Construction of exact Riemannian instanton solutions

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Abstract

We give the exact construction of Riemannian (or stringy) instantons, which are classical solutions of 2d Yang-Mills theories that interpolate between initial and final string configurations. They satisfy the Hitchin equations with special boundary conditions. For the case of $U(2)$ gauge group those equations can be written as the sinh-Gordon equation with a delta function source. Using techniques of integrable theories based on the zero curvature conditions, we show that the solution is a condensate of an infinite number of one-solitons with the same topological charge and with all possible rapidities.
1 Introduction

In this paper we intend to prove the existence of Riemannian or stringy instantons. The name is due to the fact that these are classical solutions of a 2D $U(N)$ YM theory that interpolate between initial and final string configurations. In other words they describe Riemann surfaces with punctures, where the latter represent the asymptotic entering and exiting strings. In the simplest case ($N = 2$), proving the existence of Riemannian instantons amounts to finding exact solutions of the sinh–Gordon equation with a delta–function source (see below for more details).

To be definite let $a = a(z)$ be a polynomial in the complex variable $z$, with distinct roots, and let us introduce a new variable $\zeta$ defined by $\frac{\delta\zeta}{\delta w} = \sqrt{a}$, where $z = e^w$, $w$ being the coordinate on an infinite cylinder. The equation we want to solve is

$$\partial_\zeta \partial_{\bar{\zeta}} u - 2g^2 \sinh 2u = -\frac{\pi}{4} \delta(a)(\partial_\zeta a)(\partial_{\bar{\zeta}} \bar{a}).$$ (1.1)

which has to be understood in the sense of complex distribution theory. In this equation $g$ is a constant coupling. An equivalent way to state (1.1) is to write the usual sinh–Gordon equation

$$\partial_\zeta \partial_{\bar{\zeta}} u - 2g^2 \sinh 2u = 0$$ (1.2)

and to look for solutions which, near the zeroes of $a$, behave like

$$u \sim -\frac{1}{2} \ln |a|, \quad \text{for} \quad a \sim 0.$$ (1.3)

We will also require that $u$ vanishes as $z \to 0$ and $z \to \infty$. This kind of equation was met for the first time in the framework of Matrix String Theory in [1, 2, 3, 4], and solutions to these equations, satisfying (1.3), were shown to exist only numerically. In particular in [4] this was done in the framework of a square lattice approximation, assuming a very simple form for $a$.

The same type of solutions appeared in the context of form factors and correlation functions for the Ising model in [5].

This paper is devoted to proving the existence of solutions to the above equations, with the desired boundary conditions, in an analytic way, and to give their closed expressions in terms of the modified Bessel function $K_0$. In fact, the validity of the solution relies on some non-linear differential identities satisfied by integrals of $K_0$, which to our knowledge, have not appeared in the literature.

The central ideas of the proof is (1) to use the Leznov-Saveliev approach and (2) to view the solution as a condensate of solitons.
More in detail, we begin by writing the sinh-Gordon equation (1.1) in terms of zero curvature conditions, which include besides the usual Lax-Zakharov-Shabat equation a second relation leading to non-local conservation laws. Such generalized zero curvature conditions follow from the ideas proposed in [6] to study integrable theories in any dimension. Once we have the equations of motion written in terms of a zero curvature condition, we utilize the Leznov-Saveliev method to construct the corresponding Riemannian instanton solution. This method uses the fact that the dynamical variables of the system are contained in the zero curvature potentials: the sinh-Gordon $\varphi$ field appears as a parameter of the group element we use in order to write the flat connection $A_\mu$ as

$$A_\mu = -\partial_\mu WW^{-1} = \text{function of a group element } \gamma, \quad \gamma = e^{\varphi H_0 + \nu C}. \quad (1.4)$$

Thus, due to the path independence encoded in $F_{\mu\nu} = 0$, we are able to write the group element $W$ in distinct forms. This, compounded with some properties of the Kac-Moody algebra, leads us to a simple algebraic relation for the “group parameters” $\varphi$ and $\nu$, i.e.

$$\langle \lambda | \gamma^{-1} | \lambda \rangle = \langle \lambda | \gamma_+ N_+ M_-^{-1} \gamma_- | \lambda \rangle \quad (1.5)$$

where the elements $\gamma_\pm$, $N_+$, and $M_-$ have nice properties in terms of Kac-Moody algebra representations. To determine the two parameters $\varphi$ and $\nu$, we make use of two highest weight representations. This provides us with a relation for $\varphi$ in terms of the expected values of the group elements $N_+$ and $M_-$. So, solving the sinh-Gordon equation is equivalent to furnishing these two elements. Once this is done, we conveniently choose the parameters in such a way that the boundary conditions (1.3) are satisfied.

The key point in this construction is that in order to obtain the desired solution we must choose the constant group elements of the solitonic specialization of the Leznov–Saveliev construction as an infinite product of exponentials of vertex operators. The product is in fact a continuous one, since it involves all possible values of the rapidities of the one-solitons. In addition, all one-solitons entering the expansion have the same topological charge. This leads us to interpret such configuration as a condensate of solitons. As it was realized in [7], this type of solution can be written as a Fredholm determinant, and our solution is similar to the one found in [5], where correlation functions of the Ising model were obtained in terms of $\tau^{(N)}$ functions of the sinh-Gordon model, with $N \to \infty$. In order to arrive at the true solution a continuum limit must be taken for the condensate of solitons and the Fredholm determinant must be rewritten as
an infinite series of integrals, whose convergence conditions are studied and particular normal-
izations are fixed in order to satisfy the required boundary conditions. The solution heuristically
derived in this way is finally shown to satisfy the equation (1.1) or (1.2) plus (1.3).

Before we enter into the very existence proof it is worth reviewing the framework where the
eq. (1.1) arises and plays a fundamental role. Matrix String Theory (MST) [8, 9, 10] is the
theory that arises upon compactifying Matrix Theory [11] on a circle, [12]. It is expected to be
a nonperturbative version of type IIA string theory. An attempt to substantiate such conjecture
was started in [1, 2] and completed in [3, 4, 13], where it was shown that a correspondence
between MST and type IIA theory exists not only at the tree level, but that actually MST
contains the full perturbative expansion of type IIA string theory. It was in this context that
(1.1) appeared.

Looking for classical solutions of MST that preserve half supersymmetry, the following system
of equations was found

\begin{equation}
F_{\bar{w}w} - ig^2 [X, \bar{X}] = 0
\end{equation}

\begin{equation}
D_w \bar{X} = 0, \quad D_{\bar{w}} X = 0
\end{equation}

where \( F_{\bar{w}w} \) denotes the curvature of a connection \( A_w \) and \( A_{\bar{w}} \), while \( X \) is an \( N \times N \) matrix and
\( \bar{X} \) its hermitean conjugate. \( D_w, D_{\bar{w}} \) denotes the covariant derivatives with respect to \( A_w, A_{\bar{w}} \).
(1.6) may be called Hitchin equations, because they were discussed first by Hitchin in a different
context [14], or Riemannian instanton equations because of their geometrical interpretation. To
elucidate this terminology and the importance of these equations let us consider the simplest
case, in which the gauge group is \( U(2) \). The problem to be solved is finding a couple \((A, X)\)
that satisfies (1.6). To this end we choose the following ansatz

\begin{equation}
X = Y^{-1}MY, \quad A_w = i\partial_w Y^\dagger (Y^{-1})^\dagger
\end{equation}

where \( Y \) is a suitable matrix \( \in SL(2, \mathbb{C}) \), and \( M \) is the following \( 2 \times 2 \) matrix

\begin{equation}
M = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}
\end{equation}

where \( a \) is a function on the complex plane. As a consequence of the equation \( D_{\bar{w}} X = 0 \), it
follows that \( \partial_w a = 0 \), i.e. \( a \) is holomorphic in \( z \) (at least for finite \( z \)). As explained above, we
will assume that \( a \) is a polynomial in \( z \) with distinct roots. Now, given such an \( a \) we want to
find Y so that (1.6) is satisfied. We parametrize Y as
\[ Y = \left( e^p \ 0 \\
0 \ e^{-p} \right) \]
where
\[ p = \frac{x}{2} + \frac{1}{4} \ln |a|, \]
and u is a function to be determined. Then using (1.7) we find
\[ X = \left( \begin{array}{cc} 0 & ae^{-2p} \\
e^{2p} & 0 \end{array} \right), \quad A_w = i \partial_w p \left( \begin{array}{cc} 1 & 0 \\
0 & -1 \end{array} \right) \] (1.9)
Now it is easy to verify that the first equation in (1.6) implies
\[ 2 \partial_w \partial_{\bar{w}} p - g^2 \left( e^{4p} - |a|^2 e^{-4p} \right) = 0 \] (1.10)
Inserting the explicit form of p and the change of variable \( w \to \zeta \), s.t. \( \frac{\delta \zeta}{\delta w} = \sqrt{a} \), one can rewrite (1.10) as (1.1). If u is a smooth solution of this equation, the couple \((X, A)\) is a solution of (1.6) which is smooth everywhere except perhaps at infinity. Now, the important point is that the matrix M represents a branched covering of the z-plane. This is seen by diagonalizing M by means of a matrix in \( SL(2, \mathbb{C}) \):
\[ M = S \hat{M} S^{-1}, \quad \hat{M} = \left( \begin{array}{cc} \sqrt{a} & 0 \\
0 & -\sqrt{a} \end{array} \right), \quad S = \frac{i}{\sqrt{2}} \left( \begin{array}{cc} a^{\frac{1}{4}} & a^{-\frac{1}{4}} \\
a^{-\frac{1}{4}} & -a^{\frac{1}{4}} \end{array} \right). \] (1.11)
The two eigenvalues of \( M \) represent the two branches of the equation
\[ y^2 = a \] (1.12)
which is the defining equation of a hyperelliptic Riemann surface, with branch points corresponding to the roots of a. Therefore the solution at issue represents a Riemann surface, which justifies the adjective in the name \textit{Riemannian instanton}. The instanton nature of this solution is due to the fact that (1.6) are the two dimensional reduction of the YM self-duality equation in 4D. In [15] this example was analyzed in great detail. It was shown there that, if the u solution of (1.1) satisfies the boundary conditions stated at the beginning of this introduction, the matrix X, outside the branch points of a and when \( g \to \infty \), becomes \( \hat{M} \) up to a unitary transformation. This fact plays a crucial role in establishing the correspondence between MST and type IIA theory, see [3, 4].

It is therefore of upmost importance to establish the existence of the above solutions of (1.6) and therefore of the corresponding u solutions of (1.1). On the other hand, it is clear that showing the existence of exact solutions of (1.1) is a problem interesting in itself.

The paper is organized as follows. In section 2 we apply the Leznov–Saveliev method to our problem and define the soliton condensate. In section 3 we verify that what has been heuristically constructed in section 2 is in fact the looked for solution. A few appendices are devoted to clarify some technical points utilized in the course of the proof.
2 Construction of solution through Leznov-Saveliev algebraic method

In this section we review the Leznov-Saveliev method [16] for construction of solutions of Affine-Toda type theories, based on the zero curvature formulation of two dimensional integrable systems. Even though we work in two dimensions we stick to the point of view of higher dimensional integrable models, which can be constructed from two potentials, $A$ and $B$ [6]. Among other things such an approach leads to the construction of new conserved currents, not obtained from the usual two dimensional formalism [6].

2.1 Zero curvature condition

The eq. (1.1) admits a representation in terms of the following zero curvature conditions

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0 \quad (2.1)$$

$$D^\mu B_\mu = \partial^\mu B_\mu + [A^\mu, B_\mu] = 0 \quad (2.2)$$

In two dimensions the condition (2.1) is the well known Lax-Zakharov-Shabat equation. In dimensions higher than two the eqs. (2.1) and (2.2) were shown to be sufficient local conditions for the vanishing of the generalized zero curvature equations relevant for higher dimensional integrable theories [6]. Here we apply eqs. (2.1)-(2.2) for the two dimensional model (1.1), and show that the relation (2.2) leads to non-local conservation laws. The procedure applies equally well to a wide class of two dimensional integrable models, like the abelian and non-abelian Toda models (affine or not), possessing a representation in terms of Lax-Zakharov-Shabat eq. (2.1). That equation can be enriched by the extra conservation laws (2.2) without any further restriction in their dynamics, and we plan to analyse that in more details in a future publication.

Let $\hat{G}$ be an affine $sl(2)$ Kac-Moody algebra, with basis $H^m, T^m_\pm, D, C$, satisfying the commutation relations (B.113). We take the local zero curvature potentials as

$$A_w \equiv -\partial_w \gamma \gamma^{-1} + E_{-1} \quad \quad A_{\bar{w}} \equiv \gamma E_1 \gamma^{-1}$$

$$B_w \equiv P^{\psi} (E_{-1}) \quad \quad B_{\bar{w}} \equiv P^{\psi} (\gamma E_1 \gamma^{-1}) \quad (2.3)$$

where

$$\gamma \equiv e^{\varphi H^0 + \nu C} \quad (2.4)$$
and
\[ E_1 \equiv g \, T_+^0 + g \bar{a} \, (\bar{z}) \, T^1_+ \quad E_{-1} \equiv g \, a \, (z) \, T^-_+ + g \, T^0_+ \] (2.5)

Following [6] we have taken \( A_\mu \) and \( B_\mu \) to belong to a non-semisimple Lie algebra formed by the \( sl(2) \) Kac-Moody algebra and an abelian ideal which transforms under its adjoint representation \( P^\psi \), i.e.

\[
[T_{a}^m, T_{b}^n] = f_{ab}^c T_{c}^{m+n} + \text{Tr} (T_a T_b) C \delta_{m+n,0}
\]
\[
[T_{a}^m, P^\psi (T_{b}^n)] = P^\psi ([T_{a}^m, T_{b}^n])
\]
\[
[P^\psi (T_{a}^m), P^\psi (T_{b}^n)] = 0
\] (2.6)

With this choice, the zero curvature conditions (2.1) lead us to the system of differential equations

\[
\partial_{\bar{w}} \partial_{w} \varphi = g^2 \left( e^{2 \varphi} - |a|^2 \, e^{-2 \varphi} \right)
\] (2.7)
\[
\partial_{\bar{w}} \partial_{w} \nu = g^2 |a|^2 \, e^{-2 \varphi}.
\] (2.8)

The identification

\[
\varphi = 2p \equiv u + \frac{1}{2} \ln |a|
\] (2.9)

turns (2.7) into the equation

\[
2 \partial_{\bar{w}} \partial_{w} p = g^2 \left( e^{4p} - |a|^2 \, e^{-4p} \right)
\] (2.10)

This is exactly the sinh-Gordon equation (1.1) once we make the change of variables \( w \rightarrow \zeta \), such that (see Appendix A)

\[
\frac{d\zeta}{dw} = \sqrt{a} \quad \frac{d\bar{\zeta}}{d\bar{w}} = \sqrt{\bar{a}}
\] (2.11)

As we point out in the introduction, the sinh-Gordon equation with source is equivalent to the homogeneous equation together with the following boundary conditions

\[
u \sim -\frac{1}{2} \ln |a| \quad \varphi \sim \text{finite} \quad \text{for} \quad a \sim 0 \] (2.12)

So, \( u \) must diverge logarithmically at the zeroes of \( a \), and from (2.9) it follows that \( \varphi \) should be finite there. On the other hand, far away from any zero of \( a \) we need

\[
u \sim 0 \quad \varphi \sim \frac{1}{2} \ln |a| \quad \text{for} \quad a \sim \infty
\] (2.13)
One can check that the condition (2.2) is trivially satisfied by the potentials (2.3), i.e. it holds true for any field configuration including those which are not solutions of the equations of motion (2.7)-(2.8). However, due to (2.1) it follows that the connection $A_\mu$ is flat, and so there is a group element $W$ such that

$$A_\mu \equiv -\partial_\mu WW^{-1}. \quad (2.14)$$

Consequently, it follows from (2.2) that the currents

$$J_\mu \equiv W^{-1}B_\mu W \quad (2.15)$$

are conserved

$$\partial^\mu J_\mu = 0 \quad (2.16)$$

The group element $W$ only exists for field configurations that satisfies the equations of motion, and it is non-local in the field variables. Consequently, the currents (2.15) are non-local. We intend to study the properties of these currents, in a wide class of models, in a future publication.

### 2.2 Leznov-Saveliev construction

The Leznov-Saveliev method uses some features of the Kac-Moody algebra representations, as well as some geometrical properties due to the zero curvature condition (2.1). Since $F_{\mu\nu} = 0$, it follows that the integration necessary for obtaining the group element $W$, introduced in (2.14), is path independent. That allows us to write this group element in different ways. We choose two different paths in order to integrate (2.14), and get

$$W = g_1 = \gamma g_2 \quad \gamma = g_1 g_2^{-1} \quad (2.17)$$

This means that we have two equivalent decomposition for the potentials $A_\mu$:

$$\partial_\mu g_1 g_1^{-1} = \partial_\mu WW^{-1} = \partial_\mu \gamma \gamma^{-1} + \gamma \partial_\mu g_2 g_2^{-1} \gamma^{-1}. \quad (2.18)$$

Next we recall some aspects of the $\hat{sl}(2)$ Kac-Moody algebra. An affine Kac-Moody algebra possesses integer gradation such that $[17, 18]$ \[
\hat{G} = \hat{G}_+ \oplus \hat{G}_0 \oplus \hat{G}_- \quad (2.19)
\]

where $\hat{G}_{0,\pm}$ are the eigensubspaces of the grading operator

$$Q \equiv \frac{1}{2}H^0 + 2D; \quad \left[ Q, \hat{G}_n \right] = n \hat{G}_n \quad (2.20)$$
with
\[ \mathcal{G}_+ = \bigoplus_{n>0} \mathcal{G}_n \quad \mathcal{G}_- = \bigoplus_{n<0} \mathcal{G}_n \] (2.21)

Observe that, in terms of the eigensubspaces \( \mathcal{G}_{0,\pm} \), the potentials (2.3) have the decomposition
\[ A_w \in \mathcal{G}_0 \oplus \mathcal{G}_- \quad A_{\bar{w}} \in \mathcal{G}_+ \quad B_w \in P^\psi(\mathcal{G}_-) \quad B_{\bar{w}} \in P^\psi(\mathcal{G}_+) \] (2.22)

We shall analyse how this potentials acts on a representation of the algebra. Consider a representation of the \( \hat{sl}(2) \) Kac-Moody algebra. Inside it, there is a highest weight state \( |\lambda\rangle \) on which
\[ T^+_n |\lambda\rangle = 0 \quad n \geq 0 \quad T^-_n |\lambda\rangle = 0 \quad n > 0. \] (2.23)

The scalar and spinor representations of the affine \( \hat{sl}(2) \) Kac-Moody algebra have highest weight states \( |\lambda_0\rangle \) and \( |\lambda_1\rangle \) which satisfy
\[ C |\lambda_0\rangle = |\lambda_0\rangle \quad H^0 |\lambda_0\rangle = 0 \] (2.24)
\[ C |\lambda_1\rangle = |\lambda_1\rangle \quad H^0 |\lambda_1\rangle = |\lambda_1\rangle \] (2.25)

Therefore, using (2.18), (2.22) and (2.23) we see that the elements \( g_1, g_2 \) were taken in such a form that
\[ \partial_{\bar{w}} g_1 g_1^{-1} |\lambda\rangle = 0 \quad \langle \lambda | \partial_w g_2 g_2^{-1} = 0 \] (2.26)

This means that these group elements evaluated on a heighest weight representation are holomorphic functions of the coordinates
\[ g_1^{-1} |\lambda\rangle \equiv f(w) \quad \langle \lambda | g_2 \equiv f(\bar{w}). \] (2.27)

With these ingredients at hand, we are interested in evaluating the expectation value
\[ \langle \lambda | \gamma^{-1} |\lambda\rangle = \langle \lambda | g_2 g_1^{-1} |\lambda\rangle \] (2.28)
that descends from (2.17). To this end we perform the Gauss type decomposition
\[ g_1 \equiv N \gamma_- M_- \quad g_2 \equiv M \gamma_+ N_+ \] (2.29)
with
\[ \gamma_{\pm} \in \exp(\mathcal{G}_0) \quad N, N_+ \in \exp(\mathcal{G}_+) \quad M, M_- \in \exp(\mathcal{G}_-). \] (2.30)

So, using (2.14), (2.18), (2.29) and (2.23) we compare the gradings of the potentials and of the group elements \( g_1, g_2 \), and with these we determine the action of \( \gamma \) (2.28) on the representation states.
Comparing the gradings of the potentials $A_{\mu}$ with the ones derived from (2.18) – (2.29), we get

$$\partial \bar{w} \gamma \gamma^{-1} = 0 \quad \partial \bar{w} M M^{-1} = 0 \quad \partial \bar{w} N N^{-1} = \gamma E_{1} \gamma^{-1}$$

(2.31)

$$\partial w \gamma + \gamma^{-1} = 0 \quad \partial w N N^{-1} = 0 \quad \partial w M M^{-1} = \gamma^{-1} E_{1} \gamma$$

(2.32)

as well as

$$\partial w M M^{-1} = -\gamma^{-1} E_{1} \gamma_{-} \quad \partial \bar{w} N N^{-1} = -\gamma_{+}^{-1} E_{1} \gamma_{+}$$

(2.33)

So, as a consequence of (2.28) and (2.29) the expectation values are

$$\langle \lambda | \gamma^{-1} | \lambda \rangle = \langle \lambda | \gamma_{+} (\bar{w}) N_{+} (w) M_{-}^{-1} (w) \gamma_{-}^{-1} (w) | \lambda \rangle$$

(2.34)

This expectation values depends on two parameters $\gamma_{\pm}$. $N_{+}$ and $M_{-}$ are determined from (2.33).

Denoting

$$\gamma_{+} = \exp (\theta_{+} (\bar{w}) H^{0} + \xi_{+} (\bar{w}) C) \quad \gamma_{-} = \exp (\theta_{-} (w) H^{0} + \xi_{-} (w) C)$$

(2.35)

one gets from (2.4) and (2.34)

$$e^{-\nu} = e^{\xi_{+} - \xi_{-}} \langle \lambda_{0} \ | \ N_{+} (w) M_{-}^{-1} (w) \ | \ \lambda_{0} \rangle$$

(2.36)

$$e^{-\varphi - \nu} = e^{\theta_{+} - \theta_{-} + \xi_{+} - \xi_{-}} \langle \lambda_{1} \ | \ N_{+} (w) M_{-}^{-1} (w) \ | \ \lambda_{1} \rangle$$

(2.37)

At this point we are able to write the general solution of the model

$$e^{-\varphi} = \frac{\langle \lambda_{1} \ | \ N_{+} (w) M_{-}^{-1} (w) \ | \ \lambda_{1} \rangle}{\langle \lambda_{0} \ | \ N_{+} (w) M_{-}^{-1} (w) \ | \ \lambda_{0} \rangle} e^{\theta_{+} - \theta_{-}}$$

(2.38)

$$e^{-\nu} = \langle \lambda_{0} \ | \ N_{+} (\bar{w}) M_{-}^{-1} (w) \ | \ \lambda_{0} \rangle e^{\xi_{+} - \xi_{-}}$$

(2.39)

Notice that the group element $W$ introduced in (2.14) have to be regular, since otherwise $A_{\mu}$ will not be flat. Remember that in order for $A_{\mu}$ to satisfy the zero curvature condition it is necessary that the derivatives commute when acting on $W$. It then follows that the group elements $g_{1}, g_{2}$ and $\gamma$ have also to be regular. By regular we mean a quantity such that derivatives $\partial_{w}$ and $\partial_{\bar{w}}$ commute when acting on it. From (see [19])

$$\partial z z^{-k-1} = (-1)^{k} \frac{\pi}{k!} \delta^{(k,0)} (\bar{z}, z)$$

(2.40)

one observes that

$$\partial z \partial z z^{-k-1} = \partial z \partial z z^{-k-1}$$

(2.41)
Therefore, the fields $\varphi$ and $\nu$, as well as the parameters $\theta_{\pm}$ and $\xi_{\pm}$ can have log singularities, so that the corresponding group elements will have at most poles.

The general solution of (2.7), and so of (1.1), is given by (2.38). Therefore the Riemannian instanton should correspond to some particular choice of the parameters of it. Due to (2.5), (2.33) and (2.35) we have

$$\partial_{\bar{w}} N_+ N_+^{-1} = -g \left( e^{-2\theta_+} T_+^0 + \bar{a} (\bar{w}) e^{2\theta_+} T_+^1 \right)$$  \hspace{1cm} (2.42)

$$\partial_{w} M_- M_-^{-1} = -g \left( a (w) e^{-2\theta_-} T_-^{-1} + e^{2\theta_-} T_-^0 \right)$$  \hspace{1cm} (2.43)

This means that we have a solution to the model once we specify the parameters $\theta_\pm$, $\xi_\pm$, and solve (2.42), (2.43) for $M_-$, $N_+$.

2.3 The Riemannian instanton solutions

As we have seen, the general solution of the model (2.7), or equivalently (1.1), is given by (2.38). We now have to choose the parameters and integration constants of the general solution, in order to obtain the Riemannian instantons solutions with the properties described in the introduction.

We begin by choosing the functions $\theta_{\pm}$ as

$$\theta_+ = -\frac{1}{4} \ln \bar{a} \hspace{1cm} \theta_- = \frac{1}{4} \ln a$$  \hspace{1cm} (2.44)

That choice simplifies the integration of the elements $N_+$ and $M_-$. Indeed, (2.42) and (2.43) become

$$\partial_{\bar{w}} N_+ N_+^{-1} = -g \sqrt{\bar{a}} (\bar{w}) \ b_1 \hspace{1cm} \partial_{w} M_- M_-^{-1} = -g \sqrt{a (w)} \ b_{-1}$$  \hspace{1cm} (2.45)

The operators $b_1$ and $b_{-1}$ are elements of a Heisenberg subalgebra of the $\hat{sl}(2)$ Kac-Moody algebra [17, 18]. That is like an algebra of harmonic oscillators, i.e. they are generated by

$$b_{2n+1} \equiv T_+^n + T_-^{n+1} \hspace{1cm} [b_{2m+1}, b_{2n+1}] = C(2m + 1)\delta_{m+n+1,0}$$  \hspace{1cm} (2.46)

We can then integrate (2.45)

$$N_+ = e^{I_+ b_1 h_+} \hspace{1cm} I_+ = -g \int d\bar{w} \sqrt{\bar{a}} (\bar{w}) = -g\zeta$$

$$M_- = e^{I_- b_{-1} h_-} \hspace{1cm} I_- = -g \int dw \sqrt{a (w)} = -g\zeta$$  \hspace{1cm} (2.47)

where we used the change of variables to (2.11), and where $h_{\pm}$ are constant group elements obtained by exponentiating the affine Kac-Moody algebra (integration constants).
We now return to the general solution (2.38). With our choice of $\theta_\pm$ (2.44) and in view of (2.9) we get

$$e^{-u} = \frac{\langle \lambda_1 | N_+ M_{-1}^\dagger | \lambda_1 \rangle}{\langle \lambda_0 | N_+ M_{-1}^\dagger | \lambda_0 \rangle}$$

(2.48)

Here we come to a crucial point in the construction of the Riemannian instanton solution. As it is well known [20, 21], the one-soliton solutions are obtained by taking the integration constants $h_\pm$, such that $h_+ h_-^{-1} = e^{V(\mu)}$, where $V(\mu)$ is an element of the Kac-Moody algebra which is an eigenstate of the oscillators $b_{2n+1}$, i.e.

$$[b_{2n+1}, V(\mu)] = -2 \mu^{2n+1} V(\mu)$$

(2.49)

The construction of the operator $V(\mu)$ is explained in appendix B. Its expression in terms of a special basis of the Kac-Moody algebra is given in (B.129). However, the nice properties of such operator are best appreciated in the principal vertex operator representation of the Kac-Moody algebra. Its form in that representation is given in (B.151). An important relation satisfied by $V(\mu)$ is given in (B.158). From it we observe that

$$V(\mu)V(\nu) \to 0 \quad \text{for} \quad \mu \to \nu$$

(2.50)

That implies that the exponential $e^{V(\mu)}$ truncates in first order, and so we do not have convergence problems in our expressions. Such property is what makes the vertex operator representation to deserve the name of integral representation [17, 18]. It also explain the truncation of the Hirota’s expansion of the tau functions, since those are nothing more than special expectation values of $V(\mu)$ in the states of the vertex operator representation [21].

If one takes $a(z)$ to be constants and choose the integration constants $h_\pm$, such that $h_+ h_-^{-1} = e^{V(\mu)}$, then one obtains from (2.48) the one-soliton solution of the sinh-Gordon equation (by taking $u \to i u$ one gets the sine-Gordon one-soliton). The parameter $\mu$ is related to the rapidity $\theta$ of the soliton through $\mu \equiv \epsilon \epsilon^\theta$, with $\epsilon = \pm 1$. It is $\epsilon$ what determines the sign of the topological charge (in the case of sine-Gordon) and what makes the difference between the soliton and anti-soliton solutions.

The $n$-soliton solution is obtained by taking $h_+ h_-^{-1}$ as a product of those exponentials, i.e.

$$h_+ h_-^{-1} = \prod_{i=1}^n e^{V(\mu_i)}$$

As we now explain the Riemannian instanton solution is obtained by taking $h_+ h_-^{-1}$ to be a continuous infinite product of exponentials $e^{V(\mu_i)}$. In fact, we shall take the product in such a way that exponentials for smaller values of $\mu_i$ appear on the left, and we vary $\mu_i$ continuously from zero to $+\infty$. In addition, each value of $\mu_i$ appears only once, without
repetition. So, what we have is an $N$-soliton solution, with $N \to \infty$, where all the rapidities appear once, and we do not have a mixture of soliton and anti-solitons since the $\mu_i$’s are all positive. Therefore, we have some sort of soliton condensate\(^1\).

In order to build up the solution we start with an infinite discrete product of exponentials, and later take the continuous limit. So, we take the constant $h_+, h_-$ in (2.47), (2.47) to be

$$h_+ h_-^{-1} = \prod_{i=1}^{\infty} e^{V(\mu_i)}$$

(2.51)

From this point on we explore some useful properties of the vertex operators and its action on the representation states, which will allow us to end up with a closed expression for the solution $u$. So, once the constants $h_+, h_-$ have been chosen as in (2.51), the solution (2.48) depends on

$$\langle \lambda \mid N_+ M_-^{-1} \mid \lambda \rangle = \langle \lambda \mid \prod_{i=1}^{\infty} (1 + e^{\beta(\mu_i)} V(\mu_i)) \mid \lambda \rangle$$

$$\beta(\mu_i) \equiv 2g \left( \mu_i \bar{\zeta} + \frac{\zeta}{\mu_i} \right)$$

(2.52)

where we have used (2.47), (2.49) and (2.50). Using (B.160) and (B.152), we can expand (2.52) in terms of sums as

$$\langle \lambda_0 \mid N_+ M_-^{-1} \mid \lambda_0 \rangle = 1 + \sum_i e^{\beta(\mu_i)} + \sum_{i<j} e^{\beta(\mu_i)} e^{\beta(\mu_j)} \left( \frac{\mu_j - \mu_i}{\mu_j + \mu_i} \right)^2$$

$$+ \sum_{i<j<k} e^{\beta(\mu_i)} e^{\beta(\mu_j)} e^{\beta(\mu_k)} \left( \frac{\mu_j - \mu_i}{\mu_j + \mu_i} \right)^2 \left( \frac{\mu_k - \mu_i}{\mu_k + \mu_i} \right)^2 \left( \frac{\mu_k - \mu_j}{\mu_k + \mu_j} \right)^2 + \ldots$$

(2.53)

$$\langle \lambda_1 \mid N_+ M_-^{-1} \mid \lambda_1 \rangle = 1 - \sum_i e^{\beta(\mu_i)} + \sum_{i<j} e^{\beta(\mu_i)} e^{\beta(\mu_j)} \left( \frac{\mu_j - \mu_i}{\mu_j + \mu_i} \right)^2$$

$$- \sum_{i<j<k} e^{\beta(\mu_i)} e^{\beta(\mu_j)} e^{\beta(\mu_k)} \left( \frac{\mu_j - \mu_i}{\mu_j + \mu_i} \right)^2 \left( \frac{\mu_k - \mu_i}{\mu_k + \mu_i} \right)^2 \left( \frac{\mu_k - \mu_j}{\mu_k + \mu_j} \right)^2 + \ldots$$

Remember that we take the $\mu_i$’s to be real and positive, and that two $\mu_i$’s never coincide.

Expressions (2.53) can be written in the form of Fredholm determinants. A similar result was found in [5], where exact correlation functions of the Ising Model were shown to be related to the tau functions of the sinh-Gordon model. Following [5] we get

$$\langle \lambda_0 \mid N_+ M_-^{-1} \mid \lambda_0 \rangle = \det (1 + W)$$

$$\langle \lambda_1 \mid N_+ M_-^{-1} \mid \lambda_1 \rangle = \det (1 - W)$$

(2.54)

\(^1\)We are indebted to Olivier Babelon for pointing out to us that the Riemannian instanton solution should have such structure. His intuition came from his experience with D. Bernard on the calculations of form factors and correlation functions for the Ising model [5]
where $W$ is the matrix

$$W_{ij} \equiv e^{\beta(\mu_i)/2} \sqrt{\frac{4\mu_i\mu_j}{\mu_i + \mu_j}} e^{\beta(\mu_j)/2} \tag{2.55}$$

Using (2.48) one then gets

$$u = \ln \left( \frac{\det (1 + W)}{\det (1 - W)} \right) = \text{Tr} \ln \frac{1 + W}{1 - W} \tag{2.56}$$

where we used $\ln \det M = \text{Tr} \ln M$. Expanding the logarithm one gets

$$u = 2 \sum_{n=0}^{\infty} \frac{\text{Tr} W^{2n+1}}{2n+1} \tag{2.57}$$

### 2.4 The continuous limit

As we have said, we want to take the limit where the infinite product of exponentials in (2.51) becomes a continuous one. In order to do that we take the label $i$ of the parameter $\mu_i$ to be the rapidity $\theta$ of the soliton, and let it run from $-\infty$ to $\infty$. We then have that $\mu_i \rightarrow \mu_\theta = e^\theta$. Then

$$\sum_i \rightarrow \Lambda \int_{-\infty}^{\infty} d\theta = \Lambda \int_0^{\infty} \frac{d\mu}{\mu} \tag{2.58}$$

where $\Lambda$ is a scaling factor of the integration measure.

Then, from (2.55) it follows that

$$\text{Tr} W^N \rightarrow \Lambda^N \int_0^{\infty} \frac{d\mu_1}{\mu_1} \cdots \int_0^{\infty} \frac{d\mu_N}{\mu_N} e^{\beta(\mu_1)/2} \sqrt{\frac{4\mu_1\mu_2}{\mu_1 + \mu_2}} e^{\beta(\mu_2)/2} \sqrt{\frac{4\mu_2\mu_3}{\mu_2 + \mu_3}} e^{\beta(\mu_3)/2} \cdots e^{\beta(\mu_N)/2} \sqrt{\frac{4\mu_N\mu_1}{\mu_N + \mu_1}} \tag{2.59}$$

Therefore (2.57) becomes

$$u = 2 \sum_{n=0}^{\infty} \frac{(2\Lambda)^{2n+1}}{2n+1} I_{2n+1} \tag{2.60}$$

with

$$I_N \equiv \int_0^{\infty} \frac{d\mu_1}{\mu_1} \cdots \int_0^{\infty} \frac{d\mu_N}{\mu_N} \frac{\mu_1}{(\mu_1 + \mu_2)} \frac{\mu_2}{(\mu_2 + \mu_3)} \cdots \frac{\mu_N}{(\mu_N + \mu_1)} e^{\beta(\mu_1) + \cdots + \beta(\mu_N)} \tag{2.61}$$

In the case $N = 1$ we have that

$$I_1 = \frac{1}{2} \int_0^{\infty} \frac{d\mu}{\mu} e^{\beta(\mu)} = K_0 (4 \mid g \mid \zeta) \tag{2.62}$$

where $K_0$ is the modified Bessel function. In fact the above expression is valid for $\text{Re} (g\zeta) < 0$. However, we shall take it to be valid for $\text{Re} (g\zeta) > 0$ too (see comments below (2.68)).

\[ \text{Notice that the subindices of } \mu \text{ have a different meaning now. They label the variables giving the values of rows and columns of the matrices, and not the actual values of those as before.} \]
Notice that the integrals $I_N$ are real. Indeed, from (2.52) we have
\[ \beta^* (\mu_i) = \beta \left( \frac{1}{\mu_i} \right) \]  
(2.63)

Therefore, one can undo the complex conjugation with the change of integration variables, $\mu_i \to 1/\mu_i$, since $\int_0^\infty \frac{d\mu}{\mu}$ is left unchanged. In addition, $\mu_i/ (\mu_i + \mu_j) \to \mu_j/ (\mu_i + \mu_j)$, and so the product of those terms in the integrand of (2.61) is left invariant. Consequently, the solution $u$ given in (2.60) is real (since $\Lambda$ is real).

We now want to analyse the boundary conditions satisfied by the solution (2.60). In order to do that we perform the change of integration variables
\[ \phi_i \equiv \ln \frac{\mu_i}{\mu_{i+1}} \quad i = 1, 2, \ldots N - 1 \]
\[ \nu \equiv \left( \prod_{i=1}^{N} \mu_i \right)^{1/N} \]  
(2.64)

The integrals (2.61) become
\[ I_N = \frac{1}{2^N} \int_{-\infty}^{\infty} d\phi_1 \cdots d\phi_{N-1} \frac{1}{\cosh \left( \frac{1}{2} \phi_1 \cosh \left( \frac{1}{2} \phi_2 \right) \cdots \cosh \left( \frac{1}{2} \phi_{N-1} \right) \cosh \left( \frac{1}{2} \sum_{n=1}^{N-1} \phi_n \right) \right)} \times \]
\[ \int_{0}^{\infty} \frac{d\nu}{\nu} e^{2g(\bar{\zeta} f_N(\phi) \nu + \zeta f_N(-\phi) \frac{1}{\nu})} \]  
(2.65)

where
\[ f_N(\phi) \equiv \sum_{l=1}^{N} \exp \left( \frac{1}{N} \left( -\sum_{n=1}^{l-1} n \phi_n + \sum_{n=l+1}^{N} (N-n) \phi_n \right) \right) \]  
(2.66)

If Re $(g\zeta) < 0$, the integral in $\nu$ in (2.65) is the modified Bessel function $K_0$, and so one gets
\[ I_N = \frac{1}{2^{N-1}} \int_{-\infty}^{\infty} d\phi_1 \cdots d\phi_{N-1} \frac{K_0 \left( 4 \parallel g \parallel \zeta \parallel \sqrt{w_N} \right)}{\cosh \left( \frac{1}{2} \phi_1 \cosh \left( \frac{1}{2} \phi_2 \right) \cdots \cosh \left( \frac{1}{2} \phi_{N-1} \right) \cosh \left( \frac{1}{2} \sum_{n=1}^{N-1} \phi_n \right) \right)} \]  
(2.67)

where $w_N$ is given by
\[ w_N \equiv f_N(\phi) f_N(-\phi) = N + 2 \sum_{l=0}^{N-2} \sum_{j=1}^{N-l-1} \cosh \sum_{i=j}^{j+l} \phi_i \]  
(2.68)

However, we shall take the expression (2.67) to be also valid for Re $(g\zeta) > 0$. Such analytical continuation process will be justified later when we shall check the validity of the solution by directly replacing it into the equations of motion. Therefore, the solution (2.60) depends on $\zeta$ and $g$ through their norms only.
2.5 The boundary conditions

For large arguments the modified Bessel function $K_0$ have the following behaviour

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + O\left(\frac{1}{z}\right)\right) \quad \text{for large } z$$

(2.69)

Consequently it is clear that

$$I_N \to 0 \quad \text{for} \quad |\zeta| \to \infty$$

(2.70)

and so, the $u$ field does go to zero for large $\zeta$, as required (see (2.13)).

The analysis for small $\zeta$ is trickier. The reason is that taking $|\zeta|$ small does not guarantee that the argument of the Bessel function $K_0$ is small, since $w_N$ can be infinitely large. However, in the region where $|\zeta| |w_N|$ diverges for small $|\zeta|$ the function $K_0$ vanishes and so there is no contribution for the integral $I_N$. Therefore, we can use the following reasoning: Let $|\zeta|$ have a fixed infinitesimal value $|\zeta| = \epsilon$. We split the domain of integration in two regions, namely

- $D_0 \equiv$ region of $(\phi_1, \ldots, \phi_{N-1})$ where $\epsilon 4 |g| \sqrt{w_N} < \sqrt{\epsilon}$
- $D_1 \equiv$ region of $(\phi_1, \ldots, \phi_{N-1})$ where $\epsilon 4 |g| \sqrt{w_N} > \sqrt{\epsilon}$

(2.71)

In the region $D_0$ we use the fact that for small arguments, $K_0$ diverges as

$$K_0(z) \sim -\ln \frac{z}{2} \left(1 + O \left(z^2\right)\right) \quad \text{for small } z$$

(2.72)

and so

$$I_N = \frac{1}{2^{N-1}} \int_{D_0} d\phi_1 \ldots d\phi_{N-1} \frac{-\ln \left(2 \ |g| \ ||\zeta| \sqrt{w_N}\right)}{\cosh \left(\frac{1}{2} \phi_1\right) \cosh \left(\frac{1}{2} \phi_2\right) \ldots \cosh \left(\frac{1}{2} \phi_{N-1}\right) \cosh \left(\frac{1}{2} \sum_{n=1}^{N-1} \phi_n\right)}$$

$$+ \frac{1}{2^{N-1}} \int_{D_1} d\phi_1 \ldots d\phi_{N-1} \frac{K_0 \left(4 \ |g| \ ||\zeta| \sqrt{w_N}\right)}{\cosh \left(\frac{1}{2} \phi_1\right) \cosh \left(\frac{1}{2} \phi_2\right) \ldots \cosh \left(\frac{1}{2} \phi_{N-1}\right) \cosh \left(\frac{1}{2} \sum_{n=1}^{N-1} \phi_n\right)}$$

(2.73)

Notice that in the region $D_1$ the argument of $K_0$ never vanishes and so $K_0$ is finite there.

On the other hand, on $D_1$ we have

$$w_N > \frac{1}{4 \ |g| \sqrt{\epsilon}}$$

(2.74)

and so $w_N \to \infty$ as $\epsilon \to 0$. But from (2.68) one observes that the only way for that to happens is that at least one the $\phi_i$’s should diverge. Therefore, the denominator of the integrand of (2.73), in the $D_1$ region, diverges. So, one gets that the integral in $D_1$ in (2.73) vanishes for $\epsilon \to 0$. 

15
Consequently, the integral in $D_0$ in (2.73) implies that

$$I_N \sim -\kappa_N \ln |\zeta|$$

for $|\zeta| \to 0$ (2.75)

with

$$\kappa_N \equiv \frac{1}{2^{N-1}} \int_{-\infty}^{\infty} \frac{d\phi_1 \ldots d\phi_{N-1}}{\cosh (\frac{1}{2}\phi_1) \cosh (\frac{1}{2}\phi_2) \ldots \cosh (\frac{1}{2}\phi_{N-1}) \cosh \left( \frac{1}{2} \sum_{n=1}^{N-1} \phi_n \right)}$$

(2.76)

Performing the integration using the fact that

$$\kappa_N = \int_{-\infty}^{\infty} d\phi_1 \ldots d\phi_N \frac{\delta \left( \sum_{i=1}^{N} \phi_i \right)}{\prod_{i=1}^{N} \cosh \phi_i} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\phi_1 \ldots d\phi_N \frac{e^{ik\sum_{i=1}^{N} \phi_i}}{\prod_{i=1}^{N} \cosh \phi_i}$$

(2.77)

one gets

$$\kappa_{2n+1} = \frac{\pi^{2n} (2n-1)!!}{2^n n!}$$

(2.78)

Consequently we have from (2.60) that

$$u \sim -\frac{2}{\pi} f (2\pi \Lambda) \ln |\zeta|$$

for $|\zeta| \to 0$ (2.79)

where

$$f (x) \equiv \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n! (2n+1)} x^{2n+1}$$

(2.80)

Notice this series is convergent for $x^2 < 1$, and divergent for $x^2 > 1$. For $x^2 = 1$ the ratio test does not say anything, but one can check that it does converge there and $f (1) = \pi/2$. Therefore, we must have $|\Lambda| \leq 1/2\pi$.

Our function $a(z)$ is supposed to be a polynomial and to represent a hyperelliptic Riemann surface. As discussed in Appendix A, a good local coordinate near a branch point $z_i$ is $\xi_i = \sqrt{z - z_i}$, i.e., near $z_i$ we have $z = z_i + \xi_i^2$. So, according to (A.105), near the branching cut, we must have $\zeta \sim a^{3/2}$. Therefore,

$$u \sim -\frac{3}{\pi} f (2\pi \Lambda) \ln |a|$$

(2.81)

One can check that

$$f \left( \frac{1}{2} \right) = \frac{\pi}{6}$$

(2.82)
Consequently, in order to satisfy the boundary condition (2.12), we must set
\[ \Lambda = \frac{1}{4\pi} \]  
(2.83)

Therefore from (2.60), the desired solution to (1.1) is given by
\[ u = 2 \sum_{n=0}^{\infty} \frac{I_{2n+1}}{(2n+1)(2\pi)^{2n+1}} \]  
(2.84)

where \( I_{2n+1} \) is given in (2.62) and (2.67). Consequently, it depends only on the combination 
\[ 4 \mid g \mid \zeta \mid, \]  
and it is symmetric, on the plane \( \zeta \), under rotation around the origin \( \zeta = 0 \).

3 Check of the solution

As we have seen in (2.83), the value of the scaling factor \( \Lambda \) of the integration measure, introduced 
in (2.58), was fixed to \( 1/4\pi \). That was imposed by the behaviour of the solution at \( \zeta = 0 \) (or 
equivalently on the zeroes of \( a(z) \)). However, as we will see in this section, the solution holds 
true for any value of \( \Lambda \), outside the zeroes of \( a(z) \). That is a consequence of very interesting 
non-linear differential equations satisfied by the integrals \( I_{2n+1} \) defined in (2.67). Therefore, in 
order to emphasize those properties, we shall not fix \( \Lambda \) in this check of the solution.

We begin by looking at the convergence of the series (2.60). As we have seen in (2.70), the 
integrals \( I_{2n+1} \) go to zero for large arguments. In fact, from the numerical data in tables 1 and 2 
of appendix C, one sees that they decay faster for larger index \( 2n+1 \). Therefore, we should not 
have problems of convergence of the series (2.60) for large arguments. For \( \zeta \) close to zero one 
can use (2.75) and (2.78) and the ratio test to check that the series converges for \( \Lambda < \sqrt{2}/2\pi \). 
Therefore, the series (2.84) for the final solution should converge everywhere.

We begin by calculating the derivatives of the field \( u \) close to \( \zeta = 0 \). From (2.79) it follows 
that
\[ \partial_{\zeta} u \sim -\frac{1}{\pi} f(2\pi\Lambda) \frac{1}{\zeta} \]  
(3.85)
and so using (2.40)
\[ \partial_{\zeta} \partial_{\zeta} u \sim -f(2\pi\Lambda) \delta(\zeta, \tilde{\zeta}) \]  
(3.86)

Some care must be taken here since we have been working with delta functions in different 
coordinate frames. In order avoid misunderstandings which can lead to inconsistencies in fixing 
\( \Lambda \), we devote a discussion on this point in Appendix A. From (A.112) we see that
\[ \partial_{\zeta} \partial_{\zeta} u \sim -f(2\pi\Lambda) \delta(\zeta, \tilde{\zeta}) = -f(2\pi\Lambda) \frac{3}{2} \delta(a, \tilde{a}) \partial_{\zeta} a \partial_{\tilde{\zeta}} \tilde{a}, \]  
(3.87)
and using (2.82) and (2.83) we get
\[ \partial \zeta \partial \overline{\zeta} u \sim -\frac{\pi}{6} \delta (\zeta, \overline{\zeta}) = -\frac{\pi}{4} \delta (a, \overline{a}) \partial \zeta a \partial \overline{\zeta} a. \] (3.88)

Therefore, we do have that (1.1) is satisfied at \( \zeta = 0 \).

Let us now evaluate the derivatives of \( u \) for \( \zeta \neq 0 \). Since \( I_{2n+1} \) depends on \( \zeta \) through the modified Bessel function \( K_0 \), we consider
\[ \partial \zeta \partial \overline{\zeta} K_0 (4 \mid g \mid \zeta \mid \sqrt{w_N}) = (4 \mid g \mid)^2 w_N \partial \zeta \mid \zeta \mid \partial \overline{\zeta} \mid \zeta \mid K_0'' (4 \mid g \mid \zeta \mid \sqrt{w_N}) \]
\[ + 4 \mid g \mid \sqrt{w_N} \partial \zeta \partial \overline{\zeta} \mid \zeta \mid K_0' (4 \mid g \mid \zeta \mid \sqrt{w_N}) \] (3.89)

Observe that
\[ \partial \zeta \mid \zeta \mid = \frac{1}{2} \sqrt{\frac{\zeta}{\overline{\zeta}}} \quad , \quad \partial \overline{\zeta} \mid \zeta \mid = \frac{1}{2} \sqrt{\frac{\overline{\zeta}}{\zeta}} \] (3.90)

and with help of (2.40) we obtain
\[ \partial \zeta \partial \zeta \mid \zeta \mid = \frac{1}{4 \mid \zeta \mid} + \frac{\pi}{4} \mid \zeta \mid \delta (\zeta, \overline{\zeta}) \] (3.91)

Since we are taking the point \( \zeta = 0 \) out, we can use the defining equation of \( K_0 \) namely,
\[ z^2 K_0'' (z) + zK_0' (z) - z^2 K_0 (z) = 0 \] (3.92)

and (3.90) and (3.91) to get
\[ \partial \zeta \partial \zeta K_0 (4 \mid g \mid \zeta \mid \sqrt{w_N}) = 4g^2 w_N K_0 (4 \mid g \mid \zeta \mid \sqrt{w_N}) \] (3.93)

Therefore, from (2.67) one has
\[ \partial \zeta \partial \zeta I_N = 4g^2 J_N \quad \zeta \neq 0 \] (3.94)

where
\[ J_N \equiv \frac{1}{2^{N-1}} \int_{-\infty}^{\infty} d\phi_1 \ldots d\phi_{N-1} \frac{w_N K_0 (4 \mid g \mid \zeta \mid \sqrt{w_N})}{\cosh (\frac{1}{2} \phi_1) \cosh (\frac{1}{2} \phi_2) \ldots \cosh (\frac{1}{2} \phi_{N-1}) \cosh (\sum_{n=1}^{N-1} \phi_n)} \] (3.95)

Notice that the relation (3.94) is also valid for \( N = 1 \), with
\[ J_1 = K_0 (4 \mid g \mid \zeta \mid) \] (3.96)

The reason is that from (2.62) we have that \( I_1 = K_0 (4 \mid g \mid \zeta \mid) \), and so (3.93) becomes (3.94) using the fact that \( w_1 = 1 \).
Using arguments similar to those leading to (3.93), one can check that $I_N$ and $J_N$ satisfy

$$z^2 I''_N (z) + z I'_N (z) = z^2 J_N (z) \quad (3.97)$$

where $z$ stands for the argument of those functions, i.e. $z \equiv 4 \parallel g \parallel \zeta \parallel$.

We now have, from (2.60), (3.94) and (3.87), that

$$\partial_{\zeta} \partial_{\bar{\zeta}} u = 8g^2 \sum_{n=0}^{\infty} \frac{(2\Lambda)^{2n+1}}{2n+1} J_{2n+1} - \delta (\zeta, \bar{\zeta}) f (2\pi \Lambda) \quad (3.98)$$

where $f (x)$ was defined in (2.80).

So, replacing into equation (1.1), we get

$$2g^2 \left( 4 \sum_{n=0}^{\infty} \frac{(2\Lambda)^{2n+1}}{2n+1} J_{2n+1} \right) = \delta (\zeta, \bar{\zeta}) f (2\pi \Lambda) - \frac{\pi}{4} \delta (a) \partial_{\zeta} a \partial_{\bar{\zeta}} \bar{a} \quad (3.99)$$

As we have already seen the vanishing of the r.h.s. of (3.99) fixes the value of $\Lambda$. Indeed, using (A.112) we see we need to choose $\Lambda$ such that $f (2\pi \Lambda) = \pi/6$. But that is exactly what we had done in (2.82) and (2.83) to get the right boundary conditions.

The l.h.s. of (3.99) on the other hand vanishes for any $\Lambda$. That is an amazing result and involves special properties of the Bessel function $K_0$, or more precisely of $I_N$ and $J_N$, which we could not find in the literature. Expanding the l.h.s. of (3.99) in powers of $\Lambda$ we get

$$\Lambda \rightarrow K_0 (4 \parallel g \parallel \zeta \parallel) = K_0 (4 \parallel g \parallel \zeta \parallel) \quad (I_1 = J_1)$$

$$\Lambda^3 \rightarrow J_3 = I_3 + 81 I_1^3$$

$$\Lambda^5 \rightarrow J_5 = I_5 + \frac{40}{3} I_1^2 I_3 + \frac{32}{3} I_1^5$$

$$\Lambda^7 \rightarrow J_7 = I_7 + \frac{224}{9} I_1^4 I_3 + \frac{56}{9} I_1 I_3^2 + \frac{56}{5} I_1^2 I_5 + \frac{256}{45} I_1^7$$

$$\vdots \quad \vdots \quad \vdots$$

With help of (3.97) we get that the integrals $I_{2n+1}$ satisfy the following coupled non-linear differential equations

$$I''_3 + \frac{1}{x} I'_3 - I_3 = 81 I_1^3$$

$$I''_5 + \frac{1}{x} I'_5 - I_5 = \frac{40}{3} I_1^2 I_3 + \frac{32}{3} I_1^5$$

$$I''_7 + \frac{1}{x} I'_7 - I_7 = \frac{224}{9} I_1^4 I_3 + \frac{56}{9} I_1 I_3^2 + \frac{56}{5} I_1^2 I_5 + \frac{256}{45} I_1^7$$

$$\vdots \quad \vdots \quad \vdots$$

19
So, the r.h.s. of these equations is what makes the difference between $K_0(x)$ and the $I_{2n+1}$'s (see (3.92)).

We did not find a way of checking relations (3.101) analytically. In the appendix C we make a numerical check of them up to the equation for $I_7$, and find that they are indeed true.

Therefore, the configuration (2.84) is indeed a solution of (1.1).

Acknowledgements

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A Hyperelliptic Riemann surfaces and delta function

Let us consider the equation for an hyperelliptic Riemann surface

$$y^2 = a(z) = (z - z_1)(z - z_2)...(z - z_n) \quad (A.102)$$

There are branch points at $z = z_1, ..., z = z_n$. Let us suppose they are all distinct. Also the point at infinity is a branch point when $n$ is odd. For simplicity let us suppose that $n$ is even. There are two sheets, which are two copies of the $z$-plane. Now we draw a cut from $z_1$ to $z_2$, another from $z_3$ to $z_4$ and so on. The two sheets are now attached to each other through the cuts. By proceeding along a small circle around a branch point we will pass from one sheet to another and after $4\pi$ we are back to the initial point. Passing to the string interpretation, it is evident that now we have a string of length $4\pi$ instead of $2\pi$ like initially. The string interpretation is as follows: we have initially two strings that interact successively $n - 2$ times and finally split into two separate strings as in the initial state. The Riemann surface we get in this way is an hyperelliptic one with two punctures representing the initial strings, two representing the final strings and $(n - 2)/2$ handles. Let us call it $\Sigma$.

Let us return to (A.102). $y$ and $z$ are coordinates of two complex planes, but, of course they can be considered as function over $\Sigma$. The coordinate $z$ is not a good coordinate near a branch
point. A good local coordinate near a branch point $z_i$ is $\xi_i = \sqrt{z - z_i}$. I.e., near $z_i$ we have $z = z_i + \xi_i^2$. $z$ is not a good coordinate at infinity either, it must be replaced by $w = 1/z$. After these substitution we see that $y$ is a meromorphic function, with $n$ zeroes at the branch points and a pole of order $n$ at $z = \infty$ on each sheet.

Let us now consider the differential $dz$. $dz \sim \xi_i d\xi_i$ near $z_i$, therefore $dz$ has simple zeroes at the branch points. At infinity $dz \sim w^{-2} dw$, therefore it has a double pole there, on both sheets.

Therefore the product $ydz$ is a meromorphic one–form over $\Sigma$, with a single pole of order $n + 2$ at infinity. It makes sense to integrate this form along a path, and this is what we do when we write $\zeta = \int \sqrt{a} dz$, eq. (2.47).

Let us now come to eq. (2.79)

\[ u \sim -\frac{2}{\pi} f \ln |\zeta| \]  

(A.103)

and to the behaviour

\[ u = -\frac{1}{2} \ln |a| \]  

(A.104)

which is required for consistency near a branch point. Near a generic branch point $a \sim z - z_0$, a good coordinate is $\xi = \sqrt{a}$. In terms of this coordinate we have

\[ a \sim \xi^2, \quad \zeta = \int dz \sqrt{a} \sim \int \xi^2 d\xi \sim \xi^3, \quad a \sim \xi^{2/3} \]  

(A.105)

Therefore, in order that (A.103) be consistent with (A.104) we must have $f = \frac{\pi}{6}$.

Some attention must be payed to the definition of the delta functions, in order to avoid possible inconsistencies in fixing the value of $f$. Let us see this point in detail.

Consider the good coordinate $\xi$ and

\[ \int d^2 \xi \partial_\xi \partial_{\bar{\xi}} \ln |\xi| = \frac{\pi}{2} \int d^2 \xi \delta(\xi, \bar{\xi}) \]  

(A.106)

Since the contribution to the integral is only at the origin we can restrict it to the unit disk around the origin, and proceed in another way by applying Stokes theorem

\[ \int d^2 \xi \partial_\xi \partial_{\bar{\xi}} \ln |\xi| = \frac{1}{2} \int d^2 \xi \partial_\xi \frac{1}{\xi} = \frac{1}{2} \int d\xi \frac{1}{\xi} = \frac{1}{2} \int_0^{2\pi} d\theta = \pi i \]  

(A.107)

where the contour integral extends over the unit circle around the origin and $\xi = e^{i\theta}$.

If we repeat the same calculation with the 'bad' coordinate $a$ we get

\[ \int d^2 a \partial_a \partial_{\bar{a}} \ln |a| = \frac{\pi}{2} \int d^2 a \delta(a, \bar{a}) \]  

(A.108)
and
\[ \int d^2 a \, \partial_a \partial_{\bar{a}} \ln |a| = \frac{1}{2} \int d^2 a \, \partial_a \frac{1}{a} = \frac{1}{2} \oint da \frac{1}{a} = \frac{4\pi i}{2} = 2\pi i \] (A.109)
The last steps are due to the fact that the angular integration for \( a \) extends over \( 4\pi \) since \( a \sim \xi^2 \).

In a similar way for \( \zeta \) we will get
\[ \int d^2 \zeta \, \partial_\zeta \partial_{\bar{\zeta}} \ln |\zeta| = \frac{1}{2} \int d^2 \zeta \, \partial_\zeta \frac{1}{\zeta} = \frac{1}{2} \oint d\zeta \frac{1}{\zeta} = \frac{1}{2} \int_0^{4\pi} d\theta = 3\pi i \] (A.110)

At this point it is judicious to make use of different symbols for these delta functions: \( \delta(\xi, \bar{\xi}) \), which is the usual delta function, and \( \delta_a(a, \bar{a}), \delta_\zeta(\zeta, \bar{\zeta}) \) so that, roughly speaking,
\[ \delta_a(a, \bar{a}) \sim 2\delta(\xi, \bar{\xi}), \quad \delta_\zeta(\zeta, \bar{\zeta}) \sim 3\delta(\xi, \bar{\xi}) \] (A.111)

In addition we must take into account the Jacobian factor due to the change of coordinates (a delta function transforms like the component of a one-form). In conclusion we have the relation
\[ \delta_\zeta(\zeta, \bar{\zeta}) = \frac{3}{2} \delta_a(a, \bar{a}) \partial_\zeta a \partial_{\bar{\zeta}} \bar{a} \] (A.112)

B. The affine \( \hat{sl}(2) \) Kac-Moody algebra

The commutation relations of the affine \( \hat{sl}(2) \) Kac-Moody algebra are given by

\[ [H^m, H^n] = 2C m \delta_{m+n,0} \]
\[ [H^m, T^n_+] = \pm T^{m+n}_+ \]
\[ [T^m_+, T^n_-] = H^{m+n} + C m \delta_{m+n,0} \]
\[ [D, H^m] = mH^m \]
\[ [D, T^m_+] = mT^m_+ \] (B.113)

In a highest weight representation we can take
\[ (H^n)^\dagger = H^{-n} \quad (T^m_+)^\dagger = T^{-n}_- \] (B.114)

Let \( |\lambda\rangle \) be a highest weight state. Then
\[ T^m_+ |\lambda\rangle = T^{-m}_- |\lambda\rangle = H^n |\lambda\rangle = 0 \quad m \geq 0; \quad n > 0 \] (B.115)

Therefore, the norm of the state \( T^{-m}_+ |\lambda\rangle, m > 0, \) is
\[ \langle \lambda | T^{-m}_+ T^{-m}_+ | \lambda \rangle = \langle \lambda | [T^{-m}_+, T^{-m}_+] | \lambda \rangle = \langle \lambda | -H^0 + mC | \lambda \rangle \] (B.116)
Using (2.24) and (2.25), one gets

\[ T_{+}^{-1} | \lambda_1 \rangle = 0 \quad (B.117) \]

Analogously, the norm of the state \( T_+^{-m}T_+^{-m} | \lambda \rangle, m > 0, \) is

\[ \langle \lambda | T_+^{-m}T_+^{-m}T_+^{-m-m} | \lambda \rangle = 2\langle \lambda | (-H^0 + mC)(-H^0 + mC) - (-H^0 + mC) | \lambda \rangle \]

Therefore, from (2.24)

\[ T_{+}^{-1}T_{+}^{-1} | \lambda_0 \rangle = 0 \quad (B.118) \]

Using similar arguments one gets that

\[ T_0^{-} | \lambda_0 \rangle = 0 \quad T_0^{-}T_0^{-} | \lambda_1 \rangle = 0 \quad (B.119) \]

For the applications we make in this paper it is useful to work with following basis for the affine \( \mathfrak{sl}(2) \) Kac-Moody algebra

\[ b_{2m+1} = T_{+}^{m} + T_{+}^{m+1} \quad (B.120) \]
\[ F_{2m+1} = T_{+}^{-m} - T_{+}^{-m+1} \quad (B.121) \]
\[ F_{2m} = H^m - \frac{1}{2}C\delta_{m,0} \quad (B.122) \]

in addition to the generators \( C \) (central term) and \( D \). The commutation relations are

\[ [b_{2m+1}, b_{2n+1}] = C(2m+1)\delta_{m+n+1,0} \quad (B.123) \]
\[ [b_{2m+1}, F_{2n+1}] = -2F_{2(m+n+1)} \quad (B.124) \]
\[ [b_{2m+1}, F_{2n}] = -2F_{2(m+n)+1} \quad (B.125) \]
\[ [F_{2m+1}, F_{2n+1}] = -C(2m+1)\delta_{m+n+1,0} \quad (B.126) \]
\[ [F_{2m+1}, F_{2n}] = -2b_{2(m+n)+1} \quad (B.127) \]
\[ [F_{2m}, F_{2n}] = C(2m)\delta_{m+n,0} \quad (B.128) \]

The elements which diagonalize the adjoint action of the oscillators \( b_{2m+1} \) are\(^3\)

\[ V(\mu) = -2 \sum_{n=-\infty}^{\infty} \mu^{-n}F_n \quad (B.129) \]

Indeed

\[ [b_{2m+1}, V(\mu)] = -2\mu^{2m+1}V(\mu) \quad (B.130) \]

\[^3\text{We choose the normalization factor to be } -2 \text{ because we want the expectation values in (B.131) to be } \pm 1.\]
Notice that the expectation value of such operator on a highest weight state gets contribution only from its zero mode. Indeed, using (2.24) and (2.25), one gets

\[ \langle \lambda_0 | V(\mu) | \lambda_0 \rangle = 1 \quad \langle \lambda_1 | V(\mu) | \lambda_1 \rangle = -1 \quad (B.131) \]

### B.1 Vertex Operator Construction

We now discuss the principal vertex operator representation of the \( \hat{sl}(2) \) Kac-Moody algebra. It differs from the homogeneous vertex representation, i.e. the Fubini-Veneziano vertex, in the sense that the oscillators have only odd indices and so do not have zero modes. We introduce

\[ Q(z) = \sum_{n=0}^{\infty} \left( \frac{z^{-N}}{N} - \frac{z^{N}}{N} b_{-N} \right) \quad N = 2n + 1 \quad (B.132) \]

\[ P(z) = -\sum_{n=0}^{\infty} (z^{-N} b_{N} + z^{N} b_{-N}) \quad (B.133) \]

where the oscillators satisfy the commutation relations (B.123). We then have

\[ P(z)Q(w) = :: P(z)Q(w) :: + \frac{wz}{z^2 - w^2} C |z| > |w| \]

\[ Q(w)P(z) = :: Q(w)P(z) :: - \frac{wz}{w^2 - z^2} C |w| > |z| \quad (B.134) \]

We now introduce the vertex operator

\[ \mathcal{V}(z) = :: e^{\alpha Q(z)} :: \quad (B.135) \]

and so

\[ P(z)\mathcal{V}(w) = :: P(z)\mathcal{V}(w) :: + \frac{\alpha wz}{z^2 - w^2} C \mathcal{V}(w) |z| > |w| \]

\[ \mathcal{V}(w)P(z) = :: \mathcal{V}(w)P(z) :: - \frac{\alpha wz}{w^2 - z^2} C \mathcal{V}(w) |w| > |z| \quad (B.136) \]

We expand \( \mathcal{V}(z) \) in modes as

\[ \mathcal{V}(z) = \sum_{n \in \mathbb{Z}} z^{-n} A_n \quad A_n = \oint \frac{dz}{2\pi i z} z^n \mathcal{V}(z) \quad (B.137) \]

That means that \( b_M A_n \) and \( A_n b_M \) can be written as the double integral of the same term

\[ b_M A_n = -\left( \oint \frac{d\zeta}{2\pi i \zeta} \oint \frac{dz}{2\pi i z} \mathcal{V}(z) \right)_{|z|>|\zeta|} \quad \left\{ :: P(z)\mathcal{V}(\zeta) :: + \frac{\alpha \zeta}{z^2 - \zeta^2} C \mathcal{V}(\zeta) \right\} \]

\[ A_n b_M = -\left( \oint \frac{d\zeta}{2\pi i \zeta} \oint \frac{dz}{2\pi i z} \mathcal{V}(z) \right)_{|\zeta|>|z|} \quad \left\{ :: P(z)\mathcal{V}(\zeta) :: + \frac{\alpha \zeta}{z^2 - \zeta^2} C \mathcal{V}(\zeta) \right\} \quad (B.138) \]

24
where we use
\[
\oint \frac{dz}{2\pi i z} z^M P(z) = -b_M \quad M \text{ odd} \tag{B.139}
\]

Notice that
\[
\left( \oint \frac{d\zeta}{2\pi i \zeta} \frac{d\zeta^n}{2\pi i \zeta} \right) - \left( \oint \frac{d\zeta}{2\pi i \zeta} \frac{d\zeta^M}{2\pi i \zeta} \right) = \tag{B.140}
\]

The only contribution comes from the poles on \( z = \pm \zeta \); we eliminate the poles on \( z = 0 \). So we have
\[
[b_M, A_n] = -\oint \frac{d\zeta}{2\pi i \zeta} \oint \frac{d\zeta^n}{2\pi i \zeta} \left\{ \begin{array}{c} : P(z) \mathcal{V}(\zeta) : + \frac{\alpha}{z^2 - \zeta^2} C \mathcal{V}(\zeta) \end{array} \right\} \tag{B.141}
\]

The first term in r.h.s. is normal ordered, what guarantees that there is no contribution of this part on \( z = \zeta \). The second term has a simple pole on \( z = \pm \zeta \). So, by the residue theorem we have
\[
[b_M, A_n] = -\frac{\alpha}{2} C \oint \frac{d\zeta}{2\pi i \zeta} \oint \frac{d\zeta^M}{2\pi i \zeta} \mathcal{V}(\zeta) - M A_{n+M} \tag{B.142}
\]

once \( M \) is odd and we used (B.137), (B.139). This relation corresponds to (B.124), (B.125), but in order to reproduce the algebra, we must impose
\[
\alpha C = 2 \quad \Rightarrow \quad [b_M, A_n] = -2 A_{n+M} \tag{B.143}
\]

Now we must verify the remaining commutation relations, namely \([A_m, A_n] \). To this, we must evaluate
\[
\mathcal{V}(z) \mathcal{V}(\zeta) = : \mathcal{V}(z) \mathcal{V}(\zeta) : + e^{\frac{\alpha^2}{C} \ln \left( \frac{z - \zeta}{z + \zeta} \right)} \quad |z| > |\zeta| \tag{B.144}
\]

from where we conclude that
\[
[A_m, A_n] = \oint \frac{dz}{2\pi i z} \oint \frac{d\zeta}{2\pi i \zeta} \mathcal{V}(z) \mathcal{V}(\zeta) \times \left[ \begin{array}{c} \frac{\alpha^2}{C} \ln \left( \frac{z - \zeta}{z + \zeta} \right) \quad \left| \frac{\zeta - z}{\zeta + z} \right| \\ \left| \frac{\zeta - z}{\zeta + z} \right| \ \left| \frac{\zeta - z}{\zeta + z} \right| \end{array} \right] \tag{B.145}
\]

\[25\]
Since we need double poles, we set $\frac{\alpha^2}{2} C = 2$. But from (B.143) we have $\alpha C = 2$, and so we get that $\alpha = 2$ and $C = 1$. Then

$$[A_m, A_n] = \oint \frac{dz}{2\pi i z} \oint \frac{d\zeta}{2\pi i \zeta} \, \zeta^n \zeta^{m} \, V(\zeta)V(z) : \left(\frac{z - \zeta}{z + \zeta}\right)^2$$

where we take the contour to be

According with the Cauchy’s integral formula

$$f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} \, dz$$

what implies

$$[A_m, A_n] = \oint \frac{d\zeta}{2\pi i \zeta} \, \zeta^n \zeta^{m} \, 4(-1)^m [m - \alpha P(\zeta)]$$

where we have used the fact that $V(-\zeta)V(\zeta) = 1$. Using (B.139) one gets

$$[A_{2m}, A_{2n}] = 8m \delta_{m+n,0}$$

$$[A_{2m+1}, A_{2n+1}] = -4(2m + 1) \delta_{m+n+1,0}$$

$$[A_{2m+1}, A_{2n}] = -8b_{2(m+n)+1}$$

Therefore, from (B.143) and (B.149), we see that $\pm A_n/2$ satisfy the same algebra as $F_n$, given in (B.124)-(B.128). So we have reproduced the commutation relations for the $sl(2)$ Kac-Moody algebra for $C = 1$. The arbitrariness in the sign of $A_n$ is related to the normalization of the expectation values of the operators on the highest weight states. Since the operator $V(z)$ introduced in (B.135) is normal ordered, it satisfy

$$\langle \lambda \mid V(z) \mid \lambda \rangle = 1$$

Then, in order to reproduce (B.131) we must normalize $V(z)$ differently in different highest weight representations. In fact, we have to make the identification

$$V(z) = c_\lambda V(z)$$

where

$$c_{\lambda_0} = 1 \quad c_{\lambda_1} = -1$$
B.2 Vertex Operator properties

Using (B.132)

\[ Q_>(z) = \sum_{n=0}^{\infty} z^{-N} \frac{b_N}{N} \quad Q_<(z) = -\sum_{n=0}^{\infty} z^{N} \frac{b_N}{N} \quad N = 2n + 1 \] (B.153)

\[ \mathcal{V}(z)\mathcal{V}(w) = e^{2Q_<(z)} e^{2Q_>(z)} e^{2Q_<(w)} e^{2Q_>(w)} \]

\[ = e^{2Q_<(z)} e^{2Q_>(z)} e^{2Q_<(w)} e^{-2Q_>(z)} e^{2Q_>(z)} e^{2Q_>(w)} \] (B.154)

\[ e^{2Q_>(z)} e^{2Q_<(w)} e^{-2Q_>(z)} = e^{2Q_<(z)} e^{2Q_<(w)} e^{-2Q_>(z)} \]

\[ = e^{2Q_<(w)} + [2Q_>(z), 2Q_<(w)] \]

\[ = e^{2Q_<(w)} e^{-2 \ln \left( \frac{z+w}{z-w} \right)} \] (B.155)

where we used

\[ [2Q_>(z), 2Q_<(w)] = -4 \sum_{n,m \geq 0} \frac{z^{-N}}{N} \frac{w^M}{M} [b_N, b_{-M}] \]

\[ = -4 \sum_{n,m \geq 0} \frac{z^{-N}}{N} \frac{w^M}{M} N \delta_{N,M} \]

\[ = -4 \sum_{n \geq 0} \frac{1}{N} \left( \frac{w}{z} \right)^N \]

\[ = -2 \ln \left( \frac{z+w}{z-w} \right) \] (B.156)

Then, from (B.155) we have

\[ e^{2Q_>(z)} e^{2Q_<(w)} = e^{2Q_<(w)} e^{2Q_>(z)} e^{-2 \ln \left( \frac{z+w}{z-w} \right)} \] (B.157)

So, returning to (B.154) and using (B.157)

\[ \mathcal{V}(z)\mathcal{V}(w) = e^{2Q_<(z)} e^{2Q_<(w)} e^{2Q_>(z)} e^{2Q_>(w)} e^{-2 \ln \left( \frac{z+w}{z-w} \right)} \]

\[ = :: \mathcal{V}(z)\mathcal{V}(w) :: \left( \frac{z-w}{z+w} \right)^2 \] (B.158)

By the same argument,

\[ \mathcal{V}(z)\mathcal{V}(w)\mathcal{V}(k) = :: \mathcal{V}(z)\mathcal{V}(w)\mathcal{V}(k) :: \left( \frac{z-w}{z+w} \right)^2 \left( \frac{z-k}{z+k} \right)^2 \left( \frac{w-k}{w+k} \right)^2 \] (B.159)
and so on. Therefore, from (B.151) we get that

$$\langle \lambda \mid V(\mu_1) \ldots V(\mu_n) \mid \lambda \rangle = c_n^2 \prod_{j>i=1}^{n} \left( \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right)^2$$

(B.160)

where we have used that fact that

$$\langle \lambda \mid :: V(\mu_1) \ldots V(\mu_n) :: \mid \lambda \rangle = 1$$

(B.161)

C  The numerical check of equations (3.101)

As we have seen in section 3, the existence of the exact solution relies on the validity of the non-linear differential equations (3.101) satisfied by the quantities $I_{2n+1}$. We could not find in the literature any reference to those equations, and we were not able to prove them analytically. Therefore, we present here a numerical check of the first three equations in (3.101), at some specific values of the argument of those functions.

In order to perform the numerical calculation it is better to make a change of integration variables in the expressions for $I_N$ and $J_N$. We define

$$\phi_i = \ln \frac{1 + x_i}{1 - x_i} \quad i = 1, 2, \ldots N - 1$$

(C.162)

Then

$$x_i = -1 \rightarrow \phi_i = -\infty$$

$$x_i = 1 \rightarrow \phi_i = \infty$$

(C.163)

We have that

$$d\phi_i = \frac{2}{1 - x_i^2} dx_i$$

(C.164)

In addition we have

$$\cosh \left( \frac{1}{2} \phi_i \right) = \frac{1}{\sqrt{1 - x_i^2}}$$

$$\cosh \left( \frac{1}{2} \sum_{i=1}^{N-1} \phi_i \right) = \frac{1}{2} \frac{\prod_{i=1}^{N-1} (1 + x_i) + \prod_{i=1}^{N-1} (1 - x_i)}{\prod_{i=1}^{N-1} \sqrt{1 - x_i^2}}$$

$$\cosh \left( \sum_{i=j_1}^{j_2} \phi_i \right) = \frac{1}{2} \left( \prod_{i=j_1}^{j_2} \frac{1 + x_i}{1 - x_i} + \prod_{i=j_1}^{j_2} \frac{1 - x_i}{1 + x_i} \right)$$

$$= \frac{1}{2} \left( \prod_{i=j_1}^{j_2} (1 + x_i) \right)^2 + \left( \prod_{i=j_1}^{j_2} (1 - x_i) \right)^2 \prod_{i=j_1}^{j_2} \left( 1 - x_i^2 \right)$$

(C.165)
We then get that \( I_N \) and \( J_N \) given in (2.67) and (3.95) respectively, become

\[
I_N(z) = \int_{-1}^{1} d^{N-1}x_i \frac{K_0(z\sqrt{w_N})}{v_N}
\]

\[
J_N(z) = \int_{-1}^{1} d^{N-1}x_i \frac{w_N K_0(z\sqrt{w_N})}{v_N}
\]

where

\[
w_N = N + \sum_{l=0}^{N-2} \sum_{l=1}^{N-l-1} \left( \prod_{i=j}^{j+l} \frac{1+x_i}{1-x_i} + \prod_{i=j}^{j+l} \frac{1-x_i}{1+x_i} \right)
\]

\[
v_N = \frac{1}{2} \left( \prod_{i=1}^{N-1} (1+x_i) + \prod_{i=1}^{N-1} (1-x_i) \right)
\]

In order to evaluate the multidimensional integrals we used the Monte Carlo method. The program was written in Mathematica and did not make use of the command \texttt{NIntegrate}, since we found some limitations on it regarding the number of iterations. We found it better to write a routine to do it directly, using Mathematica random number generator and its package for Bessel functions. The functions \( I_N \) and \( J_N \) are defined as \((N - 1)\)-dimensional integrals. We have evaluated them for \( N = 3, 5, \) and 7, by sampling the integration region with \( 10^6 \) points, irrespective of the dimensionality of the integrals. We evaluated those functions for the arguments \( z = 0.001, 0.01, 0.1, 0.5, 1, 5, 10, \) and 100. In order to estimate the errors in the calculations, we evaluated each integral for each value of the argument, 11 times with the \( 10^6 \) point sampling. We then calculated the average (\( \bar{v} = \sum_{i=1}^{n} v_i/n \)) and the standard deviation (\( \sigma^2 = \sum_{i=1}^{n} (v_i - \bar{v})^2 / (n - 1) \)).

Using (3.97) one can write the equations (3.101) as

\[
J_{2n+1} = L_{2n+1}
\]

where

\[
L_3 \equiv I_3 + 8I_1^3
\]

\[
L_5 \equiv I_5 + \frac{40}{3} I_1^2 I_3 + \frac{32}{3} I_1^5
\]

\[
L_7 \equiv I_7 + \frac{224}{9} I_1 I_3 + \frac{56}{9} I_1 I_3^2 + \frac{56}{5} I_1^2 I_5 + \frac{256}{45} I_1^7
\]

Therefore, we check the relation (C.169) numerically. The errors in \( L_{2n+1} \) are given by

\[
\Delta L_3 = \Delta I_3
\]

\[
\Delta L_5 = \Delta I_5 + \frac{40}{3} I_1^2 \Delta I_3
\]

\[
\Delta L_7 = \Delta I_7 + \frac{224}{9} I_1 \Delta I_3 + \frac{112}{9} I_1 I_3 \Delta I_3 + \frac{56}{5} I_1^2 \Delta I_5
\]
Since $I_1 = K_0$, and since it is evaluated with the Mathematica package for Bessel function, its error was taken to be negligible as compared to the other ones. Its values, for the arguments used in the simulations are given in table 1. In table 2 we show the results of the numerical calculations of the integrals $I_3$, $I_5$ and $I_7$, with the corresponding standard deviations.

The check of the relations (C.169), or equivalently (3.101), is given in tables 3, 4 and 5. Notice that the agreement is quite good. Except for the values of $J_5$ and $L_5$ at $z = 0.001$, which differ by 1.45 standard deviations, all the other values agree well inside one standard deviation. So, our numerical calculations strongly indicate that the relations (3.101) should hold true. It would be very interesting to find an analytical proof of those equations.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$I_1 \equiv K_0$</th>
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<tbody>
<tr>
<td>0.001</td>
<td>7.02369</td>
</tr>
<tr>
<td>0.01</td>
<td>4.72124</td>
</tr>
<tr>
<td>0.1</td>
<td>2.42707</td>
</tr>
<tr>
<td>0.5</td>
<td>0.924419</td>
</tr>
<tr>
<td>1</td>
<td>0.421024</td>
</tr>
<tr>
<td>5</td>
<td>3.6911 $10^{-3}$</td>
</tr>
<tr>
<td>10</td>
<td>1.77801 $10^{-5}$</td>
</tr>
<tr>
<td>100</td>
<td>4.65663 $10^{-45}$</td>
</tr>
</tbody>
</table>

Table 1: The numerical values of $I_1$ for some values of the argument
<table>
<thead>
<tr>
<th>$z$</th>
<th>$I_3$</th>
<th>$I_5$</th>
<th>$I_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>26.237 ± 0.026</td>
<td>155.52 ± 0.44</td>
<td>1027.0 ± 3.2</td>
</tr>
<tr>
<td>0.01</td>
<td>14.92 ± 0.01</td>
<td>73.30 ± 0.12</td>
<td>390.4 ± 1.0</td>
</tr>
<tr>
<td>0.1</td>
<td>4.455 ± 0.002</td>
<td>11.429 ± 0.011</td>
<td>30.118 ± 0.035</td>
</tr>
<tr>
<td>0.5</td>
<td>0.43063 ± 0.00044</td>
<td>0.24327 ± 0.00024</td>
<td>0.13817 ± 0.00031</td>
</tr>
<tr>
<td>1</td>
<td>0.049517 ± 0.000035</td>
<td>0.006656 ± 0.000010</td>
<td>(8.974 ± 0.028) $10^{-4}$</td>
</tr>
<tr>
<td>5</td>
<td>$(4.4348 \pm 0.0068) \times 10^{-8}$</td>
<td>$(5.548 \pm 0.018) \times 10^{-13}$</td>
<td>$(6.9 \pm 0.1) \times 10^{-18}$</td>
</tr>
<tr>
<td>10</td>
<td>$(5.25 \pm 0.02) \times 10^{-15}$</td>
<td>$(1.589 \pm 0.017) \times 10^{-24}$</td>
<td>$(4.79 \pm 0.21) \times 10^{-34}$</td>
</tr>
<tr>
<td>100</td>
<td>$(9.990 \pm 0.065) \times 10^{-134}$</td>
<td>$(2.21 \pm 0.27) \times 10^{-222}$</td>
<td>$(3.8 \pm 2.8) \times 10^{-311}$</td>
</tr>
</tbody>
</table>

Table 2: The numerical values of $I_3$, $I_5$ and $I_7$ with corresponding standard deviations

<table>
<thead>
<tr>
<th>$z$</th>
<th>$J_3$</th>
<th>$L_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>2783. ± 338.</td>
<td>2798.19 ± 0.03</td>
</tr>
<tr>
<td>0.01</td>
<td>863.9 ± 27.8</td>
<td>856.82 ± 0.01</td>
</tr>
<tr>
<td>0.1</td>
<td>118.83 ± 0.27</td>
<td>118.831 ± 0.002</td>
</tr>
<tr>
<td>0.5</td>
<td>6.7521 ± 0.0074</td>
<td>6.75033 ± 0.00044</td>
</tr>
<tr>
<td>1</td>
<td>0.64645 ± 0.00040</td>
<td>0.646569 ± 0.000035</td>
</tr>
<tr>
<td>5</td>
<td>$(4.4631 \pm 0.0063) \times 10^{-7}$</td>
<td>$(4.4665410 \pm 0.000068) \times 10^{-7}$</td>
</tr>
<tr>
<td>10</td>
<td>$(5.022 \pm 0.021) \times 10^{-14}$</td>
<td>$(5.0220 \pm 0.0022) \times 10^{-14}$</td>
</tr>
<tr>
<td>100</td>
<td>$(9.051 \pm 0.059) \times 10^{-133}$</td>
<td>$(9.0770 \pm 0.0065) \times 10^{-133}$</td>
</tr>
</tbody>
</table>

Table 3: Comparison of the numerical values of $J_3$, and $L_3$ with corresponding standard deviations
<table>
<thead>
<tr>
<th>$z$</th>
<th>$J_5$</th>
<th>$L_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>181810. ± 12383.</td>
<td>199742. ± 18.</td>
</tr>
<tr>
<td>0.01</td>
<td>29616. ± 318.</td>
<td>29529.2 ± 3.1</td>
</tr>
<tr>
<td>0.1</td>
<td>1260.3 ± 1.7</td>
<td>1259.64 ± 0.17</td>
</tr>
<tr>
<td>0.5</td>
<td>12.35 ± 0.01</td>
<td>12.3505 ± 0.0052</td>
</tr>
<tr>
<td>1</td>
<td>0.26474 ± 0.00032</td>
<td>0.264801 ± 0.000093</td>
</tr>
<tr>
<td>5</td>
<td>(1.5883 ± 0.0052) $10^{-11}$</td>
<td>(1.5919 ± 0.0014) $10^{-11}$</td>
</tr>
<tr>
<td>10</td>
<td>(4.273 ± 0.046) $10^{-23}$</td>
<td>(4.268 ± 0.011) $10^{-23}$</td>
</tr>
<tr>
<td>100</td>
<td>(5.56 ± 0.67) $10^{-221}$</td>
<td>(5.45 ± 0.46) $10^{-221}$</td>
</tr>
</tbody>
</table>

Table 4: Comparison of the numerical values of $J_5$, and $L_5$ with corresponding standard deviations

<table>
<thead>
<tr>
<th>$z$</th>
<th>$J_7$</th>
<th>$L_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>(6.40 ± 0.65) $10^{6}$</td>
<td>(6.5034 ± 0.0019) $10^{6}$</td>
</tr>
<tr>
<td>0.01</td>
<td>505613. ± 4987.</td>
<td>507197. ± 166.</td>
</tr>
<tr>
<td>0.1</td>
<td>7753. ± 11.</td>
<td>7753.3 ± 2.9</td>
</tr>
<tr>
<td>0.5</td>
<td>14.641 ± 0.027</td>
<td>14.642 ± 0.013</td>
</tr>
<tr>
<td>1</td>
<td>0.07265 ± 0.00018</td>
<td>0.072600 ± 0.000059</td>
</tr>
<tr>
<td>5</td>
<td>(3.932 ± 0.053) $10^{-16}$</td>
<td>(3.9474 ± 0.0083) $10^{-16}$</td>
</tr>
<tr>
<td>10</td>
<td>(2.54 ± 0.11) $10^{-32}$</td>
<td>(2.542 ± 0.016) $10^{-32}$</td>
</tr>
<tr>
<td>100</td>
<td>(1.9 ± 1.4) $10^{-309}$</td>
<td>(2.3 ± 1.0) $10^{-309}$</td>
</tr>
</tbody>
</table>

Table 5: Comparison of the numerical values of $J_7$, and $L_7$ with corresponding standard deviations
References


