A modern review of the two-level approximation

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Abstract

The paradigm of the two-level atom is revisited and its perturbative analysis is discussed in view of the principle of duality in perturbation theory. The models we consider are a two-level atom and an ensemble of two-level atoms both interacting with a single radiation mode. The aim is to see how the latter model can be actually used as an amplifier of quantum fluctuations to the classical level through the thermodynamic limit of a very large ensemble of two-level atoms [M. Frasca, Phys. Lett. A 283, 271 (2001)]. The thermodynamic limit can be very effective in generating both classical states and decoherence on a quantum system that evolves without dissipation. Decoherence without dissipation is indeed an effect of a single two-level atom interacting with an ensemble of two-level atoms, a situation that proves to be useful to understand recent experiments on nanoscale devices showing unexpected disappearance of quantum coherence at very low temperatures.

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I. INTRODUCTION

It is safe to say that the foundations of quantum optics are built on the concept of a few level atom. Indeed, the most important concept introduced so far in this field is the two-level atom [1]. A lot of physics can be derived by such an approximation and several recent experiments agree fairly well with a description given by the so-called Jaynes-Cumming model describing a two-level atom interacting with a single radiation mode [2]. Besides, through this understanding of radiation-matter interaction has also been possible to generate Fock states of the radiation field on demand [3].

The radiation-matter interaction currently used is based on some relevant approximations that are still well verified in the current experiments: Firstly, it is assumed that the dipole approximation holds, that is the wavelength of the radiation field being much larger than the atomic dimensions; Secondly, the rotating wave approximation (RWA) is always assumed, meaning by this that just near resonant terms are effective in describing the interaction between radiation and matter, these terms being also described as energy conserving. Indeed, it is sometimes assumed that, without these two approximations no two-level atom approximation can really holds [4]. Actually, in the optical regime, that statement can be supported and widely justifies the success of the Jaynes-Cumming model both theoretically and experimentally.

Actually, things are not so straightforward to describe radiation-matter interaction. Indeed, Cohen-Tannoudji and coworkers were forced to introduce the concept of dressed states for the two-level atom [5] as, in the regime of microwaves, also the RWA fails and a good description of the first experiments in this field were achieved through the concept of dressed states without the RWA [5].

The appearance of laser sources having large intensity has made thinkable the possibility to extend the study of a two-level atom in such fields. Recent studies seem to indicate that such an approximation can give a viable model in such a physical situation [6–12]. In view of this possibility, some methods have been devised recently to approach a solution of the two-level atom in a monochromatic field (being the laser field treated classically) [13–15]. These studies retain just the two-level and dipole approximations but give up the RWA.

In our recent analysis, it was shown that, treating the laser field classically in this situation, leaves out a relevant part of the behavior of the model [16,17]. Particularly, if one is interested in a resonant behavior, it is seen that some Rabi oscillations are neglected: These oscillations have been observed in a recent experiment with Josephson junctions [18] and originate from the formation of bands for the two levels of the atom due to the radiation field [17].

The aim of this paper is to review, using the approach of duality in perturbation theory [19], the consequences of the validity of the two-level approximation relaxing the RWA approximation. Particularly, we will see that
a single radiation mode interacting with a large number of two-level atoms, without the RWA, will provide the amplification of the quantum fluctuations of the ground state of the radiation mode producing a classical radiation field [20]. It is important to point out that this effect arises when the state of the ensemble of two-level atoms is properly prepared and the way of generating classical states by unitary evolution in the thermodynamic limit of Ref. [20] is considered. Non dissipative decoherence can also appear as interaction between an ensemble of two-level systems, evolving to produce a classical state, and a quantum system interacting with it [21]. It should be said that another approach to non dissipative decoherence has been recently proposed by Bonifacio and coworkers [22].

The paper is so structured. In section II we analyze the model from a general perspective deriving the two-level approximation. In section III we present a perturbative analysis of the two-level atom interacting with a single radiation mode, by duality in perturbation theory. In section IV we give a brief excursus of a recent proposal of appearance of classical states and decoherence by unitary evolution in the thermodynamic limit. In section V we show how a strong radiation field can be obtained by strong interaction of a single mode with an ensemble of two-level atoms. Finally, in section VI the conclusions are given.

II. A PARADIGM IN QUANTUM OPTICS: TWO-LEVEL ATOM

In this section we want to derive the two-level approximation on a general footing. So, let us consider a system described by a Hamiltonian $H_0$ such that we have a complete set of eigenstates $H_0|n\rangle = E_n|n\rangle$. We assume, for the sake of simplicity, that the set is discrete. Then, we introduce a time-independent perturbation $V$. By using the identity $I = \sum_n |n\rangle\langle n|$ we can write the Hamiltonian $H = H_0 + V$ as

$$H = \sum_n (E_n + \langle n|V|n\rangle)|n\rangle\langle n| + \sum_{m \neq n} |m\rangle\langle m|V|n\rangle. \quad (1)$$

This Hamiltonian can be rewritten by introducing the operators

$$\sigma_{nm} = |n\rangle\langle m|,$$
$$\sigma_{nm}^\dagger = |m\rangle\langle n|,$$
$$\sigma_{nm}^3 = \frac{1}{2}(|n\rangle\langle n| - |m\rangle\langle m|) \quad (2)$$

and we can build the algebra of the Pauli matrices, currently named su(2), as it is straightforward to verify that

$$\begin{align*} 
[\sigma_{nm}, \sigma_{nm}^\dagger] &= 2i\sigma_{nm}^3 \\
[\sigma_{nm}^3, \sigma_{nm}^\dagger] &= i\sigma_{nm}^1 \\
[\sigma_{nm}^3, \sigma_{nm}] &= -i\sigma_{nm}. \quad (3)
\end{align*}$$

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This permits us to prove that the Hamiltonian we have considered can be rewritten as the sum of two-level Hamiltonians. Indeed, if we change to the interaction picture by the unitary transformation (we use units $\hbar = c = 1$)

$$U_0(t) = \exp \left( -it \sum_n \tilde{E}_n |n\rangle \langle n| \right),$$

being $\tilde{E}_n = E_n + \langle n|V|n\rangle$, and we rewrite the $V$ term of the Hamiltonian as

$$V' = \sum_{m \neq n} |m\rangle \langle n| (m|V|n) = \sum_{m > n} \left[ \langle m|V|n\rangle \sigma^\dagger_{nm} + \langle n|V|m\rangle \sigma_{nm} \right],$$

we get the Hamiltonian

$$H_I = U_0^\dagger(t)V'U_0(t) = \sum_{m > n} \left[ e^{-i(\tilde{E}_m - \tilde{E}_n)t} \langle m|V|n\rangle \sigma^\dagger_{nm} + e^{-i(\tilde{E}_n - \tilde{E}_m)t} \langle n|V|m\rangle \sigma_{nm} \right].$$

that proves our assertion: The time evolution of a quantum system can be described by a Hamiltonian being the sum of $su(2)$ Hamiltonians. The reason why this problem is not generally solvable arises from the fact that, given two $su(2)$ parts $H_{I,i}$ and $H_{I,j}$ of this Hamiltonian, it can happen that $[H_{I,i}, H_{I,j}] \neq 0$ and then the time evolution is not straightforward to obtain analytically. Anyhow, the Hamiltonian $H_I$ can be used to realize some approximate study of a quantum system. Clearly, the simplest way to get an approximate solution is, indeed, the two-level approximation.

The two-level approximation can be easily justified by assuming that only nearest levels of the unperturbed atom really counts in the time evolution, that is, larger is the separation between levels and lesser important is the contribution to the time evolution of the system. This means that terms with the weakest time dependence in $H_I$ are the most important. Mathematically, this means that we assume a solution by a perturbation series and recognize as most relevant the terms where a slower time dependence is present.

We now consider the case of a single radiation mode interacting with a system having Hamiltonian $H_0$. This means in turn that we can choose

$$V = \omega a^\dagger a + e \left( \frac{\omega}{2V} \right)^\frac{1}{2} (a^\dagger + a)x$$

being $e$ the electric charge of the electron assumed as the interacting particle, $a$ and $a^\dagger$ the ladder operators of the radiation mode having frequency $\omega$ and normalization volume $V$, and $x$ the coordinate where the field is oriented, having chosen a linear polarization for it. The dipole approximation is assumed to hold. In this case we have

$$H_I = \omega a^\dagger a + e \left( \frac{\omega}{2V} \right)^\frac{1}{2} \sum_{m > n} \left[ e^{-i(\tilde{E}_m - \tilde{E}_n)t} \langle m|x|n\rangle \sigma^\dagger_{nm} + e^{-i(\tilde{E}_n - \tilde{E}_m)t} \langle n|x|m\rangle \sigma_{nm} \right] (a^\dagger + a)$$

and we are now in a position to obtain the Jaynes-Cummings model. Indeed, we can apply a new unitary transformation to the interaction picture $U_0' = \exp \left( -it\omega a^\dagger a \right)$ to have
Now, on the basis on the two-level approximation given above, we have to conclude that the only terms to retain are the ones that have no time dependence at all, and these are the resonant terms. Here, we recover the rotating wave approximation (RWA). So, if we have two resonant levels \( m = 2 > n = 1 \) and we choose the phases of the eigenstates of the unperturbed system so that \( \langle m|x|n \rangle \) is real, we can finally write the Hamiltonian of the Jaynes-Cummings model as

\[
H_{JC} = g(\sigma_{12}a^\dagger + \sigma_{12}^\dagger a) \tag{10}
\]

being \( g = \langle 2|x|1 \rangle e \left( \frac{\omega}{2V} \right)^{1/2} \) and the resonance condition \( E_2 - E_1 = \omega \). This gives a proper understanding of the success of the two-level atom approximation in quantum optics when weak fields are involved. It is important to note that also a small detuning can be kept, in agreement with the above discussion.

The Jaynes-Cummings model is good until the other terms in the Hamiltonian are truly negligible. The higher order corrections can be computed by a quite general approach as shown in Ref. [23]. These turn out to be corrections to the Hamiltonian at the resonance (e.g. Bloch-Siegert shift and/or a.c. Stark shift) plus the need to add higher orders of the small perturbation theory to the solution. In the optical regime is all negligible.

So, as the small perturbation theory plays a crucial role in this analysis, one may ask what one can say if the perturbation \( V \) becomes strong. Again, by assuming that only a \( \text{su}(2) \) component really contributes to the Hamiltonian (9) we need to treat the Hamiltonian

\[
H'_S = \omega a^\dagger a + g \left[ e^{-i(\tilde{E}_1 - \tilde{E}_2)t}\sigma_{12}^\dagger + e^{-i(\tilde{E}_2 - \tilde{E}_1)t}\sigma_{12} \right] (a^\dagger + a). \tag{11}
\]

By removing the interaction picture transformation, this Hamiltonian can be rewritten as

\[
H_S = \omega a^\dagger a + \frac{\Delta}{2}\sigma_3 + g\sigma_1(a^\dagger + a), \tag{12}
\]

having set \( \sigma_{12} + \sigma_{12}^\dagger = \sigma_1, \sigma_3 = 2\sigma_{12}^3 \) and \( \Delta = E_2 - E_1 \). Neither small perturbation theory nor rotating wave approximation apply. Our aim in the next section is to discuss the perturbative solution of the Schrödinger equation with this Hamiltonian. But, while for the case of the Jaynes-Cummings Hamiltonian we have a fully theoretical justification for our approximation, in the strong coupling regime, the two-level approximation can be satisfactorily justified only by experiment, unless it is exact.

### III. PERTURBATIVE ANALYSIS OF AN INTERACTING TWO-LEVEL ATOM

In this section we will give a brief overview of the perturbative solution for a system described by the Hamiltonian (12) in the strong coupling regime.
This approach has been described in Ref. [17]. To agree about what a strong coupling regime should be, one should properly define the weak coupling regime. Indeed, if one has the Hamiltonian

$$H = H_0 + \tau V$$

being \( \tau \) an ordering parameter, the weak coupling regime is the one when \( \tau \) is very small (\( \tau \to 0 \)), while the strong coupling regime is the one having \( \tau \) very large (\( \tau \to \infty \)). The duality principle in perturbation theory as devised in Ref. [19] permits to do perturbation theory in both the cases, if one is able to find the eigenstates of \( V \), supposing known those of \( H_0 \). Indeed, small perturbation theory by the usual Dyson series gives (we set \( \tau = 1 \) as this parameter is arbitrary)

$$|\psi(t)\rangle = U_0(t)T \exp \left[ -i \int_0^t V_I(t) \right]$$

being \( T \) the time-ordering operator,

$$U_0(t) = \exp (-itH_0)$$

the time evolution of the unperturbed Hamiltonian, and

$$V_I(t) = U_0^\dagger(t)VU_0(t)$$

the transformed perturbation. The choice of a perturbation and an unperturbed part is absolutely arbitrary. So, we can interchange the role of \( H_0 \) and \( V \), obtaining the dual Dyson series

$$|\psi(t)\rangle = U_F(t)T \exp \left[ -i \int_0^t H_{0F}(t) \right]$$

being \( T \) the time-ordering operator,

$$U_F(t) = \exp (-itV)$$

the time evolution of the unperturbed Hamiltonian, and

$$H_{0F}(t) = U_F^\dagger(t)H_0U_F(t)$$

the transformed perturbation. The duality principle states that, when this interchange is done, restating \( \tau \), the series one obtains have the ordering parameters \( \tau \) and \( \frac{1}{\tau} \) respectively. One is the inverse of the other. So, if we have the eigenstates of \( V \) as \( |v_n\rangle \) and eigenvalues \( v_n \), one can write

$$U_F(t) = \sum_n e^{-iv_nt} |v_n\rangle \langle v_n|.$$  

If \( V \) is time dependent one should formally rewrite the above as the adiabatic series introducing the geometric phases of the eigenvectors that now could be time dependent themselves [19].
Coming back to the Hamiltonian (12), we realize that small perturbation theory can be recovered if the unperturbed part is that of the two-level atom, otherwise one has a strong coupling perturbation series with an unperturbed Hamiltonian to be studied given by

\[ V = \omega a^\dagger a + g\sigma_1(a^\dagger + a). \]  

(21)

The dressed states originating by diagonalizing this Hamiltonian are well known [5] and are given by

\[ |v_{n,\lambda}\rangle = |\lambda\rangle e^{\frac{\Delta}{2}\lambda(a-a^\dagger)}|n\rangle \]  

(22)

being \( \sigma_1|\lambda\rangle = \lambda|\lambda\rangle \) with \( \lambda = \pm 1 \) and, \( |n\rangle \) the Fock number states that are displaced by the exponential operator [24]. The eigenvalues are \( E_n = n\omega - \frac{g^2}{\omega} \) and are degenerate with respect to \( \lambda \). So, one has

\[ U_F(t) = \sum_{n,\lambda} e^{-iE_n t}|v_{n,\lambda}\rangle \langle v_{n,\lambda}| \]  

(23)

and the transformed Hamiltonian becomes

\[ H_{0F} = U_F^\dagger(t)\sigma_3 U_F(t) = H'_0 + H_1. \]  

(24)

Using the relation [24]

\[ \langle l|e^{\frac{\Delta}{2}\lambda(a-a^\dagger)}|n\rangle = \sqrt{\frac{n!}{l!}} \left( \frac{\lambda g}{\omega} \right)^{l-n} e^{-\lambda^2 \frac{g^2}{\omega^2}} L_n^{(l-n)} \left( \frac{\lambda^2 g^2}{\omega^2} \right) \]  

(25)

with \( l \geq n \) and \( L_n^{(l-n)}(x) \) the associated Laguerre polynomial, one gets

\[ H'_0 = \frac{\Delta}{2} \sum_n e^{-\frac{2g^2}{\omega^2}} L_n \left( \frac{4g^2}{\omega^2} \right) \left[ ||n; \alpha_1|| \langle n; \alpha_1 | 1 \rangle + \langle n; \alpha_1 | n; \alpha_1 \rangle \right] \]  

(26)

being \( L_n \) the n-th Laguerre polynomial, \( ||n; \alpha_1|| = e^{\frac{\Delta}{2}\lambda(a-a^\dagger)}|n\rangle \), and

\[ H_1 = \frac{\Delta}{2} \sum_{m,n,m\neq n} e^{-i(n-m)\omega t} \left[ \langle n|e^{-\frac{2g^2}{\omega^2}(a-a^\dagger)}|m\rangle \langle n; \alpha_1 || m; \alpha_1 \rangle \langle m; \alpha_1 | 1 \rangle - \langle m; \alpha_1 | m; \alpha_1 \rangle \langle n; \alpha_1 | 1 \rangle \right]. \]  

(27)

The Hamiltonian \( H'_0 \) can be straightforwardly diagonalized with the eigenstates

\[ |\psi_n; \sigma\rangle = \frac{1}{\sqrt{2}} \left[ \sigma |n; \alpha_1\rangle |1\rangle + |n; \alpha_{-1}\rangle |1\rangle - 1 \right] \]  

(28)

and eigenvalues

\[ E_{n,\sigma} = \frac{\sigma \Delta}{2} e^{-\frac{2g^2}{\omega^2}} L_n \left( \frac{4g^2}{\omega^2} \right) \]  

(29)
being $\sigma = \pm 1$. We see that two bands of levels are formed and two kind of transitions are possible: interband (between levels of the two bands) and intraband (between the levels of a band). This cannot happen if we consider the radiation mode being classical as, the intraband transitions would be neglected. So, looking for a solution in the form

$$|\psi_F(t)\rangle = \sum_{\sigma,n} e^{-iE_{n,\sigma}t} a_{n,\sigma}(t)|\psi_n;\sigma\rangle$$

one gets the equations for the amplitudes [17]

$$i\dot{a}_{m,\sigma}(t) = \frac{\Delta}{2} \sum_{n \neq m, \sigma} a_{n,\sigma}(t) e^{-i(E_{n,\sigma} - E_{m,\sigma})t} e^{-i(m-n)\omega t} \left[ \langle m|e^{-\frac{2g}{\hbar}(a-a^\dagger)}|n\rangle \frac{\sigma'}{2} + \langle m|e^{\frac{2g}{\hbar}(a-a^\dagger)}|n\rangle \frac{\sigma}{2} \right].$$

(31)

This equations can also display Rabi oscillations between the eigenstates $|\psi_n;\sigma\rangle$ that can be seen as macroscopic quantum superposition states, both for interband and intraband transitions [17]. States like these could prove useful for quantum computation. These kind of Rabi oscillations in Josephson junctions have been recently observed [18].

At this stage it is very easy to do perturbation theory in the strong coupling regime for this model. One has to rewrite the initial condition $|\psi(0)\rangle$ by the eigenstates $|\psi_n;\sigma\rangle$ obtaining in this way the amplitudes $a_{n,\sigma}(0)$. Then, one has to solve eq.(31) perturbatively as done routinely in the weak coupling regime. In case of a resonance one should apply the RWA obtaining Rabi oscillations. In this way we see that the dual Dyson series, as it should be expected, displays all the features of the standard weak coupling expansion.

IV. CLASSICAL STATES AND DECOHERENCE BY UNITARY EVOLUTION

An ensemble of independent two-level systems can behave classically. This has been proven in Ref. [20]. Indeed, let us consider a Hamiltonian

$$H_c = \frac{\Delta}{2} \sum_{i=1}^{N} \sigma_{3i}.$$

(32)

Assuming distinguishable systems, we can take for the initial state the one given by

$$|\psi(0)\rangle = \prod_{i=1}^{N} (\alpha_i |\downarrow_i\rangle + \beta_i |\uparrow_i\rangle)$$

(33)

with $|\alpha_i|^2 + |\beta_i|^2 = 1$ and $\sigma_{3i} |\downarrow_i\rangle = -|\downarrow_i\rangle$ and $\sigma_{3i} |\uparrow_i\rangle = |\uparrow_i\rangle$. The time evolution gives us

$$|\psi(t)\rangle = \prod_{i=1}^{N} (\alpha_i e^{i\frac{\Delta}{\hbar}t} |\downarrow_i\rangle + \beta_i e^{-i\frac{\Delta}{\hbar}t} |\uparrow_i\rangle).$$

(34)
For the Hamiltonian is easy to verify that

\[
\langle H_c \rangle = \langle \psi(0) | H_c | \psi(0) \rangle = \frac{\Delta}{2} \sum_{i=1}^{N} (|\beta_i|^2 - |\alpha_i|^2) = \frac{\Delta}{2} k_H N
\]  

(35)

being \(k_H\) a fixed number between \(-1\) and \(1\). So, in a similar way, it easy to obtain the fluctuation

\[
(\Delta H_c)^2 = \langle \psi(0) | H_c^2 | \psi(0) \rangle - \langle \psi(0) | H_c | \psi(0) \rangle^2 = \Delta^2 k'_H N.
\]  

(36)

being \(k'_H = \sum_{i=1}^{N} |\beta_i|^2 (1 - |\beta_i|^2)/N\) and it is easy to see that \(k'_H\) is a finite number independent on \(N\). So, as it happens in statistical thermodynamics, in the thermodynamic limit \(N \to \infty\) we see that quantum fluctuations are not essential, that is

\[
\frac{\Delta H_c}{H_c} \propto \frac{1}{\sqrt{N}}.
\]  

(37)

The “laws of thermodynamics” are obtained by the Ehrenfest’s theorem and are the classical equations of motion. That is, the variables \(\Sigma_x = \sum_{i=1}^{N} \sigma_{xi}\), \(\Sigma_y = \sum_{i=1}^{N} \sigma_{yi}\) and \(\Sigma_z = \sum_{i=1}^{N} \sigma_{zi}\) follow, without any significant deviation, the classical equations of motion, when the thermodynamic limit is considered and the time evolution is computed averaging with the above \(|\psi(t)\rangle\). So, we have found a classical object out of the quantum unitary evolution. The main point here is that classical objects can be obtained by unitary evolution in the thermodynamic limit depending on their initial states. Actually, one cannot apply the above argument if e.g. the state of the system is an eigenstate of the Hamiltonian \(H_c\). Besides, having a classical state obtained by unitary evolution, \(per se\), does not produce decoherence. Rather, it is interesting to see what happens when such a classical state interacts with some quantum system. This is a relevant problem that can prove quantum mechanics and its fluctuations to be just the bootstrap of a classical world: If by unitary evolution, in the thermodynamic limit, some classical objects are obtained and these are permitted to interact with other quantum objects, the latter can decohere or become classical by themselves.

As a relevant example, let us consider the interaction of the above system with a two-level atom. This model has been considered in Ref. [21] as a possible explanation of recent findings in some experiments with nanoscale devices that show unexpected decoherence in the low temperature limit [25,26]. The Hamiltonian can be written as

\[
H_D = \frac{\Omega_0}{2} \sigma_z \sigma_z + \frac{1}{2} \sum_{i=1}^{N} (\Delta_{xi} \sigma_{xi} + \Delta_{zi} \sigma_{zi}) - J \sigma_x \cdot \sum_{i=1}^{N} \sigma_{xi}.
\]  

(38)

where \(J\) is the coupling. The Hamiltonian of the two-level systems (second term in eq.(38)) is taken not diagonalized, but this does not change our argument as the above analysis still applies. Finally, \(\Omega_0\) is the parameter of the Hamiltonian of the two-level atom that we want to study. We need another
hypothesis to go on, that is we assume that the coupling $J$ is larger than any of the parameters of the two-level systems $\Delta_{x_i}, \Delta_{z_i}$, but not with respect to $\Omega_0$. By applying duality in perturbation theory, we have the leading order solution

$$|\psi(t)\rangle \approx \exp \left( -it\frac{\Omega_0}{2}\sigma_z + iJt\sigma_x \cdot \sum_{i=1}^{N} \sigma_{x,i} \right) |\psi(0)\rangle. \quad (39)$$

Now, as already seen, we have to choose the state of the two-level systems as given by the product of the lower eigenstates of each $\sigma_{x,i}$. This can be seen as a kind of “ferromagnetic” state and is in agreement with our preceding discussion. So, we take

$$|\psi(0)\rangle = |\downarrow\rangle \prod_{i=1}^{N} |\!-\!1\rangle_i \quad (40)$$

being $\sigma_z |\downarrow\rangle = -|\downarrow\rangle$ and, similarly, $\sigma_z |\uparrow\rangle = |\uparrow\rangle$. The state of the ensemble of two-level systems agrees fairly well with the one of eq.(33). So, one has, by tracing away the state of the ensemble of two-level systems being not essential for our aims,

$$|\psi'(t)\rangle \approx \exp \left( -it\frac{\Omega_0}{2}\sigma_z + iJNt\sigma_x \right) |\downarrow\rangle \quad (41)$$

that defines a spin coherent state [27,28]. The point we are interested in is the thermodynamic limit. When $N$ is taken to be large enough, the contribution $\Omega_0$ can be neglected and we have a reduced density matrix

$$\rho'(t) = \exp (iJNt\sigma_x) |\downarrow\rangle \langle\downarrow| \exp (-iJNt\sigma_x) \quad (42)$$

giving rise to

$$\rho'_{\uparrow\uparrow}(t) = \frac{1 - \cos(2Njt)}{2} \quad (43)$$
$$\rho'_{\uparrow\downarrow}(t) = -\frac{1}{2} \sin 2Njt \quad (44)$$
$$\rho'_{\downarrow\uparrow}(t) = \frac{1}{2} \sin 2Njt \quad (45)$$
$$\rho'_{\downarrow\downarrow}(t) = \frac{1 + \cos(2Njt)}{2} \quad (46)$$

where we have oscillating terms with a frequency $NJ$ that goes to infinity in the thermodynamic limit. The only meaning one can attach to such a frequency is by an average in time (see [21] and Refs. therein) and decoherence is recovered. So, when the ensemble of two-level systems strongly interacts with a quantum system produces decoherence and quantum behavior disappears, in the thermodynamic limit. The ensemble of two-level systems should evolve unitarily, producing a classical behavior. Higher order corrections have also been studied in Ref. [21]. It is important to stress that this behavior should be expected at zero temperature as quantum coherence is lost otherwise.
Such a behavior, having a characteristic decoherence time scale depending on the number of two-level systems that interact with the quantum one, has been recently observed in quantum dots [26]. In this case, the ensemble of two-level systems can be given by the spins of the electrons that are contained in the two dimensional electron gas in the dot. Another source of decoherence in quantum dots could be given by the spins of the nuclei interacting through an hyperfine interaction with the spin of the conduction electrons [31]. The nuclei are contained in the heterostructures forming the dot. In this case we have a similar spin-spin interaction but isotropic. The mechanism that produces the decay of the off-diagonal parts of the density matrix, also in this case, appears to be the same, being the decoherence produced dynamically and dependent on the initial state.

V. AMPLIFICATION OF QUANTUM FLUCTUATIONS TO THE CLASSICAL LEVEL

Spontaneous emission can be seen as a very simple example of decoherence in the “thermodynamic limit” of the number of radiation modes. Indeed, we can consider a two-level system interacting with \( N \) radiation modes and being resonant with one of it. In the limit of a small coupling between radiation and two-level system and very few spectator modes, one has Rabi oscillations, a clear example of quantum coherence. When the number of spectator modes is taken to go to infinity, a description with continuum is possible and this gives rise to decay, i.e. spontaneous emission. This representation of the process of decay is very well described in Ref. [5].

Here, we want to consider the opposite situation, that is, a single radiation mode strongly interacting with an ensemble of \( N \) two-level systems. We are going to show that, when the ensemble of two-level systems behaves as a classical object if left alone, the radiation field, supposed initially in the ground state, will have the zero point fluctuations amplified to produce a classical field having intensity dependent on \( N \), the number of two-level systems.

As done in Ref. [20], we modify the model of eq. (12) to consider \( N \) two-level systems interacting with a single radiation mode, as

\[
H_S = \omega a^\dagger a + \frac{\Delta}{2} \sum_{i=1}^{N} \sigma_{3i} + g \sum_{i=1}^{N} \sigma_{1i}(a^\dagger + a). \tag{47}
\]

Then, the strong coupling regime amounts to consider the Hamiltonian \( \frac{\Delta}{2} \sum_{i=1}^{N} \sigma_{3i} \) as a perturbation, as already done in sec.III for a single two-level atom. We take as initial state of the full system \( |\psi(0)\rangle = |0\rangle \prod_{i=1}^{N} | - 1 \rangle_i \), so that, the ensemble of two-level systems is again in a kind of “ferromagnetic” state representing its ground state. Besides, no photon is initially present. It is a well known matter that the fluctuations of the radiation mode are not zero in this case. The unitary evolution at the leading order gives us

\[
|\psi(t)\rangle \approx \exp \left[ -it\omega a^\dagger a - itg \sum_{i=1}^{N} \sigma_{1i}(a^\dagger + a) \right] |0\rangle \prod_{i=1}^{N} | - 1 \rangle_i \tag{48}
\]
that, by use of a known disentangling formula [28], produces

$$|\psi(t)\rangle = e^{i\xi(t)}e^{-i\omega a^\dagger a}D[\alpha(t)]|\psi(0)\rangle,$$

being

$$\xi(t) = \frac{N^2g^2}{\omega^2}(\omega t - \sin(\omega t)),$$

$$\alpha(t) = -\frac{Ng}{\omega}(1 - e^{i\omega t}),$$

and

$$D[\alpha(t)] = \exp[\alpha(t)a^\dagger - \alpha^*(t)a].$$

We conclude that, at the leading order, the radiation mode evolves as a coherent state with a parameter given by

$$\hat{\alpha}(t) = -\frac{Ng}{\omega}(e^{-i\omega t} - 1) = \alpha(t)e^{-i\omega t}.$$

In this way, we have amplified the quantum fluctuations of the field, being the fluctuation of the number of photons proportional to $N$, but, as the average of the number of photons is proportional to $N^2$, this ratio goes to zero as the thermodynamic limit $N \to \infty$ is taken. As it is well known [29], this produces a classical field with increasing intensity as the number of two-level systems increases, proving our initial assertion. We can see that the amplification of quantum fluctuations gives rise to a classical object, as initially no radiation field is present.

Higher order corrections have been studied in Ref. [30], showing that are not essential in the thermodynamic limit. So, this effect will prove to be a genuine example of production of a classical object by unitary evolution in the thermodynamic limit with possible technological applications.

VI. DISCUSSION AND CONCLUSIONS

We have presented a brief review about some new views on two-level systems. These appear to be even more important today with a lot of new effects to be described and experimentally observed. Paramount importance is acquiring the decoherence due to an ensemble of two-level systems as it is becoming ubiquitous to different fields of application as quantum computation and nanotechnology, fields that maybe could merge. To face these new ways to see the two-level approximation, we have exposed new mathematical approaches to analyze models in the strong coupling regime. This regime has been pioneered by Bender and coworkers [32] in quantum field theory in the eighties, but, with our proposal of duality in perturbation theory, a possible spreading of such ideas to other fields is now become possible. Indeed, a lot
of useful results, as those presented here, are obtained by this new approach and, hopefully, the future should deserve some other interesting results.

The idea of a non dissipative decoherence is also relevant due to the recent findings in the field of nanodevices, where unexpected lost of quantum coherence has appeared in experiments performed at very low temperatures. In these cases, it appears as the standard idea of decoherence, meant as interaction of a quantum system with an external environment, seems at odds with some experimental results, even if an interesting proposal through the use of quantum fluctuations has been put forward by Büttiker and coworkers [33].

The conclusion to be drawn is that, today, a lot of exciting work at the foundations of quantum mechanics is expecting us, giving insight toward new understandings and methods and, not less important, applications.
REFERENCES


