OF-shell effects and consistency of many-body treatments of dense matter
has the form
\[ V_1 = C_0 + C_2 p^2, \]  
where \( p^2 / M \) is the relative kinetic energy of the nucleons. The LO coupling constant \( C_0 \) is of order \( Q^{-1} \) (where \( Q \) is a generic low-energy scale) in KSW counting and so it should be treated nonperturbatively [7]. The NLO coupling \( C_2 \) is proportional to the effective range, and is thus of order \( Q^2 \) and can be treated perturbatively in this counting scheme. Nonetheless, to show an example of the principles discussed above, we solve the Lippmann-Schwinger (LS) equation with this potential to all orders in \( C_2 \), and find the vacuum \( T \)-matrix
\[ T_1 = \frac{C_0 + p^2 C_2}{1 + \frac{M}{8\pi^2} C_0 (p^2 + p^2 C_2)(ip + \mu)}. \]  
Here we have used a subtractive renormalization procedure [7, 14]. (All coupling constants here should be understood as renormalized ones which depend on \( \mu \) to ensure that the scattering amplitude is \( \mu \)-independent.)

More generally, the effective Lagrangian can also include interactions with space derivatives of the nucleon fields and this leads to a potential that depends on momentum as well as energy. The most general NLO potential has the form
\[ V_2 = C_0 + C_2 p^2 + \frac{1}{2} C_3 (k^2 + k'^2 - 2p^2), \]  
where \( k \) and \( k' \) denote the initial and final relative momenta of the nucleons. The coupling \( C_3 \) describes a purely off-shell interaction. Solving the LS equation we get
\[ T_2 = T_1 \left[ 1 + \frac{1}{2(C_0 + p^2 C_2)} \left( C_3 (k^2 + k'^2 - 2p^2) - \frac{M}{8\pi^2} C_0 (p^2 C_0 - k^2)(p^2 - k'^2)(ip + \mu) \right) \right], \]  
where \( T_1 \) is given by Eq. (2). From this we can see that the two \( T \)-matrices coincide on-shell (\( k^2 = k'^2 = p^2 \)) and so the scattering observables are indeed independent of the off-shell behaviour of the potential as required by the equivalence theorem.

The situation becomes much less trivial in the presence of the nuclear medium. An in-medium \( T \)-matrix [15] can be obtained by solving the Feynman-Galitskii (FG) equation,
\[ T^\text{nm} = V + V G^F T^\text{nm}, \]  
where \( G^F \) denotes the two-nucleon propagator which contains both particle-particle (pp) and hole-hole (hh) states. This \( T \)-matrix can be thought of as an extension of the more familiar \( G \)-matrix [16, 17] to include hh as well as pp ladders. It is convenient to represent the FG equation graphically in terms of the Hugenholtz diagrams of Fig. 1 (a) and (b). These are versions of Feynman diagrams which explicitly incorporate antisymmetry of the interactions. Internal lines represent Feynman propagators which describe both particles and holes. The arrows represent the flow of quantum numbers such as baryon number. Each topologically distinct diagram should be multiplied by a symmetry factor to take account of the number of ways it can be constructed from the antisymmetric vertices. More details of these diagrams and the rules for evaluating them can be found in the textbooks [16, 17].

The solution of the FG equation is rather straightforward in the case of zero total momentum of the nucleons. For the potential \( V_1 \) it takes the form
\[ T^\text{nm}_1 = \frac{1}{T_1 + \frac{M}{8\pi^2} \left[ \log \frac{p + p_\pi}{p - p_\pi} - 2pp \right]}, \]  
where \( p_\pi \) is the Fermi momentum. In the same way we can solve the FG equation for the potential \( V_2 \). We shall assume that the \( C_2 \) term can be treated as a perturbation. For simplicity we omit the energy-dependent term \( C_2 \) from now on. Although this term makes a physically important contribution to the energy of the two-particle amplitude, it does not take part in the cancellation of off-shell effects which is of interest here. To first order in \( C_2 \) the in-medium \( T \)-matrix can be written
\[ T^\text{nm}_2 = T^\text{nm}_1 - T^\text{nm}_1 \frac{C_2}{C_0} (2p^2 - k^2 - k'^2) - 2(T^\text{nm}_1)^2 \frac{C_2}{C_0} \frac{M}{6\pi^2} p_\pi. \]  

If we now evaluate \( T^\text{nm}_2 \) at the on-shell point, we see that it does not agree with \( T^\text{nm}_1 \) since the last term does not vanish. This indicates that calculations of nuclear matter based on the in-medium \( T \)-matrix (or similarly the \( G \)-matrix) do not satisfy the requirement of reparametrization invariance. Alternatively, in more traditional nuclear physics language, results for in-medium observables depend on the off-shell behaviour assumed for the NN potential. Such a dependence is unphysical and should not be present. A clue to how the dependence may be removed comes from the form of the final term in Eq. (7), which is proportional to the density. Its structure is thus similar to that arising from a three-body contact interaction. This suggests that it may be possible to trade off the off-shell dependence against a three-body force. As shown below, this can be done, provided our approach includes more than just ladder diagrams.
Before exploring what additional physics is needed to remove the off-shell dependence, we should note that there is no clear separation of scales in strongly interacting, dense matter. This is an unsolved problem for the application of EFT’s: no power counting has been found which leads to a consistent expansion. One should really solve the many-body theory for $C_0$ exactly, by constructing the full in-medium NN vertex, $\Gamma$. Nonetheless simpler approximations are commonly used in nuclear physics, typically replacing the full NN vertex by a $G$- or $T$-matrix. Including $ph$ rings as well as $pp$ and $hh$ ladders leads to the parquet approximation [16, 18]. We examine here the consistency of these approximations with reparametrization invariance.

At LO in $C_0$ the contributions to the ground-state energy of matter are shown in Fig. 2, where the solid dot denotes an in-medium NN vertex. If $C_0$ were weak enough we could expand these diagrams perturbatively to get a contribution of order $M C_0 C_2$. The resulting diagrams have a similar structure to those in Fig. 4 below, except that none of the propagators are dressed. As shown in Ref. [11], they can be exactly cancelled against the LO contribution of a contact three-body interaction with strength $D_0 = 12 M C_0 C_2$. This is as required by the equivalence theorem, since the off-shell term and three-body force with this strength are both generated from a Lagrangian which contains neither by the same field redefinition. The details are given in Ref. [11]. For definiteness we repeat the relevant Feynman rules here: the two-body vertices, represented by an open circle and a triangle, respectively, are

\[ -i C_0 S_2 \quad \text{and} \quad i C_2 (\Delta_i + \Delta_j + \Delta_j^*) S_2, \]

where $\Delta_i = M p_i^0 - (p_i)^2 / 2$, $p_i$ being the four-momentum of the $i$th nucleon, and the spin-isospin structure is given by $S_2 = \delta_{ii} \delta_{jj} - \delta_{ij} \delta_{ji}$. The three-body vertex (an open square) is

\[ -i D_3 [\delta_{ii} \delta_{jj} \delta_{kk} - \delta_{ij} \delta_{jk} \delta_{kk}] + \text{cyclic}(i, j, k). \]

When the leading two-body vertex is resummed we get an effective vertex $\Gamma(p_i, p_j; p_k)$ which is denoted by a filled circle [20].

Treating $C_0$ nonperturbatively, the same three-body force gives rise to the diagrams shown in Fig. 3. Each of these diagrams gives a contribution equal to a distinct integral multiplied by $D_0$ and a degeneracy factor. The detailed forms of these integrals, which we denote by $I_1, I_2, \ldots$, are not needed here. However we can use these integrals to classify the structures which arise from the diagrams of Fig. 2. To evaluate them, we note that the off-shell vertex can be written as a sum of four pieces, each of which can cancel a bare propagator on one “leg”:

\[ G_0(q) (M q_0 - q^2 / 2) C_2 = i M C_2, \]

where $G_0(q)$ is the bare single-particle propagator. The diagrams in Fig. 2 give rise to many different contributions, which can be identified by calculating the equations for the in-medium NN vertex and dressed propagator to pull out a bare propagator ending on a lowest-order vertex $C_0$ on any of the lines in the original diagrams. When the bare propagator is cancelled against the off-shell vertex, as in Eq. (10), the result is one of the integrals $I_1$ multiplied by $M C_0 C_2$ and a numerical factor. Thus we can examine the cancellation of off-shell dependence for each integral individually.

We consider first the Brueckner-Hartree-Fock approximation (BHF) [16, 17], in which propagators are dressed and the in-medium NN vertex is obtained by iterating the potential in the $pp$ and $hh$ channels, as shown in Fig. 1. The contributions proportional to $I_0$ from Figs. 2(a) and (b) are shown in Fig. 4. Except for the dressing of the propagators, these have the same structures as the perturbative diagrams considered in Ref. [11] and they can be shown to cancel with Fig. 3(a) in the same way.
FIG. 6: (a) Contribution from Fig. 2(b) proportional to the integral \( I_0 \) in the BHF approximation. (b-c) Extra contribution in the parquet approximation. (d) Diagram containing a non-parquet contribution. Diagrams which can be obtained from those shown by simply reversing all the arrows are not shown separately.

Fig. 2(a) gives one other contribution, shown in Fig. 5(a), which is proportional to \( I_0 \). In the ladder approximation to the NN vertex, Fig. 2(b) also gives contributions proportional to \( I_0 \), shown in Fig. 5(b-c). The sum of Figs. 5(a-c) is \(-2g(g - 1)(2g - 3)MC_{g2} C_{2} I_0\), where \( g \) is the spin-isospin degeneracy factor (\( g = 4 \) for symmetric nuclear matter). In contrast, the three-body force gives \( g(g - 1)(g - 2)D_{0} I_0/2 \) from Fig. 3(b). We see that the degeneracy coefficients do not agree and cancellation does not occur. For example, the off-shell dependence is nonzero for neutron matter (\( g = 2 \)) where the Pauli principle forbids a contact three-body force.

There is one other structure proportional to \( I_0 \), Fig. 5(d). However this cannot be generated from the diagrams of Fig. 2 if the potential is iterated in the pp and \( hh \) channels only; it requires iteration in the \( ph \) channel as well. When this contribution is included, the degeneracy factors agree and the off-shell dependence proportional to \( I_0 \) is indeed cancelled by the three-body force with \( D_{0} = 12MC_{g2} C_{2} \).

The crucial point to note is that the cancellation requires diagrams which can only be obtained by iterating the two-body potential in the \( ph \) channel. These are not contained in the ladder or BHF approximations and so any approach based on a \( G \)- or \( T \)-matrix cannot satisfy the equivalence theorem. Observables calculated in these approaches will have an unphysical off-shell dependence which cannot be absorbed into a three-body force.

The need for diagrams with iteration in the \( ph \) channels suggests that one should try a more complete approach. One such, which treats all two-body channels in a symmetric way, is the parquet approximation [16, 18]. If we interpret the solid circles in Fig. 2 as parquet NN vertices constructed from \( C_{0} \), then all of the contributions in Fig. 5 can be generated by iterating the parquet equations. (Note that the parquet self-energy can still be expressed in the form of Fig. 1(b) [19], and so the discussion of \( I_0 \) above is unchanged.)

Turning now to terms proportional to \( I_2 \), which ought to cancel with the three-body graph Fig. 3(c), we find one self-energy contribution, Fig. 6(a). This would be present even in the BHF approximation for the \( T \)-matrix but, not unexpectedly, this does not provide the cancellation. In the parquet approximation, Fig. 2(b) gives additional contributions, shown in Fig. 6(b,c), and part of (d). Only when the non-parquet contributions are included (see Table 3 of Ref. [18]), i.e., with a full set of diagrams, do we match the result from Fig. 3(c). Thus we conclude that the parquet approximation also violates reparametrization invariance!

It may well be possible to include the necessary structures and extend the parquet approximation along the lines discussed in Ref. [16], starting from a basic vertex which is a sum of diagrams which are two-particle irreducible in all channels. However if, as suggested there, these structures are simply added in perturbatively, they will not generate the full in-medium vertices needed for the diagrams of Fig. 5.

In summary, our results demonstrate that the requirement of reparametrization invariance, which would require the off-shell dependence to be cancelled by a three-body force, is not satisfied by any of the commonly used truncations of the two-body scattering amplitude such as ladder, BHF or parquet approximations. The violations show up in structures with higher numbers of insertions of the in-medium NN vertices for the more sophisticated truncations, but as the interaction is strong this does not provide a consistent expansion scheme. Finding such a scheme remains essential for practical applications of EFT’s to dense, strongly interacting matter.

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[16] J. P. Blaizot and G. Ripka, Quantum theory of finite sys-


[20] Beyond the ladder approximation the NN vertex also has a piece which is symmetric in spin-isospin and antisymmetric in momenta. In the integrals we are considering the rest of the momentum dependence is symmetric, and so the other structure does not contribute.