Experiences with the multi-level algorithm.

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Small expectation values are difficult to measure in Monte Carlo calculations as they tend to get swamped by noise. Recently an algorithm has been proposed by Lüscher and Weisz [1] which allows one to measure expectation values which previously could not be measured reliably in Monte Carlo simulations. We will test our implementation of this algorithm by looking at Polyakov loop correlators and then explore ways of applying it for measuring large Wilson loops.

1. Introduction

The physical picture of confinement is often given in terms of formation of a flux tube between a quark and an anti-quark. This picture is also supported by lattice simulations. Thinking of the flux tube as a source, the Wilson loop $W(r, T)$ can be considered as correlation between sources of length $r$ propagating for time $T$. The theory of this flux tube has been proposed as an effective string model [2].

This is an effective long distance theory and therefore to test it, one has to measure physically large Wilson loops. The problem however is that large Wilson loops have small expectation values. That makes it difficult to measure them in simulations. To circumvent this problem, recent efforts to measure properties of the flux tube, have relied on asymmetric lattices [3].

Throughout this article, we will work in 3 dimensions and with SU(2). Even in this restricted case, $\langle W \rangle \sim 10^{-7}$ will take about 1 year to measure to an accuracy of 1% with conventional algorithms on a 1Ghz PIII processor. Here we explore a new algorithm which reduces this time to about a day.

2. The Algorithm

The algorithm is based mainly on the factorization property of the partition function. This makes it possible to obtain really small expectation values as products of expectation values of intermediate orders of magnitude. This works for the Wilson action to which we restrict ourselves, but can be generalized to other actions as long as the factorization property holds. We do not go into further discussion of the algorithm but just present the operational details of our implementation of the algorithm. For a detailed discussion we refer the reader to [1].

Suppose we want to measure a Wilson loop of extent $r$ and $T$ in $x$ and $t$ directions respectively. We choose the time extent of the lattice to be a multiple of $T$ and divide the lattice into time slices of thickness $d$. At $\beta = 4/g_0^2 = 5$ ($g_0$ is the bare coupling), we found $d = 2$ to be adequate. Higher $\beta$ might warrant a higher value of $d$.

First we create a 2-link operator (fig.1A)

$$T_{\alpha \beta \gamma \delta} = U_1^{* \alpha \beta} U_2^{\gamma \delta}. \quad (1)$$

Next take a product of two $^2$ of these $T$’s to get,

$$\hat{T}_{\alpha \beta \gamma \delta} = T_1^{\alpha \mu \gamma \nu} T_2^{\mu \beta \nu \delta}. \quad (2)$$

$\langle \hat{T} \rangle$’s are now estimated by an averaging procedure in which only the time slices are updated holding the space-like links at the boundary of the slices fixed. They are then multiplied together to form the averaged propagator $\hat{T}$. Finally the sources $L_1$ and $L_2$ are calculated and the full Wilson loop is obtained as (fig.1B)

$$W = L_2^{* \alpha \gamma} T_{\alpha \beta \gamma \delta} L_1^{\beta \delta}. \quad (3)$$

$^2$It can be any number depending on $T$ and $d$
3. Optimization

Let us now try to optimize the parameters of this algorithm. Probably the most important one is the number of sub-lattice updates ($i_{upd}$) and we concentrate on that. Among the others, we use three over-relaxation steps to every heat-bath, choose the basic time slice as two lattice spacings and keep the level of averaging at one.

For Polyakov loop correlators ($\langle P^* P \rangle$), the optimal value of $i_{upd}$ was estimated in [1] as the one which minimized $\sqrt{\text{time}} \times \langle |P^* P| \rangle$. We follow a slightly different procedure. The 2-link operator $\hat{T}$ is the main quantity which is affected by the sub-lattice averaging. We propose to define the norm of this operator as

$$N(r) = \sqrt{\sum_{\alpha\beta\gamma\delta \in \{1, 2\}} \langle \hat{T}(r) \rangle_{\alpha\beta\gamma\delta}^2}.$$  \hspace{1cm} (4)

As seen in fig.2, $N(r)$ versus no. of $i_{upd}$, $r= 2 (+), 4 (x), 6 (*) and 8 (□). \beta = 3$. As seen in fig.2, $N(r)$ depends quite strongly on $r$. Moreover the propagator $T$ is a number $O(N(r)^{T/2})$. Thus the optimal value of $i_{upd}$ depends not only on $\beta$, but also on the size of the loop itself. We estimate the optimal value of $i_{upd}$ to be the one which minimizes $N(r)^{T/2} \times \sqrt{i_{upd}}$. (Note $i_{upd} \propto \text{time}$.) $N(r)^{T/2}$ is a number $O(\langle |P^* P| \rangle)$, but measuring $N(r)$ is cheaper. We checked that both methods give values of $i_{upd}$ which are similar. We also find the optimal value of $i_{upd}$ to be reasonably flat and not too sensitive to the exact definition of $N(r)$.

For Wilson loops we have to deal in addition with the sources which lie entirely in the space slices and whose fluctuation is not reduced by this algorithm. Note that it is incorrect to apply multi-hit to any of the spatial links in this algorithm. Fluctuation of the sources greatly reduce the optimal number of $i_{upd}$ for Wilson loops. We do not have any quantitative estimate on how to choose an optimal $i_{upd}$ for the Wilson loop apriori, but merely point out that if we can estimate the order of magnitude of $\langle W \rangle$ then it is best to choose $i_{upd}$ so that each measurement yields a value which is of the same order of magnitude.

In fig.3 we plot the error on a 12 $\times$ 12 Wilson loop against $i_{upd}$. This plot indicates that the optimal value of $i_{upd}$ is $\sim 200$. In contrast the corresponding value for $\langle P^* P \rangle$ is more than 1600.

4. Results

4.1. Polyakov Loop Correlators

As a test of our program, we compute the string tension from $\langle P^* P \rangle$. Neglecting higher eigenstates, the static potential between a $\bar{q}q$ pair is

$$V(r) \approx -\frac{1}{T} \ln(P^*(r, T) P(0, T)).$$  \hspace{1cm} (5)
In the string picture, for large $r$, $V(r) \approx V_0 + \alpha r - c/r$ where $V_0$ is a constant, $\alpha$ is the string tension and $c = (\pi/24)$ is a universal number. Next we define the force $F$, and $c$ in terms of $V$ as

$$F(r) = [V(r + 1) - V(r - 1)]/2$$

and

$$c(r) = [V(r + 1) - 2V(r) + V(r - 1)]r^2/2.$$ 

We tabulate our results below. For each measurement, all values of $r$ were obtained from the same sample. Hence various $r$’s were correlated. Therefore $V(r)$, $F(r)$ and $c(r)$ were calculated from each sample and the averages and errors obtained using the jackknife method. The finite volume corrections and higher state contributions are negligible.

To get the asymptotic value for $\alpha$ and $c$, we plot $F(r)$ vs $1/r^2$. The slope and intercept of this plot gives $c$ and $\alpha$ at $r = \infty$. We get $r = 0.134(5)$. This is very close to the string value of $\pi/24$. The string tension $\alpha \sqrt{\alpha} = 0.313(3)$, compares very well with Teper’s value [4] of 0.3129(20).

The above measurement took 10 days on a 1.2 GHz AMD Athlon processor.

4.2. Wilson Loops

Polyakov loops are very good for getting the ground state of the flux tube or the string tension, but for the excited states and states of higher angular momentum, it is more useful to evaluate Wilson loops [5].

With the view of computing the spectrum of the string, we looked at correlation matrices $C_{ij}(r, T)$ between sources $ij$. As a first step we define the source $s_i = \mathcal{P} \left( \prod_{k=1}^{n} (U_k + \alpha_i \sum S_k) \right)$, with only a single smearing step. $U_k$ is the original link variable, $\sum S_k$ is the sum over all the space-like staples and $\mathcal{P}$ denotes projection back to SU(2). $\alpha_i$ is the smearing parameter and takes values $\{0.1, 0.3, 0.5\}$.

The energies are extracted by

$$E_\alpha(r) = -\frac{1}{T_1 - T_2} \ln \frac{\lambda_\alpha(r, T_1)}{\lambda_\alpha(r, T_2)}$$

$T_1 > T_2$ and $\lambda_\alpha(r, T)$ are the eigenvalues of $C_{ij}$. We still do not have enough statistics to get the excited states which are highly suppressed, and are presently using higher smearing levels and better wave functions to extract them. Here we present some $T$ dependence for the ground state.

The apparent downward trend of the data might be due to the error [6] introduced for using smaller values of $T_1$ and $T_2$. The energy determined from $\langle P^* P \rangle$, which has $T = 24$, is 0.5717(5).

5. Discussion

With the multilevel algorithm it is possible to measure $\langle W \rangle \sim 10^{-6}$ with 1% error in about a day, even on a PC. Even though the algorithm works better for Polyakov loops, the improvement over multi-hit for $\langle W \rangle < 10^{-6}$ is quite significant. It is possible to couple this algorithm with other techniques like smearing to reduce fluctuation of the sources and study correlations among them.

REFERENCES

3. K. J. Juge, J. Kuti and C. Morningstar, hep-lat/0207004