Abstract

I study the weak basis CP-violating invariants in supersymmetric models, in particular those which cannot be expressed in terms of the Jarlskog–type invariants, and find basis–independent conditions for CP conservation. With an example of the \( K - \bar{K} \) mixing, I clarify what are the combinations of supersymmetric parameters which are constrained by experiment.

1 Introduction.

The Standard Model possesses only one CP–odd quantity invariant under a quark basis transformation (apart from \( \bar{\theta}_{QCD} \)), which is known as the Jarlskog invariant \[1\]:

\[
J = \text{Im} \left( \text{Det} \left[ Y_u^a Y_u^{a\dagger}, Y_d^b Y_d^{b\dagger} \right] \right)
\propto (m_t^2 - m_u^2)(m_c^2 - m_u^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2)
\times \text{Im}(V_{11}V_{22}V_{12}^*V_{21}^*) ,
\]

where \( Y_{ij} \) are the Yukawa matrices and \( V_{ij} \) is the CKM matrix. In supersymmetric models, there are many additional sources of CP violation as well as new flavor structures [2]. In this paper, I will concentrate on the quark–squark sector and will ignore leptonic effects for simplicity. Then, the relevant superpotential and the soft SUSY breaking terms are written as follows:

\[
\Delta W = -\hat{H}_2 Y_u^a \bar{Q}_i \hat{U}_j + \hat{H}_1 Y_d^b \bar{Q}_i \hat{D}_j - \mu \hat{H}_1 \hat{H}_2 ,
\]

\[
\Delta V_{\text{s.b.}} = M_{ij}^2 \bar{q}_i \bar{q}_j^* + M_{ij}^2 \bar{u}_i \bar{u}_j^* + M_{ij}^2 \bar{d}_i \bar{d}_j^* + \frac{1}{2} \sum_i M_i \lambda_i \lambda_i .
\]

Clearly, all quantities with flavor indices can contain flavor–dependent and, possibly, flavor–independent CP violating phases. The latter (e.g. overall phases of the A-terms,
\(\mu, B\mu, M_i\) have been studied scrupulously in the past, while the former (e.g. the off-diagonal phases of \(M^2_{qL}\), etc.) have not received as much attention. One of the reasons besides cumbersomeness is that such phases are basis-dependent and thus should be treated with care. In the case of the Standard Model, the flavor-dependent CKM phase can be expressed in terms of the Jarlskog invariant. An important question to address is what is the generalization of the Jarlskog invariant for supersymmetric models and how it is related to the SUSY CP phases.

The class of supersymmetric models under consideration possesses the following symmetries

\[
U(3)_{Q_L} \times U(3)_{\tilde{U}_R} \times U(3)_{\tilde{D}_R}
\]

acting on the quark superfields, which preserve the structure of the supergauge interactions. The CP transformation acts on the Yukawa and mass matrices as the complex conjugation:

\[
M \overset{\text{CP}}{\rightarrow} M^*,
\]

where \(M = \{Y^u, Y^d, M^{2q_L}, M^{2u_R}, M^{2d_R}, A^u, A^d\}\). If this can be “undone” with the symmetry transformation (3), the physical flavor-dependent CP phases vanish. Supersymmetric models also possess the Peccei-Quinn and R symmetries \(U(1)_{\text{PQ}}\) and \(U(1)_{\text{R}}\), which allow us to eliminate two of the flavor-independent phases (see e.g. [3]). Then, in order for CP to be conserved, the invariant CP-phases

\[
\text{Arg} \left[ (B\mu)^* \mu M_i \right], \quad \text{Arg} \left[ A^*_\alpha M_i \right]
\]

have to vanish too. Here \(\text{Arg}[A_\alpha], \alpha = u, d\), denotes the “overall” phases of the A-terms which can be defined in a basis-independent way as

\[
\text{Arg} [A_\alpha] = \frac{1}{3} \text{Arg} \left( \text{Det} \left[ A_\alpha Y^{u\dagger}_\alpha \right] \right),
\]

provided this determinant is non-zero. The flavor-independent phases are not affected by the quark superfield basis transformation and thus are physically meaningful. The discussion of the flavor-dependent phases is much more involved. The main subject of this paper is to find the physical CP-phases, i.e. those which are invariant under phase redefinitions of the quark superfields in analogy with the CKM phase, and the corresponding basis-invariant quantities similar to the Jarlskog invariant.

The paper is organized as follows. In section 2 I build up necessary techniques to handle the issues of CP violation in theories with many flavor structures. In section 3 I apply these methods to the Minimal Supersymmetric Standard Model and provide some examples of how observable quantities can be written in manifestly reparametrization invariant form.

## 2 Auxiliary Construction.

### 2.1 The case of three matrices.

In the Standard Model, the CP-odd invariant is built on the hermitian quantities \(Y^u Y^{u\dagger}\) and \(Y^d Y^{d\dagger}\) which transform in the same way under a basis transformation, i.e. \(Y^u Y^{u\dagger} \rightarrow \)
Suppose, in addition to these, we have another quantity with the same transformation property, for instance, \( M^q_L \). Denoting \( A \equiv Y^u Y^u\dagger \), \( B \equiv Y^d Y^d\dagger \), and \( C \equiv M^q_L \), we have

\[
A \rightarrow U_L A U_L\dagger , \quad B \rightarrow U_L B U_L\dagger , \quad C \rightarrow U_L C U_L\dagger , \quad (7)
\]

where \( U_L \) is a \( U(3) \) quark superfield transformation \( \hat{Q}_L \rightarrow (U_L)^T \hat{Q}_L \) (clearly, these quantities are invariant under the right–handed superfield transformations). What are the invariant CP–violating quantities and the physical CP–phases in this case?

Taking advantage of the unitary symmetry, let us go over to the basis where one of the matrices, say \( A \), is diagonal. In this basis,

\[
A = \begin{pmatrix}
    a_1 & 0 & 0 \\
    0 & a_2 & 0 \\
    0 & 0 & a_3
\end{pmatrix}, \quad B = \begin{pmatrix}
    b_{11} & b_{12} & b_{13} \\
    b_{12}^* & b_{22} & b_{23} \\
    b_{13}^* & b_{23}^* & b_{33}
\end{pmatrix}, \quad C = \begin{pmatrix}
    c_{11} & c_{12} & c_{13} \\
    c_{12}^* & c_{22} & c_{23} \\
    c_{13}^* & c_{23}^* & c_{33}
\end{pmatrix}. \quad (8)
\]

The residual symmetry is associated with the \( U(3) \) generators commuting with the diagonal matrix \( A \). These are two \( SU(3) \) Cartan subalgebra generators and the generator proportional to the unit matrix. This means that \( B \) and \( C \) are defined up to the phase transformation \( B,C \rightarrow U_1 B,C U_1\dagger \) with

\[
U_1 = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}) . \quad (9)
\]

Under this phase transformation the matrix elements transform as

\[
b_{ij} \rightarrow b_{ij} e^{i(\delta_i-\delta_j)} \quad (10)
\]

and similarly for \( c_{ij} \). Physically this freedom corresponds to the arbitrariness in the choice of the quark superfield phases. The quark and squark fields are to be transformed with the same phases in order not to pick up CP phases in the interaction vertices such as \( \bar{q}q \bar{g}g \). In our basis, \( A = \text{diag}(m^2_u, m^2_c, m^2_t)/v_2^2 \), \( B = V\text{diag}(m^2_d, m^2_s, m^2_b) V\dagger /v_1^2 \), where \( V \) is the CKM matrix and \( v_{1,2} \) are the Higgs VEVs. The supergauge vertices are diagonal and the flavor mixing is contained in the propagators (and the non–gauge vertices). The residual rephasing symmetry implies that all physical quantities must be invariant under a phase redefinition of the quark superfields.

If we have only two matrices \( A \) and \( B \), the only reparametrization–invariant CP phase we can construct is the CKM–type phase

\[
\phi_0 = \text{Arg}(b_{12} b_{13}^* b_{23}) . \quad (11)
\]

In the case of three matrices, there are 3 additional non–CKM–type invariant phases

\[
\phi_i = \epsilon_{ijk} \text{Arg}(b_{jk} c_{jk}^*) , \quad (12)
\]

i.e. \( \text{Arg}(b_{12} c_{12}^*) \), etc. In the non–degenerate case, the other physical CP–phases can be expressed in terms of these 4 phases. For instance, the CKM–type phase for the matrix \( C \), \( \phi'_0 \equiv \text{Arg}(c_{12} c_{13}^* c_{23}) \), is given by \( \phi'_0 = \phi_0 - \phi_1 - \phi_2 - \phi_3 \) (yet, this is not true in the
degenerate case, e.g. when some $b_{ij} = 0$). The number of physical phases can also be computed by a simple parameter counting. Three hermitian matrices have nine phases. A $U(3)$ transformation has six phases, of which one leaves all hermitian matrices invariant. Thus the number of non-removable phases is 9−5=4. An interesting feature here is that a new class of reparametrization–invariant CP–phases, not expressible in terms of those of the CKM type, arises.

Consequently, the necessary and sufficient conditions for CP conservation are

$$\phi_0 = \phi'_0 = \phi_i = 0 \mod \pi \quad (i = 1, 2, 3). \quad (13)$$

Having identified a set of the physical CP phases, one may ask what are the weak basis CP–odd invariants associated with such phases. In the case of two matrices, there is only one independent CP–odd invariant which can be written as

$$J_{AB} = \text{ImTr}[A, B]^3. \quad (14)$$

It is proportional to the Jarlskog invariant of Eq.(1) and $\sin \phi_0$. In the case of three matrices with the same transformation properties, one can construct a number of CP–odd invariants such as

$$K_{ABC}(p, q, r) = \text{ImTr}[A^p, B^q]C^r \quad (15)$$

with integer $p, q, r$. This invariant can also be written as $\text{ImTr}A^p[B^q, C^r]$ or an imaginary part of the trace of the completely antisymmetric product of $A^p, B^q,$ and $C^r$. For $p = q = r = 1$ it is proportional to a linear combination of $\sin \phi_i \ (i = 1, 2, 3)$:

$$K_{ABC}(1, 1, 1) = 2(a_1 - a_2)|b_{12}c_{12}| \sin \phi_3 + 2(a_2 - a_3)|b_{23}c_{23}| \sin \phi_1 + 2(a_3 - a_1)|b_{13}c_{13}| \sin \phi_2. \quad (16)$$

It is worth emphasising that the $K$-invariants are entirely new objects which cannot be expressed in terms of the Jarlskog invariants and vice versa. The simplest way to see that is to imagine that $A$ and $B$ (and maybe $C$) have 2 degenerate eigenvalues. Then all Jarlskog invariants $J_{AB}, J_{BC},$ and $J_{CA}$ vanish. Yet, the $K$-invariants can be nonzero. And conversely, suppose that $A$ is proportional to the unit matrix. Then all $K$-invariants vanish, while $J_{BC}$ can be nonzero.

It is instructive to express these invariants in terms of the eigenvalues and the mutual CKM–type matrices. That is, write Eq.(8) as

$$A = \text{diag}(a_1, a_2, a_3), \quad B = V \text{ diag}(b_1, b_2, b_3) V^\dagger, \quad C = U \text{ diag}(c_1, c_2, c_3) U^\dagger. \quad (17)$$

Then,

$$K_{ABC}(p, q, r) = \sum_{ijkl}(a_i^p - a_j^p)b_k^qb_l^rc_i^rV_{ik}V_{jk}^*U_{ij}U_{il}^*. \quad (18)$$

Note the appearance of the rephasing invariant quantities $V_{ik}V_{jk}^*U_{ij}U_{il}^*$ which generalize the CKM–type combination $V_{ik}V_{jk}^*V_{ji}V_{il}^*$. To be exact, there are three independent invariant

*The commutator under the trace can also be raised to an odd power. This will not provide independent invariants and I omit its discussion for brevity.
quantities

\[ \phi_1 = \text{Arg} \left( \sum_i b_i V_{2i} V_{3i}^* \right) \left( \sum_i c_i U_{2i} U_{3i}^* \right)^* , \]

\[ \phi_2 = -\text{Arg} \left( \sum_i b_i V_{1i} V_{3i}^* \right) \left( \sum_i c_i U_{1i} U_{3i}^* \right)^* , \]

\[ \phi_3 = \text{Arg} \left( \sum_i b_i V_{1i} V_{2i}^* \right) \left( \sum_i c_i U_{1i} U_{2i}^* \right)^* , \] (19)

while the other are functions of these and \( \phi_0 \). Indeed, let us consider an example of \( K_{ABC}(1, 2, 1) \). The arising invariant quantity is, e.g.

\[ \text{Im} \left( \sum_i b_i^2 V_{1i} V_{2i}^* \right) \left( \sum_i c_i U_{1i} U_{2i}^* \right)^* . \] (20)

Rewriting

\[ \sum_i b_i^2 V_{1i} V_{2i}^* = \sum_{ijk} (b_i V_{1i} V_{k_i}^*) (b_j V_{k_j} V_{2j}^*) , \] (21)

and extracting terms with different \( k \), Eq.(20) can be brought to the form

\[ \sin \phi_3 \sum_i b_i \left[ |V_{1i}|^2 + |V_{2i}|^2 \right] \left| \sum_i b_i V_{1i} V_{2i}^* \right| \left| \sum_i c_i U_{1i} U_{2i}^* \right| + \]

\[ \sin(\phi_3 - \phi_0) \left| \sum_i b_i V_{1i} V_{3i}^* \right| \left| \sum_i b_i V_{3i} V_{2i}^* \right| \left| \sum_i c_i U_{1i} U_{2i}^* \right| . \] (22)

This can be seen even more easily in terms of the original matrix elements \( b_{ij} \equiv \sum_k b_k V_{ik} V_{jk}^* \) and \( c_{ij} \equiv \sum_k c_k U_{ik} U_{jk}^* \).

We therefore see that, as expected, not all of the \( K \)-invariants are independent. From the above exercise it is clear that, in the non-degenerate case, one can choose

\[ J_{AB} , K_{ABC}(1, 1, 1) , K_{ABC}(2, 1, 1) , K_{ABC}(1, 2, 1) \] (23)

as the basis of the CP–odd invariants. Given the values of these four invariants, the mixing angles, and the eigenvalues, one can solve for the physical CP–phases \( \phi_i \) \((i = 0..3)\). A complication here, compared to the Standard Model, is that the \( K \)-invariants generally are non–trivial functions of these CP–phases. The necessary and sufficient conditions for CP conservation can be expressed as

\[ \text{ImTr}[A, B]^3 = \text{ImTr}[B, C]^3 = \text{ImTr}[C, A]^3 = \text{ImTr}[A^p, B^q] C^r = 0 \] (24)

for any \( p, q, r \).
2.1.1 Degenerate case.

So far we have been assuming that all of the physical phases are non–zero. It is important to find out under which circumstances some of them vanish. To do that, let us go back to Eq.(8). Now, suppose that two eigenvalues of matrix $A$ are degenerate, $a_1 = a_2$. In this case, one of the physical phases will disappear. Indeed, the residual symmetry in this case is $U(2) \times U(1)$. Using this symmetry the upper left 2\times2 block of matrix $B$ can be diagonalized:

$$U_2 \begin{pmatrix} b_{11} & b_{12} \\ b_{12}^* & b_{22} \end{pmatrix} U_2^\dagger \longrightarrow \begin{pmatrix} b_{11}' & 0 \\ 0 & b_{22}' \end{pmatrix},$$

where $U_2$ is a $U(2)$ matrix. As a result, the invariant phase $\phi_3$ vanishes. Of course, the CKM–type phase $\phi_0$ also vanishes, but it can be replaced with an analogous phase built from the matrix elements of $C$: $\phi_0' = \text{Arg}(c_{12}c_{13}^*c_{23})$. Thus, the basis for the remaining physical phases can be chosen as

$$\phi_0', \phi_1, \phi_2.$$

Now suppose that two of the eigenvalues of matrix $B$ are also degenerate. By a unitary (permutational) transformation these can be made $b_1$ and $b_2$. This introduces an additional $U(2)$ symmetry, so the CKM matrix between $A$ and $B$ is now defined only up to a $U(2) \times U(1)$ biunitary transformation, i.e.

$$A = \text{diag}(a, a, a_3) , \quad B = U_2 V \tilde{U}_2^\dagger \text{diag}(b, b, b_3) \tilde{U}_2 V^\dagger U_2^\dagger,$$

where $U_2$ and $\tilde{U}_2$ have a $U(2)$ block in upper left corner and a phase in the (33) position. Since any matrix can be diagonalized by a biunitary transformation, the upper left block of $V$ can be brought to a diagonal form, i.e. $V_{12}' = V_{21}' = 0$. The unitarity of $V'$ then requires $V_{13}' = V_{31}' = 0$ and $|V_{11}'| = 1$ (or $V_{23}' = V_{32}' = 0$ and $|V_{22}'| = 1$),

$$V' = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & V_{22}' & V_{23}' \\ 0 & V_{32}' & V_{33}' \end{pmatrix}.$$

This results in $b_{12} = b_{13} = 0$. We therefore see that the residual symmetry allows us to eliminate two of the physical phases. The remaining CP phases are

$$\phi_0', \phi_1.$$

Finally, if two eigenvalues of $C$ are also degenerate, $\phi_0' = 0$ and the only physical phase is

$$\phi_1.$$

Note that in this case $\text{Tr}[A, B]C \propto \sin \phi_1$.

Now let us briefly discuss the case of three degenerate eigenvalues. If $A \propto I$, the residual symmetry is $U(3)$ which allows us to diagonalize another matrix, say $B$. Then the only invariant phase is

$$\phi_0'.$$
and \( J_{BC} \propto \sin \phi_0 \). It is clear that if either \( B \) or \( C \) have degenerate eigenvalues, no CP violation occurs.

These results can also be obtained by a naive parameter counting. \( U(2) \) has an extra phase parameter compared to \( U(1) \times U(1) \), so enlarging the residual symmetry from \( U(1) \times U(1) \) to \( U(2) \) will reduce the number of physical phases by one. Similarly, \( U(3) \) will allow us to eliminate three more phases compared to \( U(1) \times U(1) \times U(1) \).

The number of the physical phases also reduces if some of the mixings are zero. The mixing angles are defined through the following parametrization of a unitary matrix \( V \) (up to a phase transformation) [4]:

\[
V = \begin{pmatrix}
C_{12}C_{13} & S_{12}C_{13} & S_{13}e^{-i\delta_{13}} \\
-S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\delta_{13}} & C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta_{13}} & S_{23}C_{13} \\
S_{12}S_{23} - C_{12}C_{23}S_{13}e^{i\delta_{13}} & -C_{12}S_{23} - S_{12}C_{23}S_{13}e^{i\delta_{13}} & C_{23}C_{13}
\end{pmatrix},
\]

where \( \delta_{13} \) is a phase; \( S_{ij} = \sin \theta_{ij}, \ C_{ij} = \cos \theta_{ij} \), and \( \theta_{12}, \theta_{13}, \theta_{23} \) are the mixing angles. It is easy to see that one vanishing mixing angle annihilates one phase, say the CKM–type phase \( \phi_0 \). Another vanishing mixing eliminates \( \phi_0 \) which is equivalent to saying \( \phi_1 + \phi_2 + \phi_3 = 0 \).

If both of these mixings are in the same matrix, say \( V \), then this implies that two of the elements \( \{b_{12}, b_{13}, b_{23}\} \) vanish, so again two of the physical phases disappear. Further, the third zero mixing would eliminate another phase. The next step, however, is nontrivial. If one matrix contains three zero mixings and the other – one, then no CP violation is possible. Indeed, this means that two matrices, e.g. \( A \) and \( B \), are diagonalizable simultaneously so that all \( K– \)invariants vanish, whereas the \( J– \)invariants vanish due to a single zero mixing. On the other hand, if \( V \) and \( U \) have two zero mixings each, then CP violation is still possible. As mentioned above, in this case two of \( \{b_{12}, b_{13}, b_{23}\} \) and two of \( \{c_{12}, c_{13}, c_{23}\} \) vanish, so that one can have

\[
A = \begin{pmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{pmatrix}, \quad B = \begin{pmatrix}
b_{11} & b_{12} & 0 \\
b_{12} & b_{22} & 0 \\
0 & 0 & b_{33}
\end{pmatrix}, \quad C = \begin{pmatrix}
c_{11} & c_{12} & 0 \\
c_{12}^* & c_{22} & 0 \\
0 & 0 & c_{33}
\end{pmatrix}.
\]

The surviving invariant phase is \( \phi_3 = \text{Arg}(b_{12}c_{12}^*) \) and \( \text{Tr}[A, B]C \propto \sin \phi_3 \). If the non–vanishing off–diagonal entries are misaligned, CP is conserved. No CP violation can occur if five of the mixing angles are zero.

### 2.2 Generalization to more than three matrices.

Suppose we have \( N \) hermitian matrices \( H_1, H_2, \ldots, H_N \) with the same transformation properties, \( H_i \rightarrow U_L H_i U_L^\dagger \). How can the results of the previous subsection be generalized for this case?

The generalization is quite straightforward. Using the unitary freedom, we bring \( H_1 \) to the diagonal form:

\[
H_1 = \begin{pmatrix}
(H_1)_{11} & 0 & 0 \\
0 & (H_1)_{22} & 0 \\
0 & 0 & (H_1)_{33}
\end{pmatrix}, \quad H_2 = \begin{pmatrix}
(H_2)_{11} & (H_2)_{12} & (H_2)_{13} \\
(H_2)_{21}^* & (H_2)_{22} & (H_2)_{23} \\
(H_2)_{31}^* & (H_2)_{32}^* & (H_2)_{33}
\end{pmatrix}, \ldots
\]
The reparametrization invariant phases can be constructed by taking cyclic products of the elements of the same matrix or by taking products of elements in the same positions in different matrices. In the non-degenerate case, the \(3N - 5\) independent phases can be chosen as

\[
\phi_0 = \text{Arg} \left[ (H_2)_{12} (H_2)_{13}^* (H_2)_{23} \right], \quad \phi_i^a = \epsilon_{ijk} \text{Arg} \left[ (H_2)_{jk} (H_a)_{jk}^* \right], \quad a = 3...N. \tag{35}
\]

Any other physical phase can be expressed in terms of these basis phases. The corresponding weak basis invariants are

\[
J_{H_1 H_2}, \quad K_{H_1 H_2 H_3 (p, q, r), a = 3...N}, \tag{36}
\]

with \((p, q, r) = \{(1, 1, 1); (2, 1, 1); (1, 2, 1)\}\)..

In the degenerate case, the discussion of the previous subsection equally applies. Any additional \(\mathbb{U}(2)\) symmetry, i.e. the presence of two degenerate eigenvalues, eliminates one physical phase which can be taken to be the CKM–type phase for this matrix. An extra \(\mathbb{U}(3)\) eliminates three physical phases, for instance \(\phi_{1,2,3}^a\). A vanishing mixing angle typically entails one vanishing phase, yet there are subtleties discussed above.

The necessary and sufficient conditions for CP conservation can be written as

\[
\text{ImTr}[H_i, H_j] = 0 \tag{37}
\]

for any \(i, j, k\) and \(p, q, r\). Here the square brackets denote antisymmetrization with respect to the indices. These conditions amount to

\[
\text{Arg} \left[ (H_\alpha)_{12} (H_\beta)_{13}^* (H_\alpha)_{23} \right] = \epsilon_{ijk} \text{Arg} \left[ (H_\alpha)_{jk} (H_\beta)_{jk}^* \right] = 0 \mod \pi \tag{38}
\]

for all \(\alpha, \beta, \) and \(i\).

One may wonder whether it is possible to construct CP–odd \(K\)–type invariants with more than three matrices under the trace. This is certainly possible, yet they will be functions of the basic reparametrization invariant phases (35) or, in a more general case, (38), and thus will not provide independent CP violating quantities.

### 3 The Minimal Supersymmetric Standard Model.

The general technology of the previous section can be applied (with some reservations) to the case of the Minimal Supersymmetric Standard Model (MSSM). The MSSM has a number of flavor structures which transform under the \(\mathbb{U}(3)_{Q_L} \times \mathbb{U}(3)_{U_R} \times \mathbb{U}(3)_{D_R}\) symmetry. In particular, the transformation properties are given by

\[
Y^u \rightarrow U_L Y^u U_{u_R}^\dagger, \\
Y^d \rightarrow U_L Y^d U_{d_R}^\dagger, \\
A^u \rightarrow U_L A^u U_{u_R}^\dagger, \\
A^d \rightarrow U_L A^d U_{d_R}^\dagger, \\
M_{2q_L} \rightarrow U_L M_{2q_L} U_{L}^\dagger, \\
M_{2u_R} \rightarrow U_{u_R} M_{2u_R} U_{u_R}^\dagger, \\
M_{2d_R} \rightarrow U_{d_R} M_{2d_R} U_{d_R}^\dagger. \tag{39}
\]
To construct the weak basis invariants, it is necessary to identify hermitian objects which transform under one of the unitary groups. These are given in Table 1. Not all of them are, however, independent. In particular, only three matrices out of

$$A^u A^u\dagger, \ A^u\dagger A^u, \ A^u Y^u\dagger + \text{h.c.}, \ A^{u\dagger} Y^u + \text{h.c.}$$  \hspace{1cm} (40)$$

contain independent phases in the off-diagonal elements. This can be seen as follows. Let us go over to the basis where $Y^u$ is diagonal, $Y^u \rightarrow U_L Y^u U_R^\dagger = \text{diag}(m_u, m_c, m_t)/\sqrt{2}$. Given $A^u A^u\dagger$ and $A^{u\dagger} A^u$, we can find $A^u$ up to a phase transformation. Indeed, $A^u A^u\dagger$ and $A^{u\dagger} A^u$ fix the diagonalization matrices of $A^u$:

$$A^u A^u\dagger \rightarrow \bar{U}_L A^u A^{u\dagger} \bar{U}_L^\dagger = \text{diag}(a_1, a_2^u, a_3^u),$$

$$A^{u\dagger} A^u \rightarrow \bar{U}_{u_R} A^{u\dagger} A^u \bar{U}_{u_R}^\dagger = \text{diag}(a_1, a_2^u, a_3^u),$$  \hspace{1cm} (41)$$

so that $A^u$ is given by

$$A^u = \bar{U}_{u_R}^\dagger \text{diag}(a_1, a_2^u, a_3^u) \bar{U}_L .$$  \hspace{1cm} (42)$$

Note that both $\bar{U}_{u_R}$ and $\bar{U}_L$ are only defined up to a diagonal phase transformation,

$$\bar{U}_{u_R} \sim \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}) \bar{U}_{u_R}, \ \ \bar{U}_L \sim \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}) \bar{U}_L .$$  \hspace{1cm} (43)$$

This introduces a phase ambiguity in the matrix elements of $A^u$. A phase transformation with $\delta_i = \phi_i$ leaves $A^u$ intact, whereas that with $\delta_i = -\phi_i$ changes it. This remaining ambiguity is eliminated by fixing $A^u Y^u\dagger + \text{h.c.}$ such that $A^u$ and $A^{u\dagger} Y^u + \text{h.c.}$ can be determined unambiguously. This can also be understood by parameter counting: $A^u$ has nine phases which, in the non-degenerate case, can be found from the nine phases of the three hermitian matrices $A^u A^{u\dagger}$, $A^{u\dagger} A^u$, and $A^u Y^u\dagger + \text{h.c.}$. Of course, similar considerations apply to $A^{d\dagger} Y^d + \text{h.c.}$.

This argument can be generalized to an arbitrary number of generations $N$. $N^2$ phases of $A^u$ can be found via $N(N-1)$ phases of $A^u A^{u\dagger}$ and $A^{u\dagger} A^u$, and $N$ phases of $A^u Y^u\dagger + \text{h.c.}$. Although $A^u Y^u\dagger + \text{h.c.}$ has $N(N-1)/2$ phases, only $N$ of them are independent and correspond to the residual phase freedom with $\bar{U}_L = \bar{U}_{u_R}^\dagger = \text{diag}(e^{i\delta_1}, ..., e^{i\delta_N})$. This, however, does not work if $N > N(N-1)/2$ in which case $\delta_i$ cannot be determined unambiguously from the off-diagonal phases of $A^u Y^u\dagger + \text{h.c.}$ So, for $N = 1$ and $N = 2$, additional information besides the hermitian quantities is needed which can be, for instance, the anti-hermitian matrix $A^u Y^u\dagger - \text{h.c.}$.

Let us now identify the physical CP phases. First of all, the physical phases must be invariant under the $U(1)_{\text{PQ}}$ and $U(1)_{\text{R}}$ symmetries. By an $R$-rotation, the gluino mass can be made real, so henceforth the phases of the $A$-terms will be assumed to be relative to the gluino phase. Similarly, by a Peccei–Quinn transformation the $B\mu$ term can be made real. Thus we have three CP phases in the flavor independent objects $\mu, M_1$, and $M_2$. The phases of the flavor-dependent objects can be easily counted in the super-CKM basis, i.e. the basis where the quark mass matrices are diagonal. These will include the CKM phase, 18 phases of $A^u$ and $A^{d\dagger}$, and 9 phases of $M^{2u_R}$, $M^{2u_R}$, and $M^{2d_R}$, i.e. 28 phases total. These can be expressed in terms of the reparametrization invariant phases
<table>
<thead>
<tr>
<th>$U(3)\hat{Q}_R$</th>
<th>$U(3)\hat{U}_R$</th>
<th>$U(3)\hat{D}_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y^d Y^u$</td>
<td>$Y^d Y^u$</td>
<td>$Y^d Y^d$</td>
</tr>
<tr>
<td>$Y^d Y^d$</td>
<td>$Y^d Y^d$</td>
<td>$M^2_{u_R}$</td>
</tr>
<tr>
<td>$M^2_{d_L}$</td>
<td>$A^u_{u_R}$</td>
<td>$A^d_{d_R}$</td>
</tr>
<tr>
<td>$A^u A^u$</td>
<td>$A^u Y^u + \text{h.c.}$</td>
<td>$A^d Y^d + \text{h.c.}$</td>
</tr>
<tr>
<td>$A^d A^d$</td>
<td>$A^d Y^d + \text{h.c.}$</td>
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<td>$A^u Y^u + \text{h.c.}$</td>
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<tr>
<td>$A^d Y^d + \text{h.c.}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Hermitian objects of the MSSM transforming under the unitary flavor symmetries.

of the hermitian matrices. The three columns of Table 1 form three separate sequences. Using the unitary freedom, the first matrix in each column can be made diagonal. Then the strategy of section 2 can be applied. Each column has $3N - 5$ physical phases, where $N$ is the number of matrices in the column. However, as I argued above, the two matrices $A^u Y^u + \text{h.c.}$ and $A^d Y^d + \text{h.c.}$ are not independent. In particular, that means that, in the second column, the CKM–type phase for $A^u Y^u + \text{h.c.}$ is not an independent phase, whereas the phase differences of the off–diagonal elements of $A^u Y^u + \text{h.c.}$ and $M^2_{u_R}$, and $A^u Y^u + \text{h.c.}$ and $A^d A^d$ are independent. Therefore, instead of 7 phases in the second column we should only count 6, and similarly for the third column. Thus, we end up with $16+6+6=28$ physical independent phases, as expected.

In the basis where $Y^u Y^u$, $Y^u Y^u$, and $Y^d Y^d$ are diagonal, the 28 independent physical phases can be chosen as follows:

$$
\begin{align*}
\phi_0 &= \text{Arg} \left[ \left( Y^d Y^d \right)_{12} \left( Y^d Y^d \right)_{13} \left( Y^d Y^d \right)_{23} \right], \\
\phi_i^a &= \epsilon_{ijk} \text{Arg} \left[ \left( Y^d Y^d \right)_{jk} \left( F^a \right)_{jk}^* \right], \\
\chi_i^a &= \epsilon_{ijk} \text{Arg} \left[ \left( A^u Y^u + \text{h.c.} \right)_{jk} \left( G^a \right)_{jk}^* \right], \\
\xi_i^a &= \epsilon_{ijk} \text{Arg} \left[ \left( A^d Y^d + \text{h.c.} \right)_{jk} \left( H^a \right)_{jk}^* \right],
\end{align*}
$$

where

$$
\begin{align*}
F^a &= \{ M^2_{u_L}, A^u A^u, A^d A^d, A^u Y^u + \text{h.c.}, A^d Y^d + \text{h.c.} \}, \\
G^a &= \{ M^2_{u_R}, A^d A^u \}, \\
H^a &= \{ M^2_{d_R}, A^d A^d \}.
\end{align*}
$$

Here I have assumed that there are no degenerate eigenvalues and the mixing angles are non–zero. The corresponding weak basis CP–odd invariants are given by Eq.(36).

In the degenerate case, i.e. when some mixing angles are zero and/or there are degenerate eigenvalues, the situation becomes more complicated. In particular, the intrinsically non–hermitian objects such as $A^a$ may have CP–phases which cannot be “picked up” by
the hermitian quantities in the degenerate case. This occurs, for instance, when only a 2×2 block of $A^\alpha$ is non–zero such that we effectively deal with two generations. Another example is the case when $A^u$ and $Y^u$ can be diagonalized simultaneously. Then, in the basis where they are diagonal, the reparametrization invariant CP–phases are

$$\rho_i^u = \text{Arg}(A_{ii}^u Y_{ii}^{u*})$$

and similarly for $A^d$. On the other hand, the hermitian matrices $A^u A^u\dagger, A^d A^d\dagger$ and $A^u Y^{u\dagger} + \text{h.c.}$ are diagonal (and real), so there are no CP phases associated with them. The CP violating invariants corresponding to the phases $\rho_i^u$ are based on the anti–hermitian matrices:

$$L_{A^u Y^u}(p) = \text{ImTr} \left[ (A^u Y^u\dagger)^p - \text{h.c.} \right],$$

where $p$ is an integer. For $p = 1$, this becomes

$$L_{A^u Y^u}(1) = 2 \sum_i |A_{ii}^u m_{ii}^u / v_2 | \sin \rho_i^u.$$

Quantities of the type $\text{ImTr}(A^{\alpha\dagger} Y^{\alpha})^p$ do not provide independent CP–violating invariants. Further, if all $A_{ij}^u$ are non–zero in the basis where $Y^u$ is diagonal, $\rho_i^u$ are not independent and are functions of the phases (44).

The necessary and sufficient conditions for CP conservation are

$$\text{ImTr}[M_i, M_j]^3 = \text{ImTr}M_i^p M_j^q M_k^r = 0,$$

$$\text{ImTr}(A^{\alpha\dagger} Y^{\alpha})^p = 0$$

for any $i, j, k$ and $p, q, r; \alpha = \{u, d\}$. Here $M_i$ are hermitian matrices belonging to the same column of Table 1 and these conditions are to be satisfied for each column. In addition, one, of course, has to require that the gaugino and the $\mu$-term phases vanish. In this case, the full MSSM will conserve CP. If all of the physical phases (44) and (46) are much smaller than one, CP is an approximate symmetry (yet, this is unrealistic as the CKM phase is of order one experimentally). In terms of basis–independent quantities, this implies that, when the mixing angles and the eigenvalues are fixed, the $J$-, $K$-, and $L$-invariants are close to zero (in the appropriate units).

If we are to require that the CKM phase be the only source of CP violation, all invariants apart from $\text{ImTr}[Y^u Y^{u\dagger}, Y^d Y^{d\dagger}]^3$ must vanish. This occurs, for instance, when all SUSY flavor structures are diagonal in the basis where the Yukawa matrices are diagonal and the $A$–terms are real. It is important to note that it is not sufficient to require that the SUSY flavor structures be real in some basis because the presence of CP phases in the Yukawas can make some of the invariants, apart from the Jarlskog one, non–zero. For example, when $Y^u Y^{u\dagger}$ is diagonal, $Y^d Y^{d\dagger} = V \text{diag}(m_d^2, m_s^2, m_b^2) V\dagger / v_1^2$, and $M^{2q_L}$ is real, the invariant phases $\phi_1^d$ and $\text{ImTr}[Y^u Y^{u\dagger}, Y^d Y^{d\dagger}] M^{2q_L}$ are nonzero. This creates certain difficulties for realistic string models with low energy supersymmetry [5]. The reason is that such models generally predict non–universal $A$–terms. Then even if the SUSY breaking $F$–terms are real, the complex phases of the type (46) are generated by the basis transformation which brings the Yukawa couplings to the diagonal form. Equivalently,
this means that some of the $L$– or $K$–invariants are non–zero. As a result, large electric
dipole moments of fermions are induced, in conflict with experiment.

Finally, it should be noted that after the electroweak symmetry breaking a number of
corrections to the Lagrangian (2) will appear. In particular, $M_{2L}$ will split into $M_{2uL}$
and $M_{2dL}$ due to the isospin breaking corrections. Since no additional sources of CP
violation arise in this process, the consequent CP phases will be functions of the original
phases (44) and (46).

3.1 Examples.

As an application of this analysis, let us consider a few examples.

i. Kaon mixing. A first example is a supersymmetric contribution to the $K − \bar{K}$
mixing. This is conveniently expressed in terms of the mass insertions [6]. Suppose that we
work in the super–CKM basis, i.e. the basis where the quark mass matrices are diagonal,
and that the only non–vanishing mass insertion is $(\delta_{LL})_{12} \equiv (M_{2dL})_{12}/\tilde{m}^2$ where $\tilde{m}^2$ is
the average squark mass. Then the gluino–mediated contribution is usually written as [7]

$$(\Delta M_K)_{SUSY} \propto (\delta_{LL})_{12}^2.$$  (50)

Strictly speaking, this cannot be a physically meaningful answer as it is
not rephasing invariant. The super–CKM basis is defined only up to a phase transformation, which
physics should be independent of.

The consistent way to compute the SUSY contribution is as follows. By choosing an
appropriate $U_L$, let us go to the basis where

$$Y_{i}^d Y_{i}^d \text{diag}(m_{i}^2, m_{s}^2, m_{b}^2) / v_1^2, \quad Y_{i}^u Y_{i}^u \text{diag}(m_{u}^2, m_{c}^2, m_{t}^2) V / v_2^2,$$  (51)

where $V$ is the CKM matrix. In this basis $M_{2dL}$ is the same as in the super–CKM basis.
The reparametrization invariant CP phases are the relative phases between the matrix
elements of $Y_{i}^u Y_{i}^u$ and $M_{2dL}$ in the same position. In particular, the relevant to the
$K − \bar{K}$ mixing invariant phase is

$$\delta = \text{Arg} \left( \sum_{i} |m_{i}^u|^2 V_{1i} V_{2i}^* \right) (M_{2dL})_{12}.$$  (52)

Here I have used the absolute values of the masses to stress their invariance with respect
to the phase transformations. Therefore, the proper quantity which should appear in
Eq.(50) is

$$(\delta_{LL})_{12} \longrightarrow |(\delta_{LL})_{12}| e^{i\delta}.$$  (53)

In particular, this implies that the SUSY contribution to the $\varepsilon_K$ parameter vanishes
if the up–type quark masses are degenerate or if the CKM matrix is trivial. In this
(hypothetical) case, the relevant CP phases of $M_{2dL}$ can be transformed away.

The physical phases can be expressed in terms of the basis–independent quantities. In
particular, if $(\delta_{LL})_{12}$ is the only non–zero mass insertion then

$$\text{ImTr}[Y_{i}^d Y_{i}^d, Y_{i}^u Y_{i}^u] M_{2dL} = 2\frac{m_{s}^2 - m_{b}^2}{v_1^2 v_2^2} \sum_{i} |m_{i}^u|^2 V_{1i} V_{2i}^* (M_{2dL})_{12} \sin \delta.$$  (54)
In a more general case, the invariant phases and the magnitudes of the matrix elements of $M^{2dL}$ can be found via 3 CP–violating and 6 CP–conserving weak basis invariants

\[
\begin{align*}
&\text{ImTr}[Y^dY^d, Y^uY^u]M^{2dL}, \quad \text{ImTr}[(Y^dY^d)^2, Y^uY^u]M^{2dL}, \quad \text{ImTr}[Y^dY^d, (Y^uY^u)^2]M^{2dL}, \\
&\text{Tr} Y^uY^u M^{2dL}, \quad \text{Tr} (Y^uY^u)^2 M^{2dL}, \quad \text{Tr} Y^uY^u (M^{2dL})^2, \\
&\text{Tr} Y^dY^d M^{2dL}, \quad \text{Tr} (Y^dY^d)^2 M^{2dL}, \quad \text{Tr} Y^dY^d (M^{2dL})^2.
\end{align*}
\]

(55)

ii. EDMs. Another example is a SUSY contribution to the electric dipole moments of the quarks. For the down quark, the relevant gluino–mediated contribution is typically written as [7]

\[
(d_d)_{\text{SUSY}} \propto \text{Im}(\delta^d_{LR})_{11} M^3,
\]

(56)

with $(\delta^d_{LR})_{11} \sim v_1 A^d_{11}/\tilde{m}^2$ (omitting the $\mu$-term contribution). Again, it is clear that this result is not rephasing invariant.

To rectify this problem, one may use the strategy advocated above. For the purpose of illustration, let us assume the following simple form of $A^d$ in the super–CKM basis:

\[
A^d = \begin{pmatrix} A^d_{11} & A^d_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

(57)

The relevant physical phase can be expressed via the hermitian quantities of Table 1, column 3:

\[
\xi = \text{Arg}(A^d_{11} Y^d + \text{h.c.}) = \text{Arg} A^d_{11} m^*_d.
\]

(58)

Since $m_d$ has the same transformation properties under the left and right rephasings as $A^d_{11}$, this expression is manifestly reparametrization invariant. The corresponding weak basis invariant is $\text{ImTr}[Y^d Y^d, (A^d_{11} Y^d + \text{h.c.})] A^{d*} A^d \propto \sin \phi$. Thus, Eq.(56) is to be modified as

\[
(\delta^d_{LR})_{11} \rightarrow \left| (\delta^d_{LR})_{11} \right| e^{i\xi}.
\]

(59)

Clearly, if $m_d = 0$, the SUSY EDM contribution vanishes.

The invariant phase associated with $A^d_{12}$ can similarly be found from $A^d Y^d + \text{h.c.}$ and $Y^u Y^u$. It is important to note that if $A^d_{12}$ vanishes, $A^d$ and $Y^d$ are diagonal simultaneously, and the phase of $A^d_{11}$ cannot be extracted from hermitian quantities. In this case, the relevant phase is given by Eq.(46) and the corresponding weak basis invariant is $L_{A^d Y^d}(1)$.

4 Summary.

In this work, I have studied the CP–odd weak basis invariants in supersymmetric models and the corresponding reparametrization invariant CP phases. I have shown that, in SUSY models, a new class of CP–odd invariants, not expressible in terms of the Jarlskog–type invariants, appears. I have also obtained basis independent conditions for CP conservation and clarified the issues of rephasing invariance of observable quantities. This work was supported by PPARC.
References


