-bars. Gasser and Leutwyler give in their initial paper [1] the estimate

$$\tilde{l}_3(m_{\text{phys}}) = 2.9 \pm 2.4, \quad \tilde{l}_4(m_{\text{phys}}) = 4.3 \pm 0.9$$

\footnote{Unlike $F$, the other LO “constant” $B$ is scheme- and scale-dependent, as is $m$. In the chiral representation, only the product $mB$ appears which is RG-invariant, thus the definition (5). In order to determine $B$ and $m$ separately, a lattice computation is needed; the result of the CP-PACS study [2] along with (5) for the physical pion is

$$m_{\text{phys}}(\overline{\text{MS}}, \mu \sim 2 \text{ GeV}) \simeq 3.5 \pm 0.2 \text{ MeV}, \quad B(\overline{\text{MS}}, \mu \sim 2 \text{ GeV}) \simeq 2.8 \pm 0.15 \text{ GeV}.$$}
for real world quark masses. It turns out that even this limited information is useful, since it determines the curvature of \( M_\pi \) (or \( F_\pi \)) as a function of the quark mass. Close to the chiral limit, both \( \tilde{l}_i \) are positive and as a consequence \( M_\pi^2 \) does not rise strictly linear in \( m \) but turns right, while \( F_\pi \) has a positive second derivative in \( m \). More specifically, (12) translates into

\[
M_{\pi,\text{phys}} \simeq 139 \text{MeV}, \; F_{\pi,\text{phys}} \simeq 92.6 \text{MeV} \iff M_{\text{phys}} \simeq 141 \text{MeV}, \; F \simeq 86.6 \text{MeV}
\]

which means that the \( \pi \) mass deviates from the value it would have, if the GOR relation were exact, by -1\%, while the \( \pi \) decay constant deviates from its value in the chiral limit by +7\%.

A seemingly formal point which, in the end, proves convenient in analyzing the lattice data is the following. A naive look at (10) suggests that the typical structure of a NLO prediction is gone – rather than a \( M^4 \) and a \( M^4 \log(M^2) \) contribution, only the polynomial part seems left. The point is that this impression is entirely misleading; the IR divergencies (which are genuine to QCD in the chiral limit) are not gone, they are just hidden in the \( \tilde{l}_3 \). The situation is, in fact, opposite – the \( M^4 \) part has been eliminated in favour of a pure \( M^4 \log(M^2) \) contribution, and the last step is to make this apparent. The quark mass dependence of \( \tilde{l}_3 \) is given through

\[
\log\left( \frac{\Lambda^2}{M^2} \right) = \log\left( \frac{\Lambda_i^2}{M_{\text{phys}}^2} \right) + \log\left( \frac{M_{\text{phys}}^2}{M^2} \right) = \tilde{l}_i(m_{\text{phys}}) + \log\left( \frac{M_{\text{phys}}^2}{M^2} \right) = \tilde{l}_i(m)
\]

and together with (5), the relation takes its final form (see e.g. [3])

\[
M_{\pi}^2 = 2mB - \frac{m^2B^2}{8\pi^2F^2} \log\left( \frac{\Lambda_3^2}{2mB} \right) + O(m^3) \tag{15}
\]

\[
F_{\pi} = F + \frac{m^2B^2}{8\pi^2F^2} \log\left( \frac{\Lambda_4^2}{2mB} \right) + O(m^3) \tag{16}
\]

where \( \Lambda_4, \Lambda_3 \) represent universal low energy scales. The estimates (12) translate into

\[
\Lambda_3 = 0.6 \text{GeV} \; +1.4 \text{GeV} \; -0.4 \text{GeV}, \; \quad \Lambda_4 = 1.2 \text{GeV} \; +0.7 \text{GeV} \; -0.4 \text{GeV}
\]

which looks promising, since one would expect a low-energy scale to be of the order of \( O(1 \text{GeV}) \). It is worth emphasizing that since all four parameters \( F, M^2 = 2Bm, \Lambda_3, \Lambda_4 \) do not depend on the QCD renormalization scheme [1], i.e. they are the proper physical low-energy parameters at order \( O(p^4) \). Furthermore the \( \Lambda_i \) do not depend on the quark masses, and this means that the representation (15) is, in principle, suitable to analyze the lattice data even if they are gotten at quark masses larger than \( m \equiv (m_u + m_d)/2 \) the real world – as long as they are not beyond the regime of validity of the chiral expansion itself.

The latter point is one of the key issues in the comparison we aim at. The chiral expansion is known to be asymptotic, and this means that increasing the order will enhance the accuracy near the chiral limit – at the price of worsening the prediction for heavier masses. What is the “critical scale” beyond which the chiral expansion “explodes” is, a priori, not clear. From a formal point of view, one might argue that (10, 11) indicate that the expansion is in \( M/(4\pi F) \) and hence hope that it is good for pion masses up to 1 GeV. In this paper I will argue that watching the convergence pattern at a given quark mass gives a more trustworthy feeling for what is the permissible range. This is facilitated since the NNLO expression for \( M_{\pi}^2 \) (with \( N_f = 2 \)) is known [4, 5]. The result reads (for the presentation I follow [3])

\[
M_{\pi}^2 = 2mB \left( 1 - \frac{mB}{16\pi^2F^2} \log\left( \frac{\Lambda_3^2}{M^2} \right) + \frac{m^2B^2}{64\pi^4F^4} \left\{ \frac{17}{8} \left( \log\left( \frac{\Lambda_M^2}{M^2} \right)^2 + k_M \right) + O(m^3) \right\} \right)
\]

where \( \Lambda_M \) is implicitly defined through

\[
51 \log\left( \frac{\Lambda_3^2}{M^2} \right) = 28 \log\left( \frac{\Lambda_4^2}{\mu^2} \right) + 32 \log\left( \frac{\Lambda_3^2}{\mu^2} \right) - 9 \log\left( \frac{\Lambda_4^2}{\mu^2} \right) + 49
\]

and the mass-independent \( k_M \) accounts for the remainder at \( O(p^6) \), in particular the new counterterms. Phenomenological values for \( \Lambda_M \) and \( k_M \) will be mentioned below.
Figure 1: CP-PACS and UKQCD data converted to the form $M^2_\pi$ vs. $2m$ with masses in units of $r_0^{-1}$. Top: Perturbative renormalization with $c_{SW}$ as used in the simulations. Bottom: Same but $c_{SW} = 1$ throughout. A quadratic fit, constrained to go through zero, is applied to the renormalized AWI data. Segments of the asymptotic slope in the chiral limit highlight the curvature.
Lattice Data

We are now in a position to esteem the results by the CP-PACS and UKQCD collaborations for the quark mass dependence of the pion mass. We shall consider, out of these \( N_f = 2 \) data, only the two-fold degenerate case where both valence quarks have the same mass and where they are, at the same time, exactly degenerate with the sea quarks so that the theory is unitary.

The CP-PACS collaboration has simulated various \((\beta, \kappa)\) combinations with an RG improved gauge action and a mean-field improved clover quark action [2]. They use a grid of size \( 12^3 \times 32, 16^3 \times 32, 24^3 \times 48, 24^3 \times 48 \) at \( \beta = 1.8, 1.95, 2.1, 2.2 \), respectively, which leads to a lattice spacing, if determined through the \( \rho \) mass, between 0.215 fm and 0.087 fm and hence a spatial box size between 2.58 fm and 2.08 fm. With so much information at hand, one could, in principle, attempt a continuum extrapolation for \( M_\pi^2 \) versus the sum of the (degenerate) quark masses, \( 2m \). However, non-perturbative renormalization might be necessary, and/or the lattice spacing might be too large at the lower \( \beta \) values. For this reason, I have decided to concentrate on the \( \beta = 2.1 \) data, since here discretization effects are not supposed to be too large, and good statistics is available. Moreover, at that particular \( \beta \) value, even the lightest pion is unlikely to suffer from finite-size effects, since \( M_\pi L > 7 \), and the lattice spacing determined via the \( \rho \) mass is of order 0.1 fm and hence comparable with that in the UKQCD simulations.

The UKQCD collaboration works on a \( 16^3 \times 32 \) grid, using different actions: Wilson glue and non-perturbatively \( O(a) \) improved clover quarks [6]. Another point in which they differ from CP-PACS is a tactical one: they try to relax \( \beta \) in pushing \( \kappa_{\text{sea}} \) up (i.e. the quark mass down) such that the lattice spacing, in units of \( r_0 \), stays constant. Numerically, it is \( a \sim 0.1 \) fm for the data considered below (though the ensemble at \((\beta, \kappa_{\text{sea}}) = (5.2, 0.1355)\) is not matched any more), and the hope is that the size of discretization effects would be approximately constant. This choice implies that the physical box size stays constant, too, and the bound \( M_\pi L > 4.5 \) maintained makes one feel comfortable that finite size effects are small.

The plan of this article is to ignore that the dynamical quark masses might be too heavy for the chiral prediction to be applicable and to ignore that in principle a continuum extrapolation is needed but instead to go ahead and simply compare the two datasets on a “as-is” basis with the LO/NLO/NNLO prediction from Chiral Perturbation Theory in the continuum. With the relevant low-energy constants on the chiral side given in physical units, \( r_0^{-1} \) must be so, too. In this article, this is done through the assumption that \( r_0^{-1} \) represents a universal low-energy scale, unaffected by unquenching effects; the numerical value used is \( r_0 = 0.5 \) fm [7].

Fig. 1 shows \((M_\pi r_0)^2\) versus the sum of the valence quark masses. On the lattice, there are two definitions of the “quark mass”, one through the vector Ward-Takahashi identity (VWI mass), one through the axial identity (AWI mass). Before renormalization they do not agree, and after renormalization they would (up to \( O(a^2) \) effects) if the renormalization factors were computed non-perturbatively. With unquenched \((N_f = 2)\) data this is currently not the case. If the respective renormalization factors are computed e.g. to one-loop, the remaining discrepancy indicates the size of higher-order corrections. The appendices collect the details of a perturbative renormalization of both sets. In the case of the CP-PACS data this merely repeats their calculation [2], except that (to compare like with like) the scale is now set through the measured \( r_0 \) throughout and \( r_0 \) is derived from the measured plaquette, since in the UKQCD data (because of the “matched” strategy) there is no uniquely defined chirally extrapolated version. Fig. 1 displays \((M_\pi r_0)^2\) versus the renormalized quark masses with filled symbols, open symbols indicate the bare data to visualize the shift. In the upper graph the renormalization was performed with the \( c_{\text{SW}} \) values as they were used in the simulations.
Figure 2: LO/NLO/NNLO chiral predictions with phenomenological values for $B, F, \Lambda_3, \Lambda_M, k_M$ ($\pm 1\sigma$ variation included at each order, cf. text for details) compared to the renormalized data in the version with $c_{SW} = 1$. Note that the lines represent parameter-free predictions, not fits.

(suggested by a mean-field analysis [2] or the Alpha study [6, 8]). In the graph below the renormalization factors were computed with $c_{SW} = 1$ throughout, which is a consistent choice at one-loop order. Obviously, the overall consistency of the data is much better with this latter choice, something which is particularly obvious from the traditional quadratic fits (constrained to go through the origin) to the AWI data. In the following we shall stick with this latter choice ($c_{SW} = 1$), but it is useful to keep in mind that the difference to the plot above is another indication of the size of inherent perturbative uncertainties. In this respect it’s worth mentioning that all error-bars in this article represent only statistical errors.

We now turn to the physics content. To convey a feeling for the scales, I mention that at $2mr_0 = 1$ the sum of the valence quark masses is of order 400 MeV, i.e. about four times as much as in a physical $K$, and the corresponding “pion” weighs about 1 GeV. The physical kaon weighs 496 MeV, i.e. $(M_K r_0)^2 \approx 1.58$. If the pion would satisfy $(M_\pi r_0)^2 \approx 1.58$ this would mean that its $u$- and $d$-quarks are about half as heavy as the $s$-quark in the real world. The lightest pion in the CP-PACS and UKQCD simulations have $(M_\pi r_0)^2$ values 1.75 and 2.20, respectively, and from this we conclude that their lightest $u$- and $d$-quarks (in the unitary theory) have about 55% and 70% of the mass of the physical strange quark.

With such heavy “light” quarks it is a priori not clear whether XPT is of any use to extrapolate them to the physical $u$- and $d$-quark masses. In an attempt to shed a light on this issue, Fig. 2 shows the same data (as in the bottom part of Fig. 1) now in conjunction with the predictions from XPT at tree/one-loop/two-loop level. The low-energy constants are taken from phenomenology, i.e. these curves represent parameter-free predictions. At LO the chiral prediction is a straight line with a slope given by the parameter $B$ in (2), multiplied...
Figure 3: Relative shifts in $M^2_\pi$ versus $2m r_0$ in XPT – due to the uncertainty in $B$ at LO (light), due to NLO contribution with $B$ fixed at central value, $\Lambda_3$ varied within $1\sigma$ bound (intermediate), and due to NNLO contribution with $B$ and $\Lambda_3$ fixed at central values, $k_M \in \{0, \pm 2\}$ (dark).

by $r_0$ (the $\pm 1\sigma$ bound is indicated with dotted lines). At NLO the prediction is curved, and the numerical values of the additional parameters are taken from (13) and (17). $\Lambda_3$ is varied within its phenomenological $\pm 1\sigma$ bound (full versus dotted lines$^2$), with $B, F$ fixed at their central values. At NNLO the parameters $\Lambda_M, k_M$ in (18) are determined as follows. With $\Lambda_1 = 0.11 \pm 0.04\text{ GeV}, \Lambda_2 = 1.2 \pm 0.06\text{ GeV}$ [9] and (17) the relation beneath eqn. (18) gives

$$\Lambda_M = 0.60\text{ GeV} \quad \pm 0.032\text{ GeV} - 0.030\text{ GeV}$$ (19)

where errors have been added in quadrature. It turns out that over the range considered the uncertainty in $\Lambda_M$ is negligible compared to that coming from $k_M$. For the latter, phenomenological arguments indicate $|k_M| \sim 2$ [10], and the sum-rule estimates for the NNLO counterterms published in [9] may be converted into a more accurate determination. For the purpose of the present article it is sufficient to use $k_M = 0\pm 2$, and the associate curves (keeping $B, F, \Lambda_3, \Lambda_M$ fixed at their central values) are included in Fig. 2.

We are now in a position to gauge, on an intuitive level, the chiral convergence behaviour in $M^2_\pi$ versus $2m$. To that aim we compare the LO/NLO/NNLO predictions for a given quark mass. Fig. 3 shows the relative shift in $M^2_\pi$ when one more order is included or the new low-energy constants are varied within reasonable bounds. At first sight, the uncertainties at higher orders due to the error bars of the counterterms seem large, but one should keep in mind that the associate shifts are 100% correlated over the whole range. For instance, if the CP-PACS VVI point at $2mr_0 \sim 0.2$ sits on the $-1\sigma$ curve (upper dotted NLO line in Fig. 2),

\footnote{Lowering $\Lambda_3$ to 200 MeV yields the upper, increasing it to 2 GeV yields the lower dotted line.}
then the one at $2mr_0 \sim 0.4$ should too, if only $\Lambda_3$ needs to be adapted. This means that precise lattice data would be ideally suited to greatly reduce the error on $l_3^\Lambda$. Altogether, one is left with the impression that specifically for $M_\pi$ the chiral expansion is sufficiently well-behaved as to be practically useful for quark masses up to

$$2mr_0 \leq 0.25 \iff (M_\pi r_0)^2 \leq 2 \iff M_\pi \leq 560 \text{ MeV}, \quad (20)$$

and maybe more. This, if correct, means that current state-of-the-art simulations manage to make contact with the regime where XPT holds, but so far a non-trivial “lever arm” which is needed to make model independent predictions in the deeply chiral regime, is likely missing.

While it is clear that future simulation data will allow to test this forecast and provide, if it is correct, the lever arm needed, one might, already at this time, go ahead and simply try what comes out if one assumes that the estimate (20) is too pessimistic and hence uses the chiral ansatz to fit the data over an extended range. This is what we shall do below, but it is clear that this attempt is rather speculative and results should be taken with care. Obviously, the NNLO functional form is out of question, but already the NLO form (15) contains 3 free parameters. Since either dataset has 4 points, this means that in a non-trivial fit we will be unable to suppress the heaviest point to judge the quality of the ansatz, unless some parameters are kept fixed. Again, this underpins the cautionary remark above.

Since everything below is about the deviation from a linear relationship, it is useful to make the curvature optically visible. This is conveniently done by dividing out a factor $2mr_0$. The result is displayed in Fig. 4 where the predictions from XPT at LO/NLO/NNLO are included for completeness. In this representation, the genuine feature of the NLO curve is that it lies below the LO constant for light pions, but above if the (LO-) pion mass is larger than $\Lambda_3$. 

Figure 4: Same as Fig. 2, after dividing by $2mr_0$. Still, lines represent parameter-free chiral predictions at LO/NLO/NNLO (from light to dark colour), not fits.
Figure 5: Attempts to fit the arithmetic average of the VWI and AWI data to the NLO chiral functional form with constraints on $B, F, \Lambda_3$ (relaxing them from the phenomenological $B, F, \Lambda_3+\Delta \Lambda_3$ values). Note that there is a priori no reason why non-continuum extrapolated pseudoscalars which do not respect the bound (20) should follow the chiral prediction for light quarks in the continuum.
Table 1: Coefficients in the fits of the functional form (21) to the degenerate CP-PACS and UKQCD data with average VWI and AWI masses. Constrained values in brackets, ranges as indicated in the text. $\chi^2$ is underestimated, since the fact that errors are correlated has been ignored. Phenomenological values for comparison. Since the main uncertainty is systematic, I refrain from quoting errors.

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To fit the data, we shall make one more change to the NLO fitting function. One replaces $2mB$ by $M_\pi^2$, since in this form the data are plotted against something which is directly measured and the change is of yet higher order in the chiral counting. In other words, one uses

$$\frac{(M_\pi r_0)^2}{2m r_0} = Br_0 - \frac{(M_\pi r_0)^2 Br_0}{32\pi^2 (F_\pi r_0)^2} \log \left( \frac{(\Lambda_3 r_0)^2}{(M_\pi r_0)^2} \right)$$  (21)

and fits the 3 dimensionless parameters $Br_0, F_\pi r_0, \Lambda_3 r_0$ or a subset of them.

Attempts to fit all or the more chiral points with an average VWI and AWI definition of the quark mass (while keeping some of $Br_0, F_\pi r_0, \Lambda_3 r_0$ fixed at their phenomenological central or $+1\sigma$ values) are summarized in Tab. 1 and shown in Fig. 5. The number of points included is always such that the fit has 1 d.o.f. The overall impression is that the NLO chiral ansatz manages to describe the data (though most points do not obey (20) and the large values for $F_\pi$ in Fits-b/c look suspicious), but given the current quality of the data, it is too sophisticated; the LO functional form works just perfectly. It is tempting to note that the NLO fits favour a $\Lambda_3$ slightly larger than the phenomenological estimate (hence supporting the “standard” scenario regarding the nature of the chiral symmetry breaking in $N_f = 2$ QCD; for a brief introduction to this topic see [3]), but clearly the systematic uncertainty prevails definite conclusions.

**Discussion**

The aim of the present note has been to compare the CP-PACS and the UKQCD data for $M_\pi^2$ as a function of the quark mass to each other and to the prediction from Chiral Perturbation Theory (XPT). We have seen that, after dividing out a factor $2m r_0$ to “zoom in” on the deviation from a linear behaviour, both datasets show very little curvature. Checking whether the non-continuum-extrapolated data are consistent with (continuum) XPT, we found that either set is compatible with NLO-XPT, if one determines the chiral parameters $F, B, \Lambda_3$ from a fit. Consistency is lost, however, if these low-energy constants are held fixed at their phenomenological “default values” and the whole range is considered. That their data are barely consistent with NLO-XPT has also been noticed by the JLQCD collaboration [11, 12], even if they disregard that the counterterm $f_3^2$ is basically know. With these high-precision data the problem gets more serious, since a plot analogous to Fig. 4 seems to support no [11] or even negative [12] curvature.
At this point it is mandatory to think about possible reasons why the data might not coincide with the chiral prediction. An incomplete list is the following one:

1. They need not, because the pions are too heavy. Most of the data have been obtained in a regime of quark masses where there is no indication that XPT works. Phenomenological experience tells us that this expansion works for mesons as heavy as the physical $K$, i.e. for $(M_\pi r_0)^2 \leq 1.58$. I have argued that this condition might be relaxed to (20). Even if this is true, at best 1 point from either set survives, and hence there is, strictly speaking, no room for trying the NLO-XPT ansatz which has 3 parameters.

2. Scaling violations, in particular implications of the broken chiral symmetry might be so severe that the pattern in the data barely reflects the underlying continuum behaviour. To check one would have to perform a continuum extrapolation or attempt a dynamical simulation with fermions which obey the Ginsparg-Wilson relation.

3. For unknown reasons (e.g. an algorithmic flaw), the data might represent a partially quenched rather than the fully unquenched situation. Obviously, this is a rather remote possibility. The point is that the NLO prediction $M_\pi^2 / M^2 = 1 + \text{const} M^2 \log(M^2 / \Lambda^2)$ has no genuine $M^2$ term. In the partially quenched case, on the other hand, a true $M^2$ contribution does exist [13, 14], and hence a (negative slope) linear behaviour in a plot analogous to Figs. 4/5 need not imply unreasonable values for the low-energy constants.

4. Finite-volume effects have been argued to be small. Since UKQCD uses a smaller lattice and finite-size effects tend to enhance the effective mass (see e.g. [15] for a study in full QCD and a guide to the literature) one might suspect that their most chiral point experiences some shift. However, the size of the effect given in [15] is for their values of $M_\pi$ and $L$ much smaller than the statistical uncertainty.

Finally, I would like to highlight 3 points:

(i) So far, we have ignored systematic uncertainties. Going back to Fig. 1 one sees that they are far from being negligible. What we see there exemplifies the standard wisdom that perturbative renormalization factors which turn out to deviate from 1 by, say, 10% call for either repeating the exercise with non-perturbatively determined Z-factors or at much smaller lattice spacing. In the present context this means that the low-energy constants in Tab. 1 are not ready to be used in phenomenology – the data they are based on aren’t yet “mature”. Nonetheless, the key lesson of our attempt to fit the $N_f = 2$ data with the NLO chiral functional form is that the method is, in principle, well suited to determine NLO low-energy parameters from lattice data. It is reasonable to expect that in a few years time this fitting method will allow to determine $\Lambda_3$ (and hence the $\pi$-$\pi$ scattering length in the $N_f = 2$ framework; see [3] for the connection) far more accurately than by any experiment at that time.

(ii) In the present note the scale is set through the measured $r_0^{-1}$ throughout. In the CP-PACS studies, this scale has been found to depend strongly on $\hat{m}_{VW1}$ or $\hat{M}_\pi^2$ [2, 16]. Hence it is not clear whether normalizing all masses with $r_0$ is the best way to compare with XPT. One could use $\hat{r}_{0,0}$ (the value in the chiral limit) or $\hat{r}_{0,2}$ (the value at $(M_\pi r_0)^2 = 2$), but a preliminary investigation seemed to indicate that such an alternative choice would increase the slope in Fig. 5, but not bring in a considerable amount of curvature.

(iii) Strictly speaking, using the phenomenological $l_3$ in the comparison between the XPT formulas and the lattice data is not quite correct, since the phenomenological determination is with $m_s$ fixed at its physical value, while the lattice studies are $N_f = 2$ simulations, in other words here $m_s$ is sent to infinity. A similar problem arises when connecting the $N_f = 2$ GL
coefficients to their $N_f = 3$ counterparts $L_i^f(\mu)$, because in the former case those members of the pseudo-Goldstone octet which carry strangeness have been integrated out. The link is [17]

$$
L_3^r = 8 \left( 2L_6^r - L_4^r \right) + 4 \left( 2L_8^r - L_5^r \right) - \frac{1}{576\pi^2} \left( \log \left( \frac{\lim_{m_u,d \to 0} M_6^2}{\mu^2} \right) + 1 \right)
$$

(22)

(the limit on the r.h.s. refers to a situation with $m_s$ held fixed at its physical value), and it is tempting to use it as the starting point of a little gedanken experiment: Assume the s-quark mass would be such that even when it is doubled the chiral expansion would not break down. Eqn. (22) tells us that under these circumstances enhancing $m_s$ by a factor 2 (and hence roughly doubling $M_6^2$, too) would lower $l_3^r$ by 0.0001. Since (12) translates into $l_3^r(\mu \sim M_\rho) = 0.0008 \pm 0.0038$ such a shift would be negligible compared to the error-bar, and it seems therefore reasonable to assume that the effect of an added third flavour in the simulations, fixed at the physical s-quark mass (which would fully justify the comparison with $N_f = 2$ XPT), would be small compared to the inherent theoretical uncertainties.

Formula (22) is interesting in yet another respect, since it tells us that lattice studies which pin down $M_\pi^2$ versus $2m$ (and hence $\Lambda_3$ or $l_3^r$) have a say in another issue. In the 3-flavour theory there is the famous “Kaplan Manohar ambiguity”, i.e. the chiral Lagrangian stays invariant under a simultaneous transformation of the quark masses

$$
m_u \to m_u + \lambda m_d m_s, \quad m_d \to m_d + \lambda m_s m_u, \quad m_s \to m_s + \lambda m_u m_d,
$$

and an appropriate modification of $L_6^r, L_7^r, L_8^r$. This would, in principle, allow to tune $m_u = 0$ and hence provide a simple and elegant solution to the strong CP problem [18]. In XPT terminology it is the low-energy combination $2L_5^r(\mu) - L_3^r(\mu)$ that decides whether this is a viable option [19]. In the past XPT has been augmented by theoretical assumptions or model calculations to exclude $m_u = 0$. More recently it has been proposed to determine the $L_i^r$ from lattice simulations [20] and important steps in this program have been taken [21]. Formula (22) tells us that a lattice determination of $l_3^r$, when augmented by knowledge about $L_i^r$ and $L_6^r$, also helps to constrain $m_u = 0$.

Future efforts will be directed towards pushing the quark masses further down and towards getting control over the lattice artefacts. From a chiral perspective, one would prefer to do the continuum extrapolation first, in order to use standard XPT, in the second step, to perform the chiral extrapolation. If it turns out that this order cannot be sustained in practice, the chiral framework needs to be extended to account for the main discretization effects. For the case of unimproved fermions this has been done [22], and it is clear that this approach could be generalized to improved actions. The only disadvantage is the added number of counterterms that need to be fixed from the data. Whether this framework is sufficient to firmly establish consistency with Chiral Perturbation Theory or whether there is no alternative to precise data at smaller quark masses and with reduced discretization effects remains to be seen. The present analysis certainly emphasizes the need to compute renormalization factors non-perturbatively and to perform a continuum extrapolation with dynamical data. But the good news is that it suggest that precise data in the range $1 < (M_\sigma r_0)^2 < 2$ will be sufficient to resolve the issue.

**Acknowledgements**

This work has taken its origin in Seattle, during the INT program “Lattice QCD and Hadron Phenomenology”. Discussions with Maarten Golterman and Rainer Sommer are gratefully acknowledged, as well as useful correspondence with Derek Hepburn and Akira Ukawa. I am indebted to Heiri Leutwyler for providing me with phenomenological estimates of the NNLO constant $k_M$ and to Stefano Capitani for a check of the generic formulas in appendices A,B.
Appendix A: Renormalization of the VWI quark mass

Setup with \( m^\text{VWI}_u = \log(1 + \frac{1}{2}(\frac{1}{\kappa} - \frac{1}{\kappa_c})) \approx \frac{1}{2}(\frac{1}{\kappa} - \frac{1}{\kappa_c}) \) and \( u_0 = P^{1/4} = (\frac{1}{3} \langle \text{Tr} U_0 \rangle)^{1/4} \) [23]:

\[
m^\text{VWI}_{\text{MS}}(\mu) = Z_m(\mu a)(1 + b_m a m) m
\]

(24)

\[
m^\text{VWI}_{\text{MS}}(\mu) = \tilde{Z}_m(\mu a)(1 + \tilde{b}_m a m u_0) m
\]

(25)

From [24], naive versus tadpole-improved, with \( g^2 = 6/\beta \) and \( \tilde{g}^2 = \tilde{g}^2_{\text{MS}}(2 \text{GeV}) \):

\[
Z_m(\mu a) = 1 + \frac{g^2}{4\pi} \left( \frac{\tilde{z}_m}{3\pi} - \frac{1}{\pi} \log(a^2 \mu^2) \right)
\]

(26)

\[
\tilde{Z}_m(\mu a) = 1 + \frac{\tilde{g}^2}{4\pi} \left( \frac{\tilde{z}_m}{3\pi} - \frac{1}{\pi} \log(a^2 \mu^2) \right)
\]

(27)

From [25] (key to orig. literature: their [21-28]), for generic actions:

\[
\frac{1}{g^2_{\text{MS}}(\mu)} = \frac{1}{g^2} + d_g + d_f N_f + \frac{11 - 2N_f/3}{16\pi^2} \log(a^2 \mu^2)
\]

(28)

Wilson/Clover: \( d_g = -0.4682, d_f = 0.0314917 \) (for \( c_{SW} = 1 \)) and \( P = 1 - 1/3 \cdot g^2 \)

\[
\frac{1}{g^2_{\text{MS}}(\mu)} = \frac{1}{g^2} - 0.4682 + 0.0314917 N_f + \frac{11 - 2N_f/3}{16\pi^2} \log(a^2 \mu^2)
\]

(29)

\[
\frac{1}{g^2_{\text{MS}}(\mu)} = \frac{P}{g^2} - 0.1349 + 0.0314917 N_f + \frac{11 - 2N_f/3}{16\pi^2} \log(a^2 \mu^2)
\]

(30)

\[
z_m = 12.953 + 7.738 c_{SW} - 1.380 c_{SW}^2
\]

(31)

\[
\tilde{z}_m = z_m - \pi^2 = \left\{ \begin{array}{l}
13.0 \quad (c_{SW} \approx 2) \\
9.44 \quad (c_{SW} = 1)
\end{array} \right.
\]

(32)

Iwasaki/Clover: \( d_g = 0.1000, d_f = 0.0314917 \) (for \( c_{SW} = 1 \)) and \( P = 1 - 0.1402 g^2 \), \( R = 1 - 0.2689 g^2 \), thus \( 1 = P + 0.1402 g^2 = R + 0.2689 g^2 = 3.648P - 8 \cdot 0.331R - 0.2006 g^2 \)

\[
\frac{1}{g^2_{\text{MS}}(\mu)} = \frac{1}{g^2} + 0.1000 + 0.0314917 N_f + \frac{11 - 2N_f/3}{16\pi^2} \log(a^2 \mu^2)
\]

(33)

\[
\frac{1}{g^2_{\text{MS}}(\mu)} = \frac{P}{g^2} + 0.2402 + 0.0314917 N_f + \frac{11 - 2N_f/3}{16\pi^2} \log(a^2 \mu^2)
\]

(34)

\[
\frac{1}{g^2_{\text{MS}}(\mu)} = \frac{R}{g^2} + 0.3689 + 0.0314917 N_f + \frac{11 - 2N_f/3}{16\pi^2} \log(a^2 \mu^2)
\]

(35)

\[
\frac{1}{g^2_{\text{MS}}(\mu)} = \frac{3.648P - 2.648R}{g^2} - 0.1006 + 0.0314917 N_f + \frac{11 - 2N_f/3}{16\pi^2} \log(a^2 \mu^2)
\]

(36)

\[
z_m = 4.858 + 5.301 c_{SW} - 1.267 c_{SW}^2
\]

(37)

\[
\tilde{z}_m = z_m - 0.4206 \pi^2 = \left\{ \begin{array}{l}
5.76 \quad (c_{SW} = 1.47) \\
4.74 \quad (c_{SW} = 1)
\end{array} \right.
\]

(38)

In [26] one finds:

Wilson/Clover:

\[
b_m = -1/2 - 0.09623 g^2 + O(g^4)
\]

(39)

\[
\tilde{b}_m = b_m(1 - 1/12 \cdot \tilde{g}^2) \approx -1/2 - 0.05456 \tilde{g}^2
\]

(40)

Iwasaki/Clover:

\[
b_m = -1/2 - 0.0509 g^2 + O(g^4)
\]

(41)

\[
\tilde{b}_m = b_m(1 - 0.03505 \tilde{g}^2) \approx -1/2 - 0.0334 \tilde{g}^2
\]

(42)
Appendix B: Renormalization of the AWI quark mass

Setup with \( m^{AWI,\text{imp}} = \frac{1}{2} \langle \partial_\mu A_\mu^{\text{imp}}(x) O^a(0) \rangle / \langle P^a(x) O(0) \rangle \) and \( u_0 = P^{1/4} = (\frac{1}{3} \langle \text{Tr} U_0 \rangle)^{1/4} \) [23]:

\[
m^{AWI}_{MS}(\mu) = \frac{Z_A}{Z_P(\mu a)} \frac{1 + b_A m}{1 + b_P m} m^{AWI,\text{imp}}
\]

\[
m^{AWI}_{MS}(\mu) = \frac{\tilde{Z}_A}{\tilde{Z}_P(\mu a)} \frac{1 + \tilde{b}_A m / u_0}{1 + \tilde{b}_P m / u_0} m^{AWI,\text{imp}}
\]

From [24], naive versus tadpole-improved, with \( g^2 = 6/\beta \) and \( \tilde{g}^2 = \tilde{g}^2_{MS}(2 \text{ GeV}) \):

\[
Z_A = 1 + \frac{g^2}{4\pi} \frac{z_A}{3\pi}, \quad Z_P(\mu a) = 1 + \frac{g^2}{4\pi} \left( \frac{z_P}{3\pi} + \frac{1}{\pi} \log(a^2 \mu^2) \right)
\]

\[
\tilde{Z}_A = 1 + \frac{\tilde{g}^2}{4\pi} \frac{\tilde{z}_A}{3\pi}, \quad \tilde{Z}_P(\mu a) = 1 + \frac{\tilde{g}^2}{4\pi} \left( \frac{\tilde{z}_P}{3\pi} + \frac{1}{\pi} \log(a^2 \mu^2) \right)
\]

From [27], for generic actions:

Wilson/Clover:

\[
z_A = -15.797 - 0.248 c_{SW} + 2.251 c_{SW}^2
\]

\[
\tilde{z}_A = z_A + \pi^2 = \begin{cases} 2.581 & (c_{SW} \simeq 2) \\ -3.924 & (c_{SW} = 1) \end{cases}
\]

\[
z_P = -22.596 + 2.249 c_{SW} - 2.036 c_{SW}^2
\]

\[
\tilde{z}_P = z_P + \pi^2 = \begin{cases} -16.372 & (c_{SW} \simeq 2) \\ -12.513 & (c_{SW} = 1) \end{cases}
\]

Iwasaki/Clover:

\[
z_A = -8.192 - 0.125 c_{SW} + 1.610 c_{SW}^2
\]

\[
\tilde{z}_A = z_A + 0.4206 \pi^2 = \begin{cases} -0.746 & (c_{SW} = 1.47) \\ -2.556 & (c_{SW} = 1) \end{cases}
\]

\[
z_P = -10.673 + 1.601 c_{SW} - 1.281 c_{SW}^2
\]

\[
\tilde{z}_P = z_P + 0.4206 \pi^2 = \begin{cases} -6.936 & (c_{SW} = 1.47) \\ -6.202 & (c_{SW} = 1) \end{cases}
\]

In [26] one finds:

Wilson/Clover:

\[
b_A = 1 + 0.15219(5) g^2 + O(g^4)
\]

\[
\tilde{b}_A = b_A (1 - 1/12 \cdot \tilde{g}^2) \simeq 1 + 0.06886(5) \tilde{g}^2
\]

\[
b_P = 1 + 0.15312(3) g^2 + O(g^4)
\]

\[
\tilde{b}_P = b_P (1 - 1/12 \cdot \tilde{g}^2) \simeq 1 + 0.06979(3) g^2
\]

Iwasaki/Clover:

\[
b_A = 1 + 0.0733(5) g^2 + O(g^4)
\]

\[
\tilde{b}_A = b_A (1 - 0.03505 \tilde{g}^2) \simeq 1 + 0.0383(5) \tilde{g}^2
\]

\[
b_P = 1 + 0.0744(12) g^2 + O(g^4)
\]

\[
\tilde{b}_P = b_P (1 - 0.03505 \tilde{g}^2) \simeq 1 + 0.0394(12) \tilde{g}^2
\]

In all cases the relationship between \( z_X \) and \( \tilde{z}_X \) as well as \( b_X \) and \( \tilde{b}_X \) (\( X \in \{m, A, P\} \)) is mine.
Appendix C: Implementation with tadpole resummation

The stepwise renormalization calculation with $u_0$ defined via the plaquette is traced in Tab. 2 for the UKQCD data. Alternatively, one could define $u_0$ as $1/(8\kappa_c)$, but this would bring different perturbative coefficients than those listed in App. A/B. Notice the effect of the tadpole resummation, i.e. the difference of the coupling $\tilde{g}_{\text{MS}}^2$ computed via (29) and $\hat{g}_{\text{MS}}^2$ via (30). The statistical uncertainty for the VWI quark mass is larger than that for the AWI definition due to a limited accuracy of $\kappa_c$, which is defined in a partially quenched sense [6].

For the CP-PACS data, both the plaquette $P$ and the rectangle $R$ are published [2], and this gives, in principle, the option to define the tadpole resummation (besides the usual option $1/(8\kappa_c)$) via (35) or (36), where the latter choice reflects the specific combination used in the action. All these options would, however, imply different perturbative coefficients than those listed in App. A/B, i.e. we restrict ourselves to the choice (34) along with $u_0 \equiv P^{1/4}$. It is

<table>
<thead>
<tr>
<th>UKQCD</th>
<th>(5.20,0.1355)</th>
<th>(5.20,0.1350)</th>
<th>(5.26,0.1345)</th>
<th>(5.29,0.1340)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_\pi = M_\pi a$</td>
<td>0.294(4)</td>
<td>0.405(4)</td>
<td>0.509(2)</td>
<td>0.577(2)</td>
</tr>
<tr>
<td>$\tilde{r}_0 = r_0/a$</td>
<td>5.041(40)</td>
<td>4.754(40)</td>
<td>4.708(52)</td>
<td>4.813(45)</td>
</tr>
<tr>
<td>$(M_\pi r_0)^2$</td>
<td>2.20(09)</td>
<td>3.71(14)</td>
<td>5.74(17)</td>
<td>7.71(20)</td>
</tr>
<tr>
<td>$\kappa_c$</td>
<td>0.13645(3)</td>
<td>0.13663(5)</td>
<td>0.13709(3)</td>
<td>0.13730(3)</td>
</tr>
<tr>
<td>$\tilde{m}_{\text{VWI}} \equiv \frac{1}{2}(1 - \frac{1}{\kappa_c})$</td>
<td>0.0257(3)</td>
<td>0.0462(3)</td>
<td>0.0742(3)</td>
<td>0.0952(3)</td>
</tr>
<tr>
<td>$\tilde{m}_{\text{AWI},\text{imp}}$</td>
<td>0.536294(9)</td>
<td>0.533676(9)</td>
<td>0.539732(9)</td>
<td>0.542410(9)</td>
</tr>
<tr>
<td>$P$ [28]</td>
<td>2.1640(45)</td>
<td>2.1309(46)</td>
<td>2.0813(58)</td>
<td>2.0714(49)</td>
</tr>
<tr>
<td>$(a \text{2 GeV})^2 \equiv (5.06773/\tilde{r}_0)^2$</td>
<td>2.541(6)</td>
<td>2.510(6)</td>
<td>2.437(8)</td>
<td>2.423(7)</td>
</tr>
<tr>
<td>$\hat{g}_{\text{MS}}^2$ (2 GeV) [29]</td>
<td>1.011(16)</td>
<td>1.136(19)</td>
<td>1.159(26)</td>
<td>1.109(21)</td>
</tr>
<tr>
<td>$\hat{g}_{\text{MS}}^2$ (2 GeV) [30]</td>
<td>0.0231(3)</td>
<td>0.0462(3)</td>
<td>0.0742(3)</td>
<td>0.0952(3)</td>
</tr>
<tr>
<td>$\tilde{b}_m$ [40]</td>
<td>1.1750(4)</td>
<td>1.1728(4)</td>
<td>1.1678(5)</td>
<td>1.1669(5)</td>
</tr>
<tr>
<td>$\tilde{b}_A$ [56]</td>
<td>1.1773(4)</td>
<td>1.1752(4)</td>
<td>1.1701(6)</td>
<td>1.1691(5)</td>
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<td>$\tilde{b}_P$ [58]</td>
<td>1.02171</td>
<td>2.0171</td>
<td>1.9497</td>
<td>1.9192</td>
</tr>
<tr>
<td>$\tilde{z}_m$ [31, 32]</td>
<td>2.73099</td>
<td>2.73099</td>
<td>2.14587</td>
<td>1.88782</td>
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<tr>
<td>$\tilde{Z}_m(a \text{2 GeV})$ [27]</td>
<td>1.2799(17)</td>
<td>1.2690(18)</td>
<td>1.2569(22)</td>
<td>1.2566(18)</td>
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<td>$\tilde{Z}_A(a \text{2 GeV})$ [46]</td>
<td>1.0586(1)</td>
<td>1.0579(1)</td>
<td>1.0442(1)</td>
<td>1.0386(1)</td>
</tr>
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<td>$\tilde{Z}_P(a \text{2 GeV})$ [46]</td>
<td>0.6472(1)</td>
<td>0.6590(2)</td>
<td>0.6781(3)</td>
<td>0.6808(3)</td>
</tr>
<tr>
<td>$2m_{\text{VWI}}(2 \text{ GeV})r_0$ [25]</td>
<td>0.380(16)</td>
<td>0.603(25)</td>
<td>0.919(23)</td>
<td>1.181(24)</td>
</tr>
<tr>
<td>$2m_{\text{AWI}}(2 \text{ GeV})r_0$ [44]</td>
<td>0.381(09)</td>
<td>0.705(13)</td>
<td>1.076(19)</td>
<td>1.398(21)</td>
</tr>
<tr>
<td>$c_{\text{SW}}$ [32, 48, 50]</td>
<td>3.9441 , -3.9244 , -12.5134</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{Z}_m(a \text{2 GeV})$ [27]</td>
<td>1.2019(15)</td>
<td>1.192(16)</td>
<td>1.1852(20)</td>
<td>1.1869(17)</td>
</tr>
<tr>
<td>$\tilde{Z}_A(a \text{2 GeV})$ [46]</td>
<td>0.9158(2)</td>
<td>0.9168(2)</td>
<td>0.9192(3)</td>
<td>0.9197(2)</td>
</tr>
<tr>
<td>$\tilde{Z}_P(a \text{2 GeV})$ [46]</td>
<td>0.7322(4)</td>
<td>0.7429(4)</td>
<td>0.7516(5)</td>
<td>0.7503(5)</td>
</tr>
<tr>
<td>$2m_{\text{VWI}}(2 \text{ GeV})r_0$ [25]</td>
<td>0.357(15)</td>
<td>0.567(23)</td>
<td>0.867(21)</td>
<td>1.115(23)</td>
</tr>
<tr>
<td>$2m_{\text{AWI}}(2 \text{ GeV})r_0$ [44]</td>
<td>0.291(07)</td>
<td>0.542(10)</td>
<td>0.854(16)</td>
<td>1.123(17)</td>
</tr>
</tbody>
</table>

Table 2: Stepwise renormalization of UKQCD quark masses with $u_0$ defined via the plaquette. Error bars include statistical errors only, naive error propagation throughout. Note the disparity of $\hat{g}_{\text{MS}}^2$ defined via (29) and that via (30). It makes quite a difference whether $c_{\text{SW}}$ as used in the simulations is plugged in or $c_{\text{SW}} = 1$ which is a consistent choice at the order we are interested in.
interesting to see that the CP-PACS RG improved action achieves agreement of $g^2_{\text{MS}}$ with the “standard” $g^2_{\text{MS}}$ (via (34)), but not with (35) or (36), which are just added for comparison.

In the argument of the logarithm that converts to the $\overline{\text{MS}}$ scheme, the lattice spacing is multiplied with a physical scale, here 2 GeV. This means that $a$ must be assigned a physical value, as well. To compare like with like this is done via the measured $r_0$ in both sets, assuming $r_0 = 0.5 \, \text{fm}$ in physical units. As a consequence, our $Z$ factors for the CP-PACS data depend slightly on the quark mass, even though we work in a mass-independent scheme.

### Table 3: Stepwise renormalization of CP-PACS quark masses with $u_0$ defined via the plaquette. Error bars include statistical errors only, naive error propagation throughout. Note the similarity of $g^2_{\text{MS}}$ defined via (33) and that via (34). There is little difference whether $c_{\text{SW}}$ as used in the simulations is plugged in or $c_{\text{SW}} = 1$ which is a consistent choice at the order we are interested in.
References


