Improving the Quark Number Susceptibilities for Staggered Fermions

Rajiv V. Gavai

Department of Theoretical Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

Quark number susceptibilities approach their ideal gas limit at sufficiently high temperatures. As in the case of other thermodynamic quantities, this limit itself is altered substantially on lattices with small temporal extent, \( N_t = 4–8 \), making it thus difficult to check the validity of perturbation theory. Unlike other observables, improving susceptibilities or number densities is subject to constraints of current conservation and absence of chemical potential (\( \mu \)) dependent divergences. We construct such an improved number density and susceptibility for staggered fermions and show that they approximate the continuum ideal gas limit better on small temporal lattices.

1. INTRODUCTION

Thermodynamic observables, such as the energy density \( \epsilon \), the pressure \( P \) or the quark number susceptibility \( \chi \), approach the corresponding ideal gas values as the temperature \( T \) (or the chemical potential \( \mu \)) becomes very large. Corrections to the ideal gas behaviour can be computed in a weak coupling approximation. Such a picture of ideal or weakly interacting plasma is often used in most physical applications in heavy ion phenomenology and in cosmology of the Early Universe. Whether the quark-gluon plasma can be described by such a simple picture in the relevant range of temperatures, say \( 1 \leq T/T_c \leq 10 \), is a very important issue for these applications. \( \epsilon \) and \( P \) are known to show strong deviations from the ideal gas picture in this range which cannot be explained by naive perturbation theory up to \( O(g^5) \). Clever resummations of the perturbation series are thought to be a cure. One such approach [1] successfully describes the lattice data for \( P \) above \( \sim 3T_c \). The quark number susceptibilities provide an independent test of any such resummation scheme. Recent lattice results [2] for these exhibit a strong disagreement with both the naive perturbation theory as well as the resummed [3] one. These lattice results were obtained using staggered fermions on lattices with 4, 6 and 8 temporal sites in a temperature range \( 1 \leq T/T_c \leq 3 \). The ratio \( \chi/\chi_{\text{ideal}} \) from these calculations were extrapolated to the continuum to compare with the perturbative results.

As shown in Figure 1, the ideal gas result, \( \chi_{\text{ideal}} \), for the naive staggered fermions varies strongly for \( N_t = 4, 6 \) and 8, and deviates substantially from the continuum result, indicating the need for an improved operator for it in order to confirm the implied non-perturbative nature of the quark-gluon plasma by the results of [2]. Recall that \( \chi \) (or the number density) is obtained...
from \( \ln Z \) by taking second (or first) derivative with respect to \( \mu \). Improving it therefore needs to be done by modifying the number operator \( N \), which in turn is fixed by the (baryon) current conservation equation. Following the study [4] of improvement of the energy and pressure, we employ the Naik action [5] and the P4 action [4] and write down \( N \) for them from the corresponding current conservation equations. An additional complication for nonzero \( \mu \) is the presence of divergences even for the ideal gas. Prescriptions [6] were proposed for the naive action to eliminate them, which were shown [7] to satisfy a general condition. We demonstrate that the same condition can suffice for a class of improved actions as well. Numerical results for the ideal gas are presented to contrast \( \chi(0,T) \), \( n(0,T) \) and \( n(\mu,T) \) with the corresponding continuum results.

2. IMPROVED NUMBER DENSITY

We follow the notation of Ref. [4] for the general form of the fermion actions, which yields the naive, the Naik and the P4 action for their \( c_{1,0}, c_{3,0} \) and \( c_{1,2} \) equal to \((\frac{1}{2}, 0, 0)\), \((\frac{9}{16}, -\frac{1}{8}, 0)\) and \((\frac{1}{8}, 0, \frac{1}{2})\) respectively, where these coefficients multiply the first, third and mixed third derivatives respectively. Setting all the link variables to unity, the current conservation equation for the free theory can be easily shown to be:

\[
\sum_{\mu} \left[ c_{1,0} \Delta_{\mu} j_{\mu}^{(1)} + c_{3,0} \Delta_{\nu j_{\nu}}^{(3)} \right. + c_{1,2} \sum_{\nu \neq \mu} \left( \Delta_{\mu+2\nu} j_{\mu+2\nu}^{(1)} + \Delta_{\mu-2\nu} j_{\mu-2\nu}^{(2)} \right) = 0, \tag{1}
\]

where the currents are defined by

\[
\begin{align*}
  j_{\mu}^{(1)}(x) &= \bar{\psi}_x \gamma_{\mu} \psi_{x+\hat{\mu}} + \bar{\psi}_{x+\hat{\mu}} \gamma_{\mu} \psi_x, \\
  j_{\mu}^{(3)}(x) &= \bar{\psi}_x \gamma_{\mu} \psi_{x+3\hat{\mu}} + \bar{\psi}_{x+3\hat{\mu}} \gamma_{\mu} \psi_x, \\
  j_{\mu}^{(3)}(x) &= \bar{\psi}_x \gamma_{\mu} \psi_{x+\hat{\mu}+2\hat{\nu}} + \bar{\psi}_{x+\hat{\mu}+2\hat{\nu}} \gamma_{\mu} \psi_x, \\
  j_{\mu}^{(2)}(x) &= \bar{\psi}_x \gamma_{\mu} \psi_{x+\hat{\mu}-2\hat{\nu}} + \bar{\psi}_{x+\hat{\mu}-2\hat{\nu}} \gamma_{\mu} \psi_x. \tag{2}
\end{align*}
\]

The \( \Delta \)'s in eq. (1) denote appropriate backward difference operators. Summing eq. (1) over all spatial lattice sites \( \hat{x} \), one obtains \( \sum_{\hat{x}} N(x) = \) constant, where the “number density”

\[
N(x) = c_{1,0} j_{0}^{(1)}(x) + 3 c_{3,0} j_{0}^{(3)}(x) + 2 c_{1,2} \sum_{\mu \neq 0} j_{\mu}^{(1)}(x) - j_{\mu}^{(2)}(x) \] + c_{1,2} \sum_{\mu \neq 0} j_{\mu}^{(1)}(x) + j_{\mu}^{(2)}(x) + \Delta_{\mu} F(x). \tag{3}
\]

Adding the term \( \mu \sum_{x} N(x) \) to the \( \mu = 0 \) action, the last term drops out due to the anti-periodic boundary conditions on the \( \psi \) and \( \bar{\psi} \) in the time direction. Following Ref. [7], the improved action for \( \mu \neq 0 \) can thus be obtained by introducing functions \( f_n(\mu a) = 1 + n \mu a (g_n(\mu a) = 1 - n \mu a) \) for \( n = 1, 2, 3 \), which multiply the terms containing forward (backward) links in the time direction. The coefficients in eq. (3) dictate that \( f_n (\text{or} g_n) \) multiplies an \( n \) time-link term.

Without giving the details here, we assert that a) the improved action with these \( f_n \) and \( g_n \) does lead in general to \( \mu^2 / a^2 (\mu / a) \) divergences in the energy (number) density and b) the ansatz \( f_n = f_n^0 \) with \( g_n = g_n^0 \) with \( f_1(\mu a) = g_1(\mu a) = 1 \), eliminates them. This ansatz is a natural generalization of the results in [7] for the naive action. Thus, as in the naive fermion case [6], \( f_1(\mu a) = \exp(\mu a) \) or \( f_1(\mu a) = (1 + \mu a) / \sqrt{1 - \mu^2 a^2} \) can be used for the improved action in conjunction with the above ansatz for \( f_n \) and \( g_n \).

3. NUMERICAL RESULTS

Figure 1 compares the behaviour of the ideal gas susceptibility at \( \mu = 0 \), \( \chi_{\text{ideal}} \), for the Naik and the P4 action with that for the naive action. The results have been obtained on lattices with an aspect ratio of 3 \( (N_s = 3 N_t) \) and for a fixed quark mass \( m/T \). Since \( f'(0) = -f''(0) = g''(0) = 1 \) for all the allowed \( f \) and \( g \), the results in Figure 1 are valid for all prescriptions of \( f \). One sees that both the improved actions result in a much milder \( N_t \)-dependence in \( \chi_{\text{ideal}} \) and the results are close to continuum by \( N_t = 8 \) already. Interestingly, the improved action results approach the continuum result from the opposite direction as compared to the naive action.

As one switches on \( \mu \), the susceptibilities for the improved actions continue to be close to unity on lattices with \( N_t \geq 8 \) but they do exhibit a mild \( \mu \)-dependence, increasing above unity for \( \mu \sim T \). For small \( \mu \) one also sees a similar pattern in the number density, \( n/T^3 \), as seen in Figure 2: the improved action results are closer to the continu-
uum results than the naive action for \( N_t = 8 \). The results also show a mild \( \mu \)-dependence which may be a cause of concern for the methods which attempt to study the equation of state at \( \mu = 0 \) by using a Taylor expansion in \( \mu \) in the simulations and could result in systematic effects in the predictions for the plasma phase.

Figure 2. Comparison of the continuum number density as a function of \( \mu/T \) with the naive, Naik and P4 improved actions on \( 12^3 \times 4 \) and \( 24^3 \times 8 \) lattices.

Figure 3. Same as Fig. 2 but on \( 24^3 \times 48 \) lattices.

Figure 3 shows the results obtained for the number density of an ideal gas at very small temperatures by taking \( N_t \) very large. On a \( 24^3 \times 48 \) lattice, the number density seems to be essentially the same for small \( \mu \) for all actions, including the naive action. However, none of them has the correct continuum \( \mu \)-dependence. Only when \( \mu/T \) is large do they all approach the continuum. Improved actions seem to be closer to the continuum result for large \( \mu \). Note that even at the highest \( \mu/T \), \( \mu a \) is still only about 0.6 in Fig. 3. The deviations for small \( \mu/T \) are due to the smallness of the spatial volume.

4. SUMMARY

We have constructed an appropriate number density term, \( N \), for a set of improved actions from the current conservation equations for these actions. The Naik action and the P4 action, which correspond to specific choices of values for the action parameters, are included in this set. Adding \( \mu N \) to the action for \( \mu = 0 \), where \( \mu \) is the baryon chemical potential, leads to divergences in the energy density and the number density, as for the unimproved naive action. These can be eliminated by the same condition as for the naive action, namely, \( f(\mu) \cdot g(\mu) = 1 \), provided for each forward (backward) time link the corresponding gauge variable \( U_0(x) \) \((U_0^T(x)) \) is multiplied by \( f(\mu)(g(\mu)) \).

The quark number susceptibility and the number density for the ideal gas do approach the continuum result on smaller temporal lattices for the improved action. It would be interesting to study them in the high temperature regime of QCD. Finite size effects appear to be mildly \( \mu \)-dependent for the susceptibility and the number density which may lead to systematic effects.

REFERENCES