Induced quantum gravity on a Riemann Surface

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Abstract

Induced quantum gravity dynamics built over a Riemann surface is studied in arbitrary dimension. Local coordinates on the target space are given by means of the Laguerre-Forsyth construction. A simple model is proposed and pertubatively quantized. In doing so, the classical \(W\)-symmetry turns out to be preserved on-shell at any order of the \(\hbar\) perturbative expansion. As a main result, due to quantum corrections, the target coordinates acquire a non-trivial character.

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1 Introduction

The relationship between quantum world and gravitational force is nowadays one of the important issues in theoretical physics, and from several decades, it has been widely debated in the community. Indeed, in the classical theory, the dynamics of gravitation is strictly linked with space-time attributes only, so that any quantum extension should investigate the interactions between matter fields and space-time which ought to arise at very small length scales [1, 2, 3].

A local field theory describes the dynamics of physical phenomena with the help of a local coordinate system [4, 5]; as is well-known a physical theory turns out to be valid if any choice of the coordinate system leaves unaffected the physics. For instance string and (Mem)brane theories greatly take advantage of these aspects. But, even if these facts are widely exploited from the mathematical point of view, some new requirements (as non-commutativity) raise new questions on the role of the background with respect to the physics investigations.

In particular the physical locality requirements must be re-examined at the light of the need that any theory must also be globally defined. We here re-propose the well-known Laguerre-Forsyth construction for the coordinates [6], already used [7, 8, 9, 10, 11, 12] in bidimensional conformal field models. This scheme, at the same time, exploits the benefits of two-dimensional complex space, and describes manifolds of arbitrary dimensions, by preserving some prescriptions required both by Mathematics and Physics, that is, each quantity must be well defined from the geometrical point of view.

Then we avail of this construction to build a simple physical model in a Lagrangian framework, and to extend it at the Quantum level.

The Laguerre-Forsyth construction stems from the remark that the $s$ linearly independent solutions of the $s$-th order globally defined linear differential equations on a Riemann surface [13] endowed with an complex analytic atlas with local coordinates $(z, \overline{z})$,

$$L_s f_i (z, \overline{z}) \equiv \sum_{j=0}^{s} a_{ij}^{(s)} (z, \overline{z}) \partial^{(s-j)} f_i (z, \overline{z}) = 0, \quad i = 1, \ldots, s$$  \hspace{1cm} (1.1)

with the normalization choice

$$a_{(0)}^{(s)} (z, \overline{z}) = 1, \ \ \ \ \ (1.2)$$

make sense only if they are scalar densities of conformal weight $\frac{1-s}{2}$. The space of such densities will be denoted by $V_{\frac{1-s}{2}}$. So it is possible to construct $(s-1)$ independent scalar objects (for example $f_i (z, \overline{z})$) which define an $(s-1)$ dimensional space. More generally the operator $L_s$ maps scalar densities of conformal weight $\frac{1-s}{2}$ into those of weight $\frac{1+s}{2}$. Accordingly if

$$g(z, \overline{z}) \in V_{\frac{1-s}{2}}, \ \ \text{and} \ \ f(z, \overline{z}), f_i (z, \overline{z}) \in V_{\frac{1+s}{2}}$$  \hspace{1cm} (1.3)
the image of the differential operator $L_s$ can be constructed by means of a given basis of solutions of (1.1) by writing

$$g(z, \overline{z}) = L_s(z, \overline{z}) f(z, \overline{z}) = \begin{vmatrix} f(z, \overline{z}) & f_1(z, \overline{z}) & \cdots & f_s(z, \overline{z}) \\ \partial f(z, \overline{z}) & \partial f_1(z, \overline{z}) & \cdots & \partial f_s(z, \overline{z}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{(s-1)} f(z, \overline{z}) & \partial^{(s-1)} f_1(z, \overline{z}) & \cdots & \partial^{(s-1)} f_s(z, \overline{z}) \\ \partial f(z, \overline{z}) & \partial f_1(z, \overline{z}) & \cdots & \partial f_s(z, \overline{z}) \end{vmatrix}$$ (1.4)

where the coefficients $a_j^{(s)}(z, \overline{z})$ can be evaluated from the minors of the previous matrix Eq(1.4) [7]

$$a_j^{(s)}(z, \overline{z}) = \begin{vmatrix} \partial^{(s)} f(z, \overline{z}) & \partial^{(s)} f_1(z, \overline{z}) & \cdots & \partial^{(s)} f_s(z, \overline{z}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{(j+1)} f(z, \overline{z}) & \partial^{(j+1)} f_1(z, \overline{z}) & \cdots & \partial^{(j+1)} f_s(z, \overline{z}) \\ \partial f(z, \overline{z}) & \partial f_1(z, \overline{z}) & \cdots & \partial f_s(z, \overline{z}) \end{vmatrix}$$ (1.5)

Introducing now a $s \times s$-matrix $F(z, \overline{z})$ defined as

$$F(z, \overline{z}) \equiv \begin{pmatrix} f_1(z, \overline{z}) & \cdots & f_s(z, \overline{z}) \\ \partial f_1(z, \overline{z}) & \cdots & \partial f_s(z, \overline{z}) \\ \vdots & \ddots & \vdots \\ \partial^{(s-1)} f_1(z, \overline{z}) & \cdots & \partial^{(s-1)} f_s(z, \overline{z}) \end{pmatrix}$$ (1.6)

the normalization condition (1.2) is simply written as

$$\det F = 1$$ (1.7)

Now if the $s$ linearly independent solutions are constructed as

$$f_j(z, \overline{z}) = Z^{(j-1)}(z, \overline{z}) \omega^{-\frac{1}{2}}(z, \overline{z}), \quad \text{for} \quad j = 1, \ldots, s$$ (1.8)

with $Z^{(0)}(z, \overline{z}) = 1$, and $\omega(z, \overline{z})$ the Wronskian

$$\omega(z, \overline{z}) = \begin{vmatrix} \partial Z^{(1)}(z, \overline{z}) & \cdots & \partial Z^{(s-1)}(z, \overline{z}) \\ \vdots & \ddots & \vdots \\ \partial^{(s-1)} Z^{(1)}(z, \overline{z}) & \cdots & \partial^{(s-1)} Z^{(s-1)}(z, \overline{z}) \end{vmatrix}$$ (1.9)

in order to preserve the normalization (1.2). Note that the so introduced $s-1$ independent (Laguerre-Forsyth) coordinates $Z^{(r)}(z, \overline{z})$ specify, for each point of the two-dimensional ($z, \overline{z}$)
space, a \((s - 1)\)-uple of local complex coordinates in some complex target manifold of \(s - 1\) complex dimensions. This definition assigns to the coordinates a scalar field label; the role of the two-dimensional \((z, \bar{z})\) background must be exploited both from the mathematical and (mostly) the physical point of view. Due to (1.5) the coefficients \(a^{(s)}_{j}(z, \bar{z})\) can be written [15, 16] as well-defined functions of the \(Z^{(r)}(z, \bar{z})\) functions and their \(z\)-derivatives and as shown before the solutions \(f_{i}(z, \bar{z})\) as well.

Consider now, for a fixed point on the Riemann surface with local complex coordinates \((z, \bar{z})\), a change of local complex coordinates in the target manifold

\[
\left( Z^{(1)}(z, \bar{z}), \ldots, Z^{(s-1)}(z, \bar{z}) \right) \rightarrow \left( Z'^{(1)}(z, \bar{z}), \ldots, Z'^{(s-1)}(z, \bar{z}) \right) \tag{1.10}
\]

with the primed analogues of Eq (1.8). This gives rise to new solutions \(f'_{i}(z, \bar{z})\) of an analogous differential equation (1.1) in which the coefficients have been transformed but \(a_{0} = 1\) and \(a_{1} = 0\) are kept fixed. Accordingly this allows one to construct a new matrix \(F'(z, \bar{z})\) whose structure repeats (1.6) and (1.7).

Therefore the change of local complex coordinates on the target manifold induces a \(\text{SL}(s)\) coordinate mapping:

\[
F'(z, \bar{z}) = M(z, \bar{z})F(z, \bar{z}) \tag{1.11}
\]

with

\[
\det M(z, \bar{z}) = 1. \tag{1.12}
\]

In terms of the coordinates \(Z^{(j)}\) the mapping Eq (1.11) explicitly writes

\[
Z'^{(j)}(z, \bar{z}) = Z^{(j)}(z, \bar{z}) + \sum_{r=1}^{s} M_{1,r}(z, \bar{z}) \left( \sum_{m=1}^{s} \binom{r-1}{m} \frac{\partial^{m} Z^{(j)}(z, \bar{z})}{\partial^{r-1-m} \omega^{-1/s}(z, \bar{z})} \right) \sum_{\ell=1}^{s} M_{1,\ell}(z, \bar{z}) \partial^{\ell-1} \omega^{-1/s}(z, \bar{z}) \tag{1.13}
\]

This mapping is performed, as said before, for a fixed (but arbitrary) point with local complex coordinates \((z, \bar{z})\); for this reason we must extend the former to the whole Riemann surface endowed with a background complex structure defined by the system of local complex analytic coordinates \((z, \bar{z})\).

To sum up these conclusions we formulate the following:

**Statement 1.1** The Laguerre-Forsyth construction induces a mapping from each point of the two-dimensional \((z, \bar{z})\) plane to the \((s-1)\)-dimensional target space \(\left( Z^{(1)}(z, \bar{z}), \ldots, Z^{(s-1)}(z, \bar{z}) \right)\). The physical requirement for the equivalence of all the patching in the \((s-1)\)-dimensional target space...
space requires the double exigency that they must be independent on the choice of the point of the 
\((z, \overline{z})\) plane from which we start the construction, and each reparametrization as in Eq (1.13) 
will describe the symmetry principle of each dynamical model embedded in these spaces. As will 
be seen in the following, its infinitesimal counterpart generates a \(\mathcal{W}_s\)-symmetry.

Going on, let us introduce now the Maurer-Cartan form whose matrix valued components are

\[
\mathcal{A}_z(z, \overline{z}) = \partial \mathbf{F}(z, \overline{z})(\mathbf{F}(z, \overline{z}))^{-1} \tag{1.14}
\]

\[
\mathcal{A}_{\overline{z}}(z, \overline{z}) = \overline{\partial \mathbf{F}(z, \overline{z})(\mathbf{F}(z, \overline{z}))^{-1}} \tag{1.15}
\]

with values in the \(s\ell(s)\) Lie algebra. It automatically satisfies the flatness structure equation

\[
\partial \mathcal{A}_{\overline{z}}(z, \overline{z}) - \overline{\partial} \mathcal{A}_z(z, \overline{z}) + \mathcal{A}_{\overline{z}}(z, \overline{z}) \mathcal{A}_z(z, \overline{z}) - \mathcal{A}_z(z, \overline{z}) \mathcal{A}_{\overline{z}}(z, \overline{z}) = 0. \tag{1.16}
\]

The construction of the matrix \(\mathbf{F}(z, \overline{z})\) by means of the solutions of Eq (1.1), gives a \((1,0)\)-
component, \(\mathcal{A}_z(z, \overline{z})\) which turns out to be written in the so-called Drinfeld-Sokolov form.

With the transformation Eq (1.13) the components in Eq (1.14) Eq (1.15) will transform by
gauge transformations as:

\[
\mathcal{A}_z'(z, \overline{z}) = \left(\partial \mathbf{M}(z, \overline{z})\right)\mathbf{M}(z, \overline{z})^{-1} + \mathbf{M}(z, \overline{z}) \mathcal{A}_z(z, \overline{z}) (\mathbf{M}(z, \overline{z}))^{-1} \tag{1.17}
\]

\[
\mathcal{A}_{\overline{z}}'(z, \overline{z}) = \left(\overline{\partial \mathbf{M}(z, \overline{z})}\right)\mathbf{M}(z, \overline{z})^{-1} + \mathbf{M}(z, \overline{z}) \mathcal{A}_{\overline{z}}(z, \overline{z}) (\mathbf{M}(z, \overline{z}))^{-1} \tag{1.18}
\]

We remark that for a general linear (rigid) reparametrization

\[
\mathbf{Z}'^{(i)}(z, \overline{z}) = \sum_{j=1}^{s-1} M_{(i)}^{(j)}(\overline{z}) \mathbf{Z}^{(j)}(z, \overline{z}), \quad |\mathbf{M}(\overline{z})| \equiv \det \mathbf{M}(\overline{z}) \neq 0 \tag{1.19}
\]

which is left out in Eq (1.13), are relevant in our treatment. It is easy to verify the density
property of the wronskian

\[
\omega'(z, \overline{z}) = \omega(z, \overline{z}) |\mathbf{M}(\overline{z})|. \tag{1.20}
\]

Setting a new \(s \times s\) matrix \(\mathbf{M}'(\overline{z})\) as:

\[
\mathbf{M}'(\overline{z}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{M}(\overline{z}) \end{pmatrix} \tag{1.21}
\]

we can define the homologous \(f'_i(z, \overline{z})\) functions as:

\[
f'_i(z, \overline{z}) = \frac{1}{|\mathbf{M}'(\overline{z})|^{1/s}} \sum_j M'(\overline{z})^j_i f_j \equiv \sum_{j=1}^{s} \mathcal{M}(\overline{z})^j_i f_j(z, \overline{z}) \tag{1.22}
\]
where $\mathcal{M}(\tau)$ is thus unimodular.

It is obvious that the $f'_i(z, \tau)$ functions are still solutions of equations (1.1) and it is easy to verify, by means of the construction (1.5) that the reparametrization in Eq(1.19) leaves the coefficients $a_j^{(s)}(z, \tau)$ (and so the matrix $A_i(z, \tau)$) invariant. This shows in particular that a multiplicative renormalization of the $Z^{(r)}(\tau)$ fields never affect the matrix elements $a_j^{(s)}(z, \tau)$.

So with the help of two-dimensional conformal theory built on a Riemann surface, we can construct coordinate transformations on a (target) complex space of arbitrary dimensions with well defined symmetry recipes. The main conceptual difference between our treatment and the mainstream one, is the privilege of the complex structure with respect to the metric properties of the space.

In this context we shall construct a simple model, whose quantum extension can be carried out in a perturbative way by a B.R.S quantization approach.

The Laguerre-Forsyth construction will be the strategic trick of our approach which enables to use the features of the two-dimensional $(z, \tau)$ space to build and investigate models in an arbitrary number of dimensions.

The first constituent, we shall use as a probe, of our theoretical laboratory is a field $\mathcal{X}(z, \tau)$ which must be scalar under any coordinate transformation, and in particular under the ones given by Eq(1.13). Through this field it will be then possible to construct a well defined Classical Lagrangian invariant under the B.R.S. algebra induced by the $Z^{(r)}(\tau)$ field transformations. In particular, the $\mathcal{X}(z, \tau)$ equation of motion will define a mass-shell solution $\mathcal{X}(Z^{(1)}(z, \tau), \cdots, Z^{(s-1)}(z, \tau))$ which will describe the scalar field on a $(s-1)$-dimensional target space.

We shall see that this Classical Lagrangian depends, as regarding the $Z^{(r)}$ content, on their Beltrami coefficients only.

The Quantum extension of the model produces anomalies which spoil the symmetry. A counter-term machinery will compensate these anomalies, but will introduce new pure gravitational interactions at the quantum level. We shall show that this Lagrangian improvement will induce a kind of “on mass-shell symmetry” which could be extended at any order of the perturbative expansion.

The quantum corrections to the model will generate an $\hbar$ order to the $<Z^{(r)}(z, \tau)Z^{(s)}(z, \tau)>$ propagator whose ordering originates a non-commutative character of the $Z^{(r)}(z, \tau)$ fields-coordinates. This idea is not new, and has already been proposed, within the $\mathcal{W}$-algebra framework, by many authors [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. But in these approaches the existence of the $\mathcal{W}$-symmetry, does not imply a space symmetry or any coordinate transformation invariance in dimension greater than two.

In the approach we follow here (something similar can be found in [35]), the $\mathcal{W}_s$-symmetry originates from the SL$(s)$-symmetry in eq (1.11) in a $(s-1)$-dimensional space and not an induced symmetry coming form some OPEs. All the operators can be expressed by means of
the $Z^{(r)}(z,\bar{z})$ coordinates and their derivatives. Therefore the dynamical content considered in the paper has only to do with gravitational degrees of freedom and not with some properties of the primary fields.

In Section 2 we build the algebra as infinitesimal transformations of the $Z^{(r)}(z,\bar{z})$ coordinates and we recall many of its properties.

In Section 3, using a scalar field as a probe, a two-dimensional Lagrangian Field Theory model will be presented and its quantum extension will be performed only on-shell. The compensation of anomalies produces a non-trivial, and well-defined, pure gravitational sector. When the scalar field equation of motion is solved the former becomes a function of the $Z^{(r)}(z,\bar{z})$ coordinates, and so, it turns out to be defined in an arbitrary dimensional space. This sector is investigated by means of Ward identities, and the noncommutativity of the $Z^{(r)}(z,\bar{z})$ coordinates is put into evidence at the quantum level.

Some Appendices are devoted to some technicalities. In Appendix A we calculate in a closed form the derivatives with order greater than $s$ of the functions $f_i(z,\bar{z})$, which become necessary to give a closed form of the B.R.S. algebra of the ghost fields. In Appendix B, the compensation of anomalies depending on the external fields in the quantum extension of the model is carried out. Finally, in Appendix C the Ward identities are specialized to the investigation of the $\mathcal{W}_2$ and $\mathcal{W}_3$ cases, in order to exemplify our results.

2 \textit{\mathcal{W} Algebra in the $\mathcal{K}$ ghosts}

Following [11], the infinitesimal version of the transformation (1.11) can be casted in a B.R.S. formulation by introducing a ghost matrix $\mathcal{K}(z,\bar{z})$:

$$\delta_{\mathcal{W}} F(z,\bar{z}) = \mathcal{K}(z,\bar{z}) F(z,\bar{z}),$$

the nilpotency condition $\delta_{\mathcal{W}}^2 = 0$ requires:

$$\delta_{\mathcal{W}} \mathcal{K}(z,\bar{z}) = \mathcal{K}(z,\bar{z}) \mathcal{K}(z,\bar{z}).$$

The ghost transformation contains many properties of the algebra at hand, in particular:

$$\delta_{\mathcal{W}} \det \mathcal{K}(z,\bar{z}) = Tr \mathcal{K}(z,\bar{z}) \det \mathcal{K}(z,\bar{z}),$$

$$Tr \mathcal{K}^{2n}(z,\bar{z}) = 0,$$

$$\delta_{\mathcal{W}} Tr \mathcal{K}^{2n+1}(z,\bar{z}) = 0, \quad \forall n = 0, 1 \cdots$$

and from condition (1.7)

$$\sum_{i=1}^{s} \mathcal{K}_{i,i}(z,\bar{z}) = 0.$$
So, from the very definition, we infinitesimally get as gauge transformations
\[
\delta_W A_z(z, \bar{z}) = \partial K(z, \bar{z}) - \left[ A_z(z, \bar{z}), K(z, \bar{z}) \right],
\]  
\[
\delta_W A_{\bar{z}}(z, \bar{z}) = \bar{\partial} K(z, \bar{z}) - \left[ A_{\bar{z}}(z, \bar{z}), K(z, \bar{z}) \right].
\]  

The BRS algebraic version of the coordinate transformations in Eq (1.13) are derived from the first row of Eq (2.1) and the definition (1.8) as:
\[
\delta_W Z^{(j)}(z, \bar{z}) = A^{(j)}(z, \bar{z})
\]
\[
= \sum_{\ell=1}^{s} K_{1,\ell}(z, \bar{z}) \sum_{h=0}^{\ell-1} \left( \frac{\ell - 1}{h} \right) \delta^h Z^{(j)}(z, \bar{z}) \frac{\partial^{\ell-1-h} \omega^{-1/s}(z, \bar{z})}{\omega^{-1/s}(z, \bar{z})}.
\]  
These coordinate transformations have not to be confused with the patching laws by holomorphic changes of charts \( z \rightarrow w(z) \). The latter, on the other hand, induce a gluing rule for a holomorphic bundle of matrices with jet coordinates as entries, namely,
\[
F(w, \bar{w}) = \Psi^{-1/s}(z)F(z, \bar{z})
\]
where the matrix valued holomorphic transition function is defined to be
\[
\Psi^{1/s}(z) = \Phi^{1/s}(z)^{-1}t
\]
where \( t \) means transposition and where the invertible matrix
\[
\Phi^{1/s}_{\ell,k}(z) = \begin{cases} \sum_{r=\ell}^{k} \frac{k!}{(k-r)!} \partial^{k-r} (w')^{1/s} & \sum_{a_1 + \cdots + a_n = r} \left( \prod_{n=1}^{r} \frac{1}{a_n} \left( \frac{w(n)}{n!} \right)^{a_n} \right), & \ell \leq k \\ 0, & \ell > k \end{cases}
\]
is in \( \text{SL}(s, C) \) namely \([15, 16]\)
\[
\text{det} \Phi^{1/s}(z) = \prod_{k=0}^{s-1} \Phi^{1/s}_{k,k}(z) = 1.
\]  
Under such a holomorphic change of charts our objects transform as:
\[
w' A_w(w, \bar{w}) = \left( \partial_z \Psi^{1/s}(z) \right) \Psi^{1/s}(z)^{-1} + \Psi^{1/s}(z) A_z(z, \bar{z}) \Psi^{1/s}(z)^{-1},
\]
\[
w' A_{\bar{w}}(w, \bar{w}) = \Psi^{1/s}(z) A_{\bar{w}}(z, \bar{z}) \Psi^{1/s}(z)^{-1},
\]
\[
K(w, \bar{w}) = \Psi^{1/s}(z) K(z, \bar{z}) \Psi^{1/s}(z)^{-1}.
\]
In particular note that:
\[
K_{1,\ell}(z, \bar{z}) = (w')^{1/s} \sum_{j \geq \ell} K_{1,j}(w, \bar{w}) \Psi^{1/s}_{\ell,j}(z), \quad \ell = 1, \cdots, s - 1
\]
\[
K_{1,s}(z, \bar{z}) = (w')^{1-s} K_{1,s}(w, \bar{w}).
\]
So the $(1,s)$ entry of the ghost matrix $\mathcal{K}(z,\bar{z})$ transforms as a tensorial density of conformal weight $s - 1$.

Since the $F_{(i,j)}(z,\bar{z})$ matrix elements are derivatives of the $s$ solutions $f_i(z,\bar{z})$, the $\mathcal{K}(z,\bar{z})$ matrix elements are not all independent; in fact for $j = 0, \ldots , s - 1$:

$$\delta W \partial^i f_i(z,\bar{z}) \equiv \sum_{r=1}^{s} \mathcal{K}_{j+1,r}(z,\bar{z}) \partial^{r-1} f_i(z,\bar{z})$$

$$= \sum_{r=1}^{s} \partial^j \left[ \mathcal{K}_{1,r}(z,\bar{z}) \partial^{r-1} f_i(z,\bar{z}) \right]$$

$$= \sum_{r=1}^{s} \sum_{l=0}^{j} \left( \begin{array}{c} j \\ l \end{array} \right) \partial^{j-l} \mathcal{K}_{1,1+(s-r)}(z,\bar{z}) \partial^{s+(l-r)} f_i(z,\bar{z})$$

$$= \sum_{r=1}^{s} \sum_{l=0}^{j} \left( \begin{array}{c} j \\ l \end{array} \right) \partial^{j-l} \mathcal{K}_{1,1+(s-r)}(z,\bar{z}) \sum_{k=1}^{s-1} \mathcal{F}_{(k)}^{(r-l)}(s; (z,\bar{z})) \partial^{s-k} f_i(z,\bar{z}) \quad (2.19)$$

(the meaning of the $\mathcal{F}_{(k)}^{(r-l)}(s; (z,\bar{z}))$ fields, comes from the fact that the derivatives of the $f_i(z,\bar{z})$ functions of order greater than $s - 1$ can be expressed in terms of the ones of lower orders. This aspect is explained in Appendix A).

For this reason we have $s - 1$ independent $\Phi$-$\Pi$ charged ghosts $\mathcal{K}_{1,r}(z,\bar{z}), r = 2, \ldots , s$. We can state:

**Statement 2.1** The $\mathcal{K}_{j+1,m}(z,\bar{z}); j > 0$ are related to the $\mathcal{K}_{1,m}(z,\bar{z})$ by:

$$\mathcal{K}_{j+1,m}(z,\bar{z}) = \left[ \sum_{r=1}^{s} \sum_{l=0}^{j} \left( \begin{array}{c} j \\ l \end{array} \right) \partial^{j-l} \mathcal{K}_{1,1+(s-r)}(z,\bar{z}) \mathcal{F}_{(s-m+1)}^{(r-l)}(s; (z,\bar{z})) \right]. \quad (2.20)$$

This formula holds also for $j = 0$.

So Eq(2.6) requires:

$$\mathcal{K}_{1,1}(z,\bar{z}) = - \sum_{j=1}^{s-1} \left[ \sum_{r=1}^{s} \sum_{l=0}^{j} \left( \begin{array}{c} j \\ l \end{array} \right) \partial^{j-l} \mathcal{K}_{1,1+(s-r)}(z,\bar{z}) \mathcal{F}_{(s-j)}^{(r-l)}(s; (z,\bar{z})) \right]. \quad (2.21)$$

The same holds for the two matrices $A_z(z,\bar{z})$ and $A_{\bar{z}}(z,\bar{z})$:

**Statement 2.2**

$$(A_z)_{j+1,m}(z,\bar{z}) = \left[ \sum_{r=1}^{s} \sum_{l=0}^{j} \left( \begin{array}{c} j \\ l \end{array} \right) \partial^{j-l} (A_z)_{1,1+(s-r)}(z,\bar{z}) \mathcal{F}_{(s-m+1)}^{(r-l)}(s; (z,\bar{z})) \right], \quad (2.22)$$

$$(A_{\bar{z}})_{j+1,m}(z,\bar{z}) = \left[ \sum_{r=1}^{s} \sum_{l=0}^{j} \left( \begin{array}{c} j \\ l \end{array} \right) \partial^{j-l} (A_{\bar{z}})_{1,1+(s-r)}(z,\bar{z}) \mathcal{F}_{(s-m+1)}^{(r-l)}(s; (z,\bar{z})) \right]. \quad (2.23)$$
For all these reasons we now introduce a shorthand notation for the independent ghosts
\[
K_{1,p}(z, \overline{z}) \equiv K^{(p-1)}(z, \overline{z}); \quad p = 2, \ldots, s
\]  
(2.24)

and we can rewrite the variation (2.2) only in terms of the very independent ghosts as:
\[
\delta_W K^{(p)}(z, \overline{z}) = \sum_{r=1}^{s-1} \sum_{m=1}^{r} \left( r \atop m \right) K^{(r)}(z, \overline{z}) \partial^q K^{(p+m-r)}(z, \overline{z})
\]
\[
- \sum_{r=p}^{s-1} K^{(r)}(z, \overline{z}) \left( \sum_{q=1}^{r} \sum_{n=1}^{q} \right) \partial^{(n+r-s)} K^{(n)}(z, \overline{z})
\]
\[
+ \sum_{r=1}^{s-1} \sum_{q=1}^{r} \partial^q \left( \sum_{j=1}^{s-1} \sum_{l=1}^{j} \sum_{s=1}^{r} \sum_{r'=1}^{s-1} \sum_{q'=1}^{r} \right) \partial^{j-l} K^{(s-r')} (z, \overline{z})
\]
\[
F^{(r'-l)}(s; (z, \overline{z})) F^{(s+q-r')}(s; (z, \overline{z})).
\]  
(2.25)

Suitable transformations for the \( F^{(q)}(s; (z, \overline{z})) \) fields make the algebra to be nilpotent: in the usual approach for \( W \)-algebras [29] this produces a tower of laws for higher spin fields, in our approach this information is contained in the simultaneous use of Eq (2.9) and Eqs (A.77)(1.5).

An important remark has to be made: if we isolate in Eq(2.25) only the first derivative contribution:
\[
\delta_{W,0} K^{(p)}(z, \overline{z}) = \sum_{r=1}^{s-1} r K^{(r)}(z, \overline{z}) \partial K^{(p-r+1)}(z, \overline{z})
\]  
(2.26)

and it is readily checked that \( \delta_{W,0}^2 = 0 \). This property links the present treatment with the one given in [18, 36, 37], where a similar ghost transformation has been found within a symplectic framework, and where finite \( W \)-algebras occur due to a particular symplectomorphism breaking down mechanism. In the approach there, to which the reader is referred, the geometrical aspect plays an essential role and gives a complete representation of each element of the algebra in terms of the generating function of canonical transformations. In particular, the holomorphic ghost fields behave as ordinary tensors (and not as jets), and can be decomposed in non holomorphic sectors \( e^{(p,q)}(z, \overline{z}) \) with well defined \((p, q)\) tensorial characters and geometrical interpretation, transforming as:
\[
\delta_{W,0} e^{(p,q)}(z, \overline{z}) = \sum_{r,s=0, \ldots, r+s>0} \left( r e^{(r,s)}(z, \overline{z}) \partial_z e^{(p-r+1,q-s)}(z, \overline{z})
\]
\[
+ s e^{(r,s)}(z, \overline{z}) \partial_{\overline{z}} e^{(p-r,q-s+1)}(z, \overline{z}) \right).
\]  
(2.27)
Within this parametrization the contribution to the algebra of the underived lowest order ghosts \(c^{(1,0)}(z, \bar{z}), c^{(0,1)}(z, \bar{z})\) represents the point displacement (in the \((z, \bar{z})\) plane) of the fields, and the full transformation admits breakdown terms, needed to represent finite \(\mathcal{W}\)-algebras [37]. According to [38], the ordinary derivative operator can be in general represented by considering the Fock space of the monomials and their derivatives, as the anti-commutator of the B.R.S. operator and the derivative with respect to the first level ghost fields \(c^{(1,0)}(z, \bar{z})\) and \(c^{(0,1)}(z, \bar{z})\).

One thus has

\[
\partial = \left\{ \frac{\partial}{\partial c^{(1,0)}(z, \bar{z})}, \delta \right\}; \quad \overline{\partial} = \left\{ \frac{\partial}{\partial c^{(0,1)}(z, \bar{z})}, \delta \right\}. \tag{2.28}
\]

In this context we can rederive Eqs(2.7), (2.8) by considering the \(A_z(z, \bar{z})\) and \(A_{\bar{z}}(z, \bar{z})\) as derivatives of the \(K(z, \bar{z})\) matrix with respect the previous ghosts fields, namely

\[
A_z(z, \bar{z}) = \frac{\partial K(z, \bar{z})}{\partial c^{(1,0)}(z, \bar{z})}; \quad A_{\bar{z}}(z, \bar{z}) = \frac{\partial K(z, \bar{z})}{\partial c^{(0,1)}(z, \bar{z})}. \tag{2.29}
\]

The \(\delta_{W,0}\) operator is stable under any change of charts if and only if the ghosts behave as true tensors, so, if we want to reconstruct the whole algebra Eq(2.25) on a basis which leaves stable this ideal, we have to decompose \(K(z, \bar{z})\) in order to select its tensorial part. To do this, we introduce the Beltrami notation used in [18],

\[
\partial Z^{(b)}(z, \bar{z}) \equiv \lambda^{(b)}(z, \bar{z}), \quad \overline{\partial} Z^{(b)}(z, \bar{z}) \equiv \lambda^{(b)}(z, \bar{z})\mu(p, (z, \bar{z})), \tag{2.30}
\]

with the usual tricks we can get from the variation Eq (2.9) using the results coming from the symplectic approach:

\[
\lambda^{(j)}(z, \bar{z}) = \sum_{l=1}^{s} A_{z1,l}(z, \bar{z}) S^{(j,l)}(z, \bar{z}), \tag{2.31}
\]

\[
\lambda^{(j)}(z, \bar{z})\mu((j); (z, \bar{z})) = \sum_{l=1}^{s} A_{\bar{z}1,l}(z, \bar{z}) S^{(j,l)}(z, \bar{z}), \tag{2.32}
\]

where we have set

\[
S^{(j,l)}(z, \bar{z}) = \sum_{h=0}^{l-2} \left( \begin{array}{c} l - 1 \ h \end{array} \right) \left\{ \frac{\partial^{l-1-h} \omega^{-1/s}(z, \bar{z})}{\omega^{-1/s}(z, \bar{z})} \right\} \delta^{h} \lambda^{(j)}(z, \bar{z}). \tag{2.33}
\]

With the inversion of the previous Equations we can deduce the expression of the matrix elements of the first row of the \(A_z(z, \bar{z})\) matrix as function of only \(\lambda(z, \bar{z})'s\) and their \(\partial\) derivatives, while the Beltrami’s make their appearance in the \(A_{\bar{z}}(z, \bar{z})\) matrix.

Then the symplectic approach allows one to introduce ghost tensors \(C^{(r)}(z, \bar{z})\) with conformal weight \(-r\), \(r \geq 1\) and without loss of generality for \(1 < j < s\), we can write the matrix decomposition

\[
K(z, \bar{z}) = \sum_{p \geq 0, r \geq 1} \partial^{p} C^{(r)}(z, \bar{z}) B_{p}^{r}(z, \bar{z}). \tag{2.34}
\]
In particular, by eq(2.18) we can identify up to a constant factor
\[ K_{1,s}(z, \overline{z}) \sim C^{(s-1)}(z, \overline{z}), \tag{2.35} \]
(see Eq(C.98) in Appendix C for an explicit example).

Now Equation (2.34) also according to the results obtained in [18, 36, 37], and Eq(2.29) will imply the matrix decomposition
\[ A_{\tau}(z, \overline{z}) = \sum_{p \geq 0, r \geq 1} \partial^p \mu_{\tau}^{(r)}(z, \overline{z}) B^p_{\tau}(z, \overline{z}), \tag{2.36} \]
where \( \mu_{\tau}^{(r)}(z, \overline{z}) \) are the so-called Bilal-Fock-Kogan coefficients [39] : we can find out this expansion in the general case. With this result the \( B(z, \overline{z}) \) matrices can be computed in order to reconstruct the above expansion (2.34).

\section{3 Lagrangian fields Theory models}

The purpose of this Section is to study a dynamical problem in the \((s - 1)\)-dimensional space, with local complex coordinates \( Z^{(r)}(z, \overline{z}) \) such that the formulation of the dynamics must obey all the prescriptions given in Statement (1.1).

For these reasons and considering these local coordinates as scalar fields the symmetry transformations Eq(2.9) yield an equivalence relation for these coordinates over each point of the \((z, \overline{z})\) plane. This means that the lift of a dynamical model from the \((z, \overline{z})\) base space to the target space with local complex coordinates \( Z^{(r)} \) holds its validity if a \( W_s \)-symmetry is imposed as a dynamical constraint. Furthermore, if this symmetry is taken as a symmetry principle, the model can be constructed for any reparametrization of the \((s - 1)\)-dimensional target space.

To test this, we construct a toy Action with a scalar field \( \mathcal{X}(z, \overline{z}) \) interacting with the background in a \( W \)-symmetric way. The action reads
\[ \Gamma_S[\mathcal{X}, \mu(r)] = \int \mathcal{D}\mathcal{X}(z, \overline{z}) \wedge \overline{\mathcal{D}} \mathcal{X}(z, \overline{z}) \tag{3.37} \]
where:
\[ \mathcal{D} = \sum_{r=1}^{s-1} (dz + \mu(r, (z, \overline{z}))d\overline{z}) \mathcal{D}_{(r)}, \tag{3.38} \]
\[ \mathcal{D}_{(r)}(z, \overline{z}) = \frac{\partial - \overline{\mu}(r, (z, \overline{z}))}{1 - \mu(r, (z, \overline{z})\overline{\mu}(r, (z, \overline{z}))}, \tag{3.39} \]
such that \((\mathcal{D} + \overline{\mathcal{D}})\mathcal{X}(z, \overline{z})\) is globally defined as a 1-form on the Riemann surface.

The space-time content of this action depends only on the Beltrami coefficients \( \mu(r, (z, \overline{z})) \). This implies that the model does not exploit all the \( Z^{(r)}(z, \overline{z}) \) content since only the complex
structures of these fields appear in the Classical limit. This means that the \( \lambda^{(r)}(z, \bar{z}) \) sectors, and in particular the fields contained in the \( A_{s}(z, \bar{z}) \) matrix, do not appear in the tree Lagrangian.

We must take care of this particularity, since the Quantum corrections may excite these degrees of freedom which remain silent in the Classical level. So a suitable constraint on them could be required, as we shall see, for the quantization of the model.

Since the \( Z^{(r)}(z, \bar{z}) \) fields infinitesimally transform under a \( W_{s} \)-symmetry as BRS transformations Eq (2.9), we can derive:

\[
\delta_{W} \mu(r, (z, \bar{z})) = 1 - \mu(r, (z, \bar{z})) \frac{\partial Z^{(r)}(z, \bar{z})}{\partial \bar{Z}^{(r)}(z, \bar{z})} \frac{D_{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} \Lambda^{(r)}(z, \bar{z}), \quad (3.40)
\]

\[
\delta_{W} \Lambda^{(r)}(z, \bar{z}) = \sum_{s=1}^{s-1} \frac{\partial Z^{(r)}(z, \bar{z})}{\partial \bar{Z}^{(r)}(z, \bar{z})} \frac{D_{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} \Lambda^{(s)}(z, \bar{z}) \frac{D_{(s)}(z, \bar{z})}{\partial Z^{(s)}(z, \bar{z})} + \frac{D_{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} \Lambda^{(s)}(z, \bar{z}) \frac{D_{(s)}(z, \bar{z})}{\partial Z^{(s)}(z, \bar{z})}, \quad (3.41)
\]

and the scalar \( \mathcal{X}(z, \bar{z}) \) field transforms under \( W_{s} \) as:

\[
\delta_{W} \mathcal{X}(z, \bar{z}) = \sum_{r=1}^{s-1} \left\{ \frac{\Lambda^{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} D_{(r)}(z, \bar{z}) \mathcal{X}(z, \bar{z}) + \frac{\Lambda^{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} \bar{D}_{(r)}(z, \bar{z}) \mathcal{X}(z, \bar{z}) \right\}. \quad (3.42)
\]

The action \( \Gamma_{S} \) is invariant under the Classical B.R.S operator, defined in a functional language as:

\[
\delta_{W} = \int dz \wedge d\bar{z} \left\{ \sum_{r=1}^{s-1} \frac{\Lambda^{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} \frac{\delta}{\partial Z^{(r)}(z, \bar{z})} + \frac{\Lambda^{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} \frac{\delta}{\partial Z^{(r)}(z, \bar{z})} \right\}
\]

\[
+ \Lambda^{(r)}(z, \bar{z}) \frac{D_{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} \frac{\delta}{\partial Z^{(r)}(z, \bar{z})} + \Lambda^{(r)}(z, \bar{z}) \frac{\delta}{\partial Z^{(r)}(z, \bar{z})} \frac{\delta}{\partial Z^{(r)}(z, \bar{z})}
\]

\[
+ \delta_{W} \mathcal{K}^{(r)}(z, \bar{z}) + \delta_{W} \bar{\mathcal{K}}^{(r)}(z, \bar{z}) \right\}. \quad (3.43)
\]

Anyhow in order to clarify the reasons why we claim that this model is defined in a \((s - 1)\)-dimensional space. Indeed, the equation of motion of the \( \mathcal{X}(z, \bar{z}) \) field:

\[
0 = \frac{\delta \Gamma_{S}}{\delta \mathcal{X}(z, \bar{z})} = \sum_{l=1}^{s-1} \left[ \left( 1 - \mu(r, (z, \bar{z})) \frac{\partial}{\partial \bar{Z}^{(r)}(z, \bar{z})} \frac{D_{(r)}(z, \bar{z})}{\partial Z^{(r)}(z, \bar{z})} \right) \mathcal{X}(z, \bar{z}) \right]
\]

\[
+ \frac{\partial}{\partial \bar{Z}^{(r)}(z, \bar{z})} \left( 1 - \mu(l, (z, \bar{z})) \frac{\partial}{\partial \bar{Z}^{(r)}(z, \bar{z})} \bar{D}_{(r)}(z, \bar{z}) \right) \mathcal{X}(z, \bar{z}) \right] \quad (3.44)
\]

first gives a solution as a functional of the complex structures \( \mu(r, (z, \bar{z})) \). After replacing the latter through (2.30), the scalar field solution turns out to be a functional \( \mathcal{X}(Z^{(1)}(z, \bar{z}), \ldots, Z^{(s-1)}(z, \bar{z})) \);
\(Z^{(1)}(z, \bar{z}), \ldots, \bar{Z}^{(s-1)}(z, \bar{z})\) whose evolution in terms of the \(Z^{(r)}(z, \bar{z})\) coordinates is fixed by the
equation of motions of the fields \(Z^{(r)}(z, \bar{z})\) themselves,

\[
\frac{\delta}{\delta Z^{(r)}(z, \bar{z})} \int dz \wedge d\bar{z} \sum_{l,m=1}^{s-1} (1 - \mu(m, (z, \bar{z}))\bar{\mu}(l, (z, \bar{z}))) \times \\
D_{(m)}(z, \bar{z})X(z, \bar{z})D_{(l)}(z, \bar{z})X(z, \bar{z}) = 0,
\]

(3.45)
in which the expression \(X(Z^{(1)}(z, \bar{z}), \ldots, \bar{Z}^{(s-1)}(z, \bar{z}); \bar{Z}^{(1)}(z, \bar{z}), \ldots, \bar{Z}^{(s-1)}(z, \bar{z}))\) obtained in
Eq(3.44) must be plugged into. So the scalar field lives in a \(2(s-1)\)-dimensional space \(\left(Z^{(1)}(z, \bar{z}), \ldots, Z^{(s-1)}(z, \bar{z}); \bar{Z}^{(1)}(z, \bar{z}), \ldots, \bar{Z}^{(s-1)}(z, \bar{z})\right)\) and its dynamics is fixed by the evolution of the
coordinate-fields.

Anyhow, if we want to improve it at the quantum level, we have to work in the two-dimensional \((z, \bar{z})\); only in this way we can extend the \(\mathcal{W}\)-algebra as a coordinate symmetry. For
this reason, some external fields are introduced and a new Classical Lagrangian is defined to be:

\[
\Gamma_{\text{Classical}}[X, \mu(r), \gamma Z^{(r)}, \xi(r), \gamma X] = \Gamma_S[X, \mu(r)] + \int dz \wedge d\bar{z} \left\{ \\
\sum_{r=1}^{s-1} \left( \gamma Z^{(r)}(z, \bar{z}) \delta_{\mathcal{W}} Z^{(r)}(z, \bar{z}) + \xi(r)(z, \bar{z}) \delta_{\mathcal{W}} K^{(r)}(z, \bar{z}) + \text{c.c.} \right) + \gamma X(z, \bar{z}) \delta_{\mathcal{W}} X(z, \bar{z}) \right\},
\]

(3.46)
such that at the tree level

\[
\delta_{\mathcal{W}} \Gamma_{\text{Classical}} = \int dz \wedge d\bar{z} \left\{ \\
\sum_{r=1}^{s-1} \left[ \frac{\delta \Gamma_{\text{Classical}}}{\delta \gamma Z^{(r)}(z, \bar{z})} \delta Z^{(r)}(z, \bar{z}) + \frac{\delta \Gamma_{\text{Classical}}}{\delta \xi(r)(z, \bar{z})} \delta X^{(r)}(z, \bar{z}) \right] + \frac{\delta \Gamma_{\text{Classical}}}{\delta \gamma X(z, \bar{z})} \delta X(z, \bar{z}) \right\} = 0,
\]

(3.47)
and \(\delta_{\mathcal{W}}^2 = 0\).

So we search for an Action \(\Gamma_Q\) as a Quantum extension of \(\Gamma_{\text{Classical}}\) to improve at a Quantum
level the symmetry of Eq(3.47). So a perturbative calculation program by means, for example, a
Green functions expansion require a renormalization which could, in general, spoil the symmetry.
Anyhow the \((z, \bar{z})\) locality properties of the theory guarantees that any possible breakdown is,
at the first order of the perturbative extension, a local term. So we can begin the procedure
with the Quantum Action Principle:

\[
\delta_{\mathcal{W}} \Gamma_Q = \int \Delta_{1}^1(z, \bar{z}).
\]

(3.48)
The anomaly \(\Delta_1^1(z, \bar{z})\) has already been found in the literature \([11, 33, 40]\), with the classical
method of the descent equations.
It can be turned into a well defined expression as \([40, 33], Tr(A_\tau \delta_W A_\tau - K \partial_\tau A_\tau)\) which reads upon using the flatness condition (1.16) and the variation (2.7)

\[
\Delta_\tau^1 (z, \bar{z}) = Tr \left[ A_\tau (z, \bar{z}) \partial K (z, \bar{z}) - K (z, \bar{z}) \partial A_\tau (z, \bar{z}) \right] dz \wedge d\bar{z}. \tag{3.49}
\]

The existence of an anomaly, only dependent of the gravitational content, breaks, at the Quantum level, the equivalence relation of the \(Z(r)(z, \bar{z})\) coordinates for all points of the \((z, \bar{z})\) base space, imposed by Eq (1.13) (and Eq (2.9) in a B.R.S. framework). This dynamically induced inequivalence brings new difficulties, for example, in the coordinate ordering procedure. For this reason the anomaly must be managed in order to understand what kind of residual symmetry of the Classical one can be maintained at the Quantum level.

Now enters the compensation of the anomaly in order to restore the invariance at the quantum level by adding well-defined counter-terms. First we point out, as a trailer, the crucial step (coming from the anomaly structure) of this program, and we shall show how to get the on-shell quantum extension of the model.

Then we re-open the renormalization program with all the dynamical variables needed for its full completion, and after showing how to remove all the trivial anomalies introduced by all these new degrees of freedom, we prove the consistency of the sketch given in the trailer.

Note that the previous well-defined anomaly can be rewritten as

\[
\Delta_\tau^1 = 2 \int dz \wedge d\bar{z} Tr(A_\tau (z, \bar{z}) \delta_W A_\tau (z, \bar{z})) \tag{3.50}
\]

by dropping out the boundary term \(\partial_z Tr(K A_\tau)\).

In order to compensate the anomaly let us introduce an invariant background classical field, namely a background \((1, 0)\) type connection \(\tilde{A}_\tau (z, \bar{z})\) [41], with

\[
\delta_W \tilde{A}_\tau (z, \bar{z}) = 0, \tag{3.51}
\]

\[
(w')_{\tilde{\alpha}} w (w, \bar{w}) = \left( \partial_z \Psi_{1/2} (z) \right) \Psi_{1/2}^{-1} (z) + \Psi_{1/2} (z) \tilde{A}_\tau (z, \bar{z}) \Psi_{1/2}^{-1} (z). \tag{3.52}
\]

It is easy to show that

\[
\delta_W \int dz \wedge d\bar{z} Tr \left[ \left( A_\tau (z, \bar{z}) - \tilde{A}_\tau (z, \bar{z}) \right) A_\tau (z, \bar{z}) \right] = \frac{1}{2} \Delta_\tau^1 + \int dz \wedge d\bar{z} Tr \left[ \left( A_\tau (z, \bar{z}) - \tilde{A}_\tau (z, \bar{z}) \right) \delta_W A_\tau (z, \bar{z}) \right], \tag{3.53}
\]

and each term in the right hand side is well defined.
By adding a counter-term in order to modify the action $\Gamma_S$ into a well-defined one, $\hat{\Gamma}$,

$$\hat{\Gamma}_{\text{Tree}} = \Gamma_S - 2\hbar \int dz \wedge d\bar{z} \text{Tr} \left[ \left( A_z(z, \bar{z}) - \hat{A}_z(z, \bar{z}) \right) A_{\bar{z}}(z, \bar{z}) \right], \quad (3.54)$$

such that at the tree diagrams (but at the first order of $\hbar$) level:

$$\delta_W \hat{\Gamma}_{\text{Tree}} = \delta_W \left( \Gamma_S - 2\hbar \int dz \wedge d\bar{z} \text{Tr} \left[ \left( A_z(z, \bar{z}) - \hat{A}_z(z, \bar{z}) \right) A_{\bar{z}}(z, \bar{z}) \right] \right)$$

$$= -\hbar \int \Delta^3(z, \bar{z}) dz d\bar{z} + 2\hbar \int dz \wedge d\bar{z} \text{Tr} \left[ \left( A_z(z, \bar{z}) - \hat{A}_z(z, \bar{z}) \right) \delta_W A_{\bar{z}}(z, \bar{z}) \right]. \quad (3.55)$$

So the first term in the r.h.s. cancels the local anomaly coming from an hypothetical Feynman diagrams of the perturbative loop expansion starting from the action $\Gamma_{\text{Classical}}$. The quantum improvement thus leads to an Action $\hat{\Gamma}_Q$ such that the anomaly has been shifted to:

$$\delta_W \hat{\Gamma}_Q = 2\hbar \int dz \wedge d\bar{z} \text{Tr} \left[ \left( A_z(z, \bar{z}) - \hat{A}_z(z, \bar{z}) \right) \left( \delta_W A_{\bar{z}}(z, \bar{z}) \right) \right], \quad (3.56)$$

such that:

$$\delta_W \hat{\Gamma}_Q \bigg|_{A_z(z, \bar{z})=\hat{A}_z(z, \bar{z})} = 0. \quad (3.57)$$

and the symmetry is restored on the surface

$$A_z(z, \bar{z}) = \hat{A}_z(z, \bar{z}). \quad (3.58)$$

As already remarked in the discussion of Eq(2.31) this surface fixes only the derivatives $\partial^n Z^{(r)}(z, \bar{z})$, $n \geq 1$ and does not affect the complex structures parametrized by the Beltrami coefficients $\mu(r; (z, \bar{z}))$ which are coupled, in the tree Action, with the physical degrees of the scalar field $\mathcal{X}(z, \bar{z})$. On the other hand, this condition removes the arbitrariness on the definition of the $a_j^{(s)}(z, \bar{z})$ coefficients given by (1.19). Moreover the renormalization program, needed for the quantum improvement of the model, modifies the Action and, more important, the surface defined by Eq(3.58), so we have to bring all under tight control at each step of the quantum improvement. The presence of the $\hat{A}_z(z, \bar{z})$ fields could introduce already at the Classical limit arbitrary terms invariant under $\mathcal{W}$ transformations. The invariance under changes of charts (and the absence of the suitable connection) admits only the term $\partial \hat{A}_z(z, \bar{z})$ which is disconnected.

We shall now discuss this last subject, which is of great consequence in our treatment, to extend it at any order of a perturbative expansion.
First of all, we remind that the $A_z(z, \bar{z})$ matrix depends only on the $\lambda^{(r)}(z, \bar{z})$ fields and their $\partial_z$ derivatives, which do not appear in the tree Lagrangian in Eq (3.37), so this constraint do not alter the dynamics of the $X(z, \bar{z})$ scalar field at the Classical level.

To clarify this point in a Field Theory language, and to improve it at any order of the renormalization perturbative program, we have to put this constraint working off-shell and perform a Faddeev-Popov like construction working out Eq (3.58) by means of a functional integration.

Only the $A_{(z,(1,l))}(z, \bar{z}); l = 2, \ldots, s$ are independent (see Eq (2.22)) and are calculated by means of Eq (2.31). We impose:

$$\delta W (\circ A_z, 1, r + 1) - A_z, 1, r + 1 \equiv \int d\beta (r) \exp \left[ -\frac{1}{\hbar} \int dz \wedge d\bar{z} \beta (r) \left( \circ A_z, 1, r + 1 - A_z, 1, r + 1 \right) \right],$$

(3.59)

to which must be added a further $b_{(r)}(z, \bar{z})\delta W Z^{(r)}(z, \bar{z})$ term in order to restore the symmetry, with:

$$\delta_W b_{(r)}(z, \bar{z}) = -\beta_{(r)}(z, \bar{z}),$$

(3.60)

$$\delta_W \beta_{(r)}(z, \bar{z}) = 0.$$  \hfill (3.61)

So the symmetry is preserved on-shell only. Anyhow the quantization program must be performed off-shell and we have to check that the on-shell properties are not wasted. In this case, the Quantum extension of the model will treat on the anomaly cancellation program, by means of the counter-term machinery.

For this purpose we introduce already at the tree level the $\hbar$ order counter-terms and the Faddeev-Popov ghost term ; so the tree Action becomes:

$$\Gamma = \Gamma_S + \int dz \wedge d\bar{z} \left\{ \sum_{r=1}^{s-1} \left[ \beta_{(r)}(z, \bar{z}) \left( A_{(z,(1,r+1))}(z, \bar{z}) - \circ A_{(z,(1,r+1))}(z, \bar{z}) \right) \right. \right.

+ b_{(r)}(z, \bar{z}) A_{(z,(1,r+1))}(z, \bar{z}) + \gamma_{(r)}(z, \bar{z}) \delta W Z^{(r)}(z, \bar{z}) + \xi_{(r)}(z, \bar{z}) \delta W K^{(r)}(z, \bar{z}) + c.c. \right.

+ \left. \gamma X(z, \bar{z}) \delta_W X(z, \bar{z}) - 2hTr \left[ A_z(z, \bar{z}) - \circ A_z(z, \bar{z}) A_{\bar{z}}(z, \bar{z}) \right] \right.

- \theta Tr \left[ A_z(z, \bar{z}) - \circ A_z(z, \bar{z}) \right] \delta_W A_{\bar{z}}(z, \bar{z}) \right\}\}

≡ \Gamma_0 + h\Gamma_{1,Tree} + \theta \Gamma_\theta \hfill (3.62)

where we have introduced $\theta$ as a para-fermionic coordinate such that:

$$\delta_W \theta = \hbar$$  \hfill (3.63)
in order to control the anomalous term defined in Eq.(3.56). To perform it, and to show the completeness of the quantum perturbative expansion, we must switch to functional techniques and introduce by Legendre transformation the connected Green generating functional

\[ Z_c[\mathcal{J}_X, \mathcal{J}_{Z^{(r)}}, \mathcal{J}_{K^{(r)}}, \ldots] = \Gamma + \int dz \wedge d\bar{z} \left[ \mathcal{J}_X(z, \bar{z}) \mathcal{X}(z, \bar{z}) + \sum_{r=1}^{s-1} \mathcal{J}_{Z^{(r)}}(z, \bar{z}) \mathcal{Z}^{(r)}(z, \bar{z}) + \mathcal{J}_{K^{(r)}}(z, \bar{z}) \mathcal{K}^{(r)}(z, \bar{z}) + c.c. \right] \]

and we can construct the Green generating functional:

\[ Z[\mathcal{J}_X, \gamma_X, \mathcal{A}_z, \mathcal{J}_{K^{(r)}}, \xi^{(r)}, \gamma_{Z^{(r)}}, c.c.] = \int d\lambda \prod_{r=1}^{s-1} d\mathcal{Z}^{(r)} db^{(r)} d\beta^{(r)} d\mathcal{K}^{(r)} \exp[-\frac{1}{\hbar} Z_c]. \quad (3.65) \]

In this description the interpretation of the coordinates as quantum fields is particularly suitable to manage the several unexpected events in the quantization of the model. Anyhow the role of coordinates, understood as passive entity for events description, has to be widely amended at the quantum level, while at the classical approximation has to maintain all its geometrical properties.

The B.R.S operator acting on \( Z_c \) writes

\[ \delta Z_c = \int dz \wedge d\bar{z} \left[ \mathcal{J}_X(z, \bar{z}) \frac{\delta}{\delta \gamma_X(z, \bar{z})} + \sum_{r=1}^{s-1} \left( \mathcal{J}_{Z^{(r)}}(z, \bar{z}) \frac{\delta}{\delta \gamma_{Z^{(r)}}(z, \bar{z})} \right) \right. \]

\[ \left. - \beta^{(r)}(z, \bar{z}) \frac{\delta}{\delta b^{(r)}(z, \bar{z})} + \mathcal{J}_{K^{(r)}}(z, \bar{z}) \frac{\delta}{\delta \mathcal{K}^{(r)}(z, \bar{z})} + c.c. \right] + \hbar \frac{\delta}{\delta \theta} \right]. \quad (3.66) \]

Our program is to show now that at every order of the perturbative theory we can define a quantum improvement of both the Action and the B.R.S. operator such that:

\[ \delta Z, Z_c = 0. \quad (3.67) \]

For this reason, since we principally look for an extension of the classical Action, we introduce the linearized operator acting on \( \Gamma_0 \), which will be useful in the sequel:

\[ \delta L_0 = \int dz \wedge d\bar{z} \left\{ \frac{1}{2} \frac{\delta \Gamma_0}{\delta \gamma_X(z, \bar{z})} \frac{\delta}{\delta \gamma_X(z, \bar{z})} + \frac{1}{2} \frac{\delta \Gamma_0}{\delta \gamma_{Z^{(r)}}(z, \bar{z})} \frac{\delta}{\delta \gamma_{Z^{(r)}}(z, \bar{z})} \right. \]

\[ + \sum_{r=1}^{s-1} \left. \left[ \frac{1}{2} \frac{\delta \Gamma_0}{\delta \gamma_{Z^{(r)}}(z, \bar{z})} \frac{\delta}{\delta \gamma_{Z^{(r)}}(z, \bar{z})} \right] + \frac{1}{2} \frac{\delta \Gamma_0}{\delta \mathcal{K}^{(r)}(z, \bar{z})} \frac{\delta}{\delta \mathcal{K}^{(r)}(z, \bar{z})} \right. \]

\[ + \frac{1}{2} \frac{\delta \Gamma_0}{\delta \xi^{(r)}(z, \bar{z})} \frac{\delta}{\delta \xi^{(r)}(z, \bar{z})} + \left. \frac{1}{2} \frac{\delta \Gamma_0}{\delta \xi^{(r)}(z, \bar{z})} \frac{\delta}{\delta \xi^{(r)}(z, \bar{z})} \right. \]

\[ - \beta^{(r)}(z, \bar{z}) \frac{\delta}{\delta b^{(r)}(z, \bar{z})} + c.c. \right] + \hbar \frac{\delta \Gamma_0}{\delta \theta} \right\} = 0, \]

\[ \delta L_0^2 = 0, \quad \delta L_0 = \delta L_0^0 + \hbar \delta L_0^1. \quad (3.68) \]
If we consider the Classical level at $\hbar = 0$, we have:

$$\delta L_0 \Gamma_0 = 0. \quad (3.69)$$

The Feynman graph perturbative calculations require a local counter-term adjustment procedure which in general spoils the Classical symmetry and modifies the Classical Action. But at the same first order of these corrections, we have to add the tree contributions previously introduced. More precisely these tree contributions modify the B.R.S symmetry equation by

$$\delta L_0 \Gamma_{1,\text{Tree}} = \frac{1}{2} \int d\mathbf{z} \wedge d\mathbf{z}' \sum_{r=1}^{s-1} \left[ \frac{\delta \Gamma_{1,\text{Tree}}(z,\mathbf{z})}{\delta Z^{(r)}(z,\mathbf{z})} \frac{\delta \Gamma_0}{\delta \gamma_{Z^{(r)}}(z,\mathbf{z})} + \text{c.c.} \right]. \quad (3.70)$$

Moreover the $\theta$ dependent term of the B.R.S. operator (we shall use this trick only here, in order to keep the stability of the calculation) introduce the local term which compensates the anomaly (3.56) since:

$$\delta L_0 \Gamma_0 = -\hbar \Gamma_{\theta} \quad (3.71)$$

So if the anomaly of this new Action is still the one of Eq (3.56), (i.e. no new anomaly occurs, due to the introduction of plenty of such new fields) the counter-term will perform its duty and the anomaly will disappear. Thus if no nasty surprise appears, as we shall see in the following, the renormalization can be extended to all orders.

The $\delta(\mathcal{A}_z(z,\mathbf{z}) - \mathcal{A}_z(z,\mathbf{z}))$ constraint is found to be the equation of motion for the $\beta(z,\mathbf{z})$ field:

$$0 = \frac{\delta \Gamma}{\delta \beta_{(r)}(z,\mathbf{z})} = \mathcal{A}_{z,1,r+1}(z,\mathbf{z}) - \mathcal{A}_{z,1,r+1}(z,\mathbf{z}). \quad (3.72)$$

Note that the constraint Eq(3.72) is not B.R.S. invariant since

$$\delta L_0 \frac{\delta \Gamma_0}{\delta \beta_{(r)}(z,\mathbf{z})} = \frac{\delta \Gamma_0}{\delta b_{(r)}(z,\mathbf{z})} - \frac{1}{2} \int d\mathbf{z}' \wedge d\mathbf{z}'' \sum_{m=1}^{s-1} \frac{\delta A_{z,1,r+1}(z,\mathbf{z})}{\delta Z^{(m)}(z',\mathbf{z}'')} \times \frac{\delta \Gamma_0}{\delta \gamma_{(m)}(z',\mathbf{z}')}. \quad (3.73)$$

So $\beta_{(r)}(z,\mathbf{z})$-dependent counter-terms would a priori modify this equation of motion. In Appendix B, the analysis of the anomalies depending on the external fields is performed. It is shown there that they become trivial and can be compensated by means of local counter-terms. In particular $\beta_{(r)}(z,\mathbf{z})$-dependent anomalies can be compensated by means of counter-terms with no $\beta_{(r)}(z,\mathbf{z})$-dependence and so the condition Eq(3.58), expressed through the $\beta_{(r)}(z,\mathbf{z})$ mass-shell condition Eq(3.72) is not affected by the renormalization.
Thus the renormalization of external field independent anomalies can be carried out and it can be easily realized that the analysis and the results already encountered in Eq(3.49) can be repeated again.

The main feature of the model is that the well defined procedure for counterterms introduces pure gravitational terms of order $\hbar$ which define a local propagators $<Z^{(r)}(z,\overline{z})Z^{(s)}(z',\overline{z}')>$ also of order $\hbar$ whose the form depends on the rank of the $\mathcal{W}$-symmetry.

We shall investigate this fact, with detailed calculations in Appendix C, by means of Ward identities, in the $\mathcal{W}_2$ case (where the $(Z,\overline{Z})$ space has the same dimension of the background $(z,\overline{z})$) and in the $\mathcal{W}_3$ one, where the four-dimensional $Z^{(r)}(z,\overline{z})$ space is of particular interest.

The Ward identities can be derived, as well known, from the B.R.S. equation Eq(3.67) (or equivalent). Anyhow the inequivalent behavior of the $\mathcal{K}^{(r)}(z,\overline{z})$ (jet-) ghosts under change of charts, produces a complete lack of homogeneity between the Ward identities obtained from functional differentiation with respect these ghosts. For this reason we turn to the $\mathcal{C}^{(r)}(z,\overline{z})$ ghost fields, whose good tensorial behavior, is suitable for handling the identities. Accordingly, the decomposition Eq(2.34) must be at our disposal and this is the reason why we have to fix the order of the $\mathcal{W}$-algebra in order to push further ahead the calculation.

Indeed, the symmetry conservation and the presence of the local $\hbar$ pure-gravitational counter-term induce(as we shall see in detail in the third Appendix), in the $\mathcal{W}(2)$ case, the local two point coordinate function:

$$\frac{\delta^2 \Gamma_Q}{\delta Z'(z',\overline{z}')\delta Z(z,\overline{z})} \left\{ \begin{array}{l} \mathcal{A}_z = \mathcal{A}_z^0 \\
<\mathcal{X}> = 0 \\
\end{array} \right\} \equiv <Z'(z',\overline{z}')Z(z,\overline{z})>$$

$$= \frac{\hbar}{\partial Z(z,\overline{z})} \left[ \partial - \mu(z,\overline{z}) \partial \right] \frac{\delta T(z,\overline{z})}{\delta Z'(z',\overline{z})} \left\{ \begin{array}{l} \mathcal{A}_z = \mathcal{A}_z^0 \\
\end{array} \right\}$$

(\text{where } T(z,\overline{z}) \text{ is a projective connection [6, 7, 9] which appears as an entry in the } A(z,\overline{z}) \text{ matrix})

while in the $\mathcal{W}(3)$ case, a rather elaborated calculation allows to reach well defined results (in Appendix C the $<Z^{(1)}(z,\overline{z})Z^{(1)}(z',\overline{z}')>$ two-point function is explicitly given).

It is a matter of straightforward calculation that the $(z,\overline{z})$ locality of the model assure a noncommutative imprinting for all Green functions we can calculate by means of Ward identities. In particular in the $\mathcal{W}(2)$ case we easily derive:

$$\left[ \frac{\delta}{\delta Z'(z',\overline{z}')} \frac{\delta}{\delta Z(z,\overline{z})} \right] \Gamma_Q \left\{ \begin{array}{l} \mathcal{A} = \mathcal{A}_z^0 \\
<\mathcal{X}> = 0 \\
\end{array} \right\} \equiv <Z'(z',\overline{z}'), Z(z,\overline{z})>$$

$$= \left( \frac{\hbar}{\partial Z(z,\overline{z})} \left[ \partial - \mu(z,\overline{z}) \partial \right] \frac{\delta T(z,\overline{z})}{\delta Z'(z',\overline{z})} \right)$$
\[
- \frac{\hbar}{\partial Z'(z', \bar{z}')} \left[ \mathcal{J} - \mu'(z', \bar{z}') \delta' \right] \delta T'(z', \bar{z}') \right|_{\delta T'(z', \bar{z}') = \delta Z(z, \bar{z})} \left\{ A_z = A_{\bar{z}} \right. \}
\] (3.75)

We emphasize that this commutation rule highly depends on the \((z, \bar{z})\) background directions. In particular, along the real axis \(z = \bar{z}\) and \(z' = \bar{z}'\) with \(z \neq z'\) the commutator vanishes.

In higher dimensions the order of the \(\mathcal{W}\)-algebra must be increased, and the results acquire a rather elaborated complexity. Moreover the examples clarify the dichotomic role of \(Z^{(r)}(z, \bar{z})\) as fields and local complex coordinates. In fact when they are upgraded to quantum fields, non trivial commutation relations will qualify them as noncommutative coordinates, so the present treatment acquires some new perspectives.

4 Conclusions

The lesson we can derive from our calculations is that the \(\mathcal{W}\)-symmetry already hides many unknown aspects, and its properties are not completely used within a physical scenario.

In this context we have constructed a Lagrangian Field Theory model over a complex two-dimensional Riemannian background manifold where the “true” complex coordinates are defined à la Laguerre-Forsyth.

Using a scalar field as a probe for the coordinates, we show that the interaction of this field with the background, introduces, in the quantum extension of the model, non trivial two-point functions in the coordinate fields, which generate an induced quantum gravity over the Riemann surface. The fields become on-shell "coordinates" when the scalar field equation is solved, and the space which arises turns out, by construction, to be noncommutative.

The model can be extended to all orders in the \(\hbar\) perturbative expansion, but its on-shell peculiarity greatly limits its value.

The construction presented in the paper for the particular case of Riemann surface raises the issue whether this way of generating noncommutative space-time coordinates from a quantum field approach may be (or not) also generalizable to higher complex dimensions, one may think of the two-complex dimensional (or four-real dimensional) case. The field coordinates do not commute due to quantum corrections and one may ask whether this property can be embedded in an algebraic way which might generalize the many products (Moyal, Kontsevitch..) found in the literature.

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A Higher orders derivatives of the solutions $f_i$

Equations (1.8) say that the $s$-th order of derivative of $f_i(z, \bar{z})$ can be expressed in terms of the lowest order ones $\partial^j f_i(z)$, $j = 1 \cdots s - 1$. Furthermore if we derive at all orders the previous equations, we can express all the derivatives of $f_i(z, \bar{z})$ with order $l \geq s + 1$ with the ones with order $j = 1 \cdots s - 1$, which thus form a basis of the all arbitrary functions which can be expressed in terms of $f_i(z, \bar{z})$ and their derivatives. So we, after some work, we deduce:

**Statement A.1**

$$\partial^{m+s} f_i(z, \bar{z}) = \sum_{k=1}^{s} \mathcal{F}_k^{(-m)}(s; (z, \bar{z})) \partial^{s-k} f_i(z, \bar{z}),$$

(A.76)

where for $m \geq 0$

$$\mathcal{F}_k^{(-m)}(s; (z, \bar{z})) = \sum_{i=1}^{i} (-1)^i \left[ \prod_{l_i, j_i} \left( m - \sum_{r=1}^{i-1} (l_r + j_r) \right) \partial^{j_i} a_i^{(s)}(z, \bar{z}) \right] \left\{ \begin{array}{l} 1 \leq j_i \leq s \\ 0 \leq l_i \leq m - \sum_{r=1}^{i-1} (l_r + j_r) \\ k + m = \sum_{r=1}^{i} (l_r + j_r) \end{array} \right\}$$

(A.77)

while we can trivially extend the previous formula for $-1 \geq m \geq -s$ defining:

$$\mathcal{F}_k^{(-m)}(s; (z, \bar{z})) = \delta_{k}^{-m}.$$ 

(A.78)

We shall call this the "trivial extension" of Eq (A.76). So our definition of $\mathcal{F}_k^{(-m)}(s; (z, \bar{z}))$ will gather together both Eqs (A.77) and (A.78).

B External Field dependence of the anomalies

The Q.A.P gives, after the introduction of the $\hbar$ counterterm:

$$\delta_{L_0} \Gamma = \int dz \wedge d\bar{z} \Delta(z, \bar{z}) = \int dz \wedge d\bar{z} \left[ \Delta_0(z, \bar{z}) + \sum_{r=1}^{s-1} \left( b_{(r)}(z, \bar{z}) \Delta_{b_{(r)}}(z, \bar{z}) + \beta_{(r)}(z, \bar{z}) \Delta_{\beta_{(r)}}(z, \bar{z}) + \xi_{(r)}(z, \bar{z}) \Delta_{\xi_{(r)}}(z, \bar{z}) + \gamma_{(r)}(z, \bar{z}) \Delta_{\gamma_{(r)}}(z, \bar{z}) \right) + c.c \right]$$

(B.79)

where all the $\Delta$'s terms can be only functions of the fields $Z^{(r)}(z, \bar{z}), K^{(r)}(z, \bar{z}), \chi(z, \bar{z})$ and their derivatives.
It is possible to show, following the tricks contained in [38], that the cohomology space of \(\delta_L\) in the local functions depending on \(Z^{(r)}(z, \bar{z}), K^{(r)}(z, \bar{z}), \chi(z, \bar{z})\) and their derivatives, and with Faddeev-Popov charge less than \(s - 1\) is empty. First one gets

\[
\delta_W \Delta_{\gamma(r)}(z, \bar{z}) \equiv D^{(r)}_{(2)}(z, \bar{z}) \Delta \\
= \frac{1}{2} \int dz' \wedge d\bar{z}' \sum_{m=1}^{s-1} \left[ \frac{\delta_{\gamma(r)} Z^{(r)}(z, \bar{z})}{\delta Z^{(m)}(z', \bar{z}')} \Delta_{\gamma(m)}(z', \bar{z}') + \frac{\delta_{\gamma(r)} Z^{(r)}(z, \bar{z})}{\delta K^{(m)}(z', \bar{z}')} \Delta_{\xi(m)}(z', \bar{z}') \right]
\]

(B.80)

\[
\delta_W \Delta_{\xi(r)}(z, \bar{z}) \equiv D^{(r)}_{(3)}(z, \bar{z}) \Delta \\
= \frac{1}{2} \int dz' \wedge d\bar{z}' \sum_{m=1}^{s-1} \left[ \frac{\delta_{\xi(r)} K^{(r)}(z, \bar{z})}{\delta Z^{(m)}(z', \bar{z}')} \Delta_{\gamma(m)}(z', \bar{z}') + \frac{\delta_{\xi(r)} K^{(r)}(z, \bar{z})}{\delta K^{(m)}(z', \bar{z}')} \Delta_{\xi(m)}(z', \bar{z}') \right]
\]

(B.81)

Applying again the B.R.S operator we reach a two-dimensional system which leads to the trivial solutions:

\[
\Delta_{\xi(r)}(z, \bar{z}) = \delta_{L_0} \frac{\delta \hat{\Gamma}_1}{\delta \xi(r)(z, \bar{z})} + D^{(r)}_{(3)}(z, \bar{z}) \hat{\Gamma}_1, \\
\Delta_{\gamma(r)}(z, \bar{z}) = -\delta_{L_0} \frac{\delta \hat{\Gamma}_2}{\delta \gamma(r)(z, \bar{z})} + D^{(r)}_{(2)}(z, \bar{z}) \hat{\Gamma}_2,
\]

where the \(\hat{\Gamma}\)'s are unknown counterterms. Second, one has

\[
\delta_W \Delta_{b(r)}(z, \bar{z}) \equiv D^{(r)}_{(1)}(z, \bar{z}) \Delta \\
= \frac{1}{2} \int dz' \wedge d\bar{z}' \sum_{m=1}^{s-1} \left[ \frac{\delta_{b(r)} A_{(1,r+1)}(z, \bar{z})}{\delta Z^{(m)}(z', \bar{z}')} \Delta_{\gamma(m)}(z', \bar{z}') + \frac{\delta_{b(r)} A_{(1,r+1)}(z, \bar{z})}{\delta K^{(m)}(z', \bar{z}')} \Delta_{\xi(m)}(z', \bar{z}') \right]
\]

(B.84)

whose solution reads

\[
\Delta_{b(r)}(z, \bar{z}) = \delta_{L_0} \frac{\delta \hat{\Gamma}_3}{\delta b(r)(z, \bar{z})} + D^{(r)}_{(1)}(z, \bar{z}) \hat{\Gamma}_3.
\]

(B.85)

Finally we get:

\[
\delta_W \Delta_{\beta(r)}(z, \bar{z}) = \Delta_{\beta(r)}(z, \bar{z}) - \left[ \frac{1}{2} \int dz' \wedge d\bar{z}' \sum_{m=1}^{s-1} \frac{\delta A_{(1,r+1)}(z, \bar{z})}{\delta Z^{(m)}(z', \bar{z}')} \Delta_{\gamma(m)}(z', \bar{z}') \right]
\]

\[
= \delta_{L_0} \frac{\delta}{\delta b(r)(z, \bar{z})} \hat{\Gamma}_3 + D^{(r)}_{(1)}(z, \bar{z}) \hat{\Gamma}_3 - \left[ \frac{1}{2} \int dz' \wedge d\bar{z}' \sum_{m=1}^{s-1} \frac{\delta A_{(1,r+1)}(z, \bar{z})}{\delta Z^{(m)}(z', \bar{z}')} \left( -\delta_{L_0} \frac{\delta \hat{\Gamma}_2}{\delta \gamma(m)(z', \bar{z}')} + D^{(m)}_{(2)}(z', \bar{z}') \hat{\Gamma}_2 \right) \right]
\]

(B.86)
with general solution
\[
\Delta_{\beta(r)}(z, \bar{z}) = \delta L_0 \frac{\delta}{\delta \beta(r)(z, \bar{z})} \hat{\Gamma}_4 - D_0^{(r)}(z, \bar{z}) \hat{\Gamma}_4. \tag{B.87}
\]

It is readily seen that the counter-term \( \hat{\Gamma}_4 = \int dz \wedge d\bar{z} b_{(r)}(z, \bar{z}) \Delta_{\beta(r)}(z, \bar{z}) \) removes the anomaly without any modification of the \( \beta(r)(z, \bar{z}) \) sector of the Lagrangian, and then with no change of the \( \beta(r)(z, \bar{z}) \) equations of motions Eq(3.72). This counter-term also benefits the cancellation of \( b_{(r)}(z, \bar{z}) \) dependent anomalies (see Eq(B.85)) by means the use of Eq(B.86) consistency condition.

\section{Ward identities}

As said before, the Ward identities can be derived from the BRS variation Eq (3.67) by means of functional differentiation with respect the ghosts fields. Anyhow the different covariance behaviors under changes of charts of the \( \mathcal{K}(r)(z, \bar{z}) \) ghosts produce, if we adopt these ghosts for our procedure, various Ward identities which cannot be compared among themselves. This is mainly for this reason that the \( \mathcal{C}(r)(z, \bar{z}) \) ghosts are preferred and fit well to our requirements. However the decomposition Eq(2.34) is too difficult to be managed in a general case and so we limit our analysis to the two lowest orders of \( \mathcal{W} \)-algebras for the sake of simplicity.

\subsection{The \( \mathcal{W}_2 \) case}

In this case the \( \mathcal{K}(z, \bar{z}) \) ghost is at the top and is a \((1,0)\)-type tensor, and the \( Z(z, \bar{z}) \) transformation law together with the Eq(2.28) give the relations:
\[
\delta_{\mathcal{W}} Z(z, \bar{z}) = \mathcal{C}(z, \bar{z}) \partial Z(z, \bar{z}),
\]
\[
\left( \partial - \mu_x^x(z, \bar{z}) \partial \right) Z(z, \bar{z}) = 0, \quad \mu_x^x(z, \bar{z}) = \frac{\partial \mathcal{C}(z, \bar{z})}{\partial c(0,1)}(z, \bar{z}) \tag{C.88}
\]

The \( A_z(z, \bar{z}) \) and \( A_{\bar{z}}(z, \bar{z}) \) fields satisfying Eq(1.16) can be grouped into:
\[
A(z, \bar{z}) \equiv A_z(z, \bar{z}) dz + A_{\bar{z}}(z, \bar{z}) d\bar{z} = \begin{pmatrix} 0 & \frac{1}{2} \partial T(z, \bar{z}) \\ -\frac{1}{2} T(z, \bar{z}) & 0 \end{pmatrix} dz + \begin{pmatrix} -\frac{1}{2} \partial \mu_x^x(z, \bar{z}) \\ -\frac{1}{2} \mu_x^x(z, \bar{z}) T(z, \bar{z}) - \frac{1}{2} \partial^2 \mu_x^x(z, \bar{z}) \end{pmatrix} d\bar{z}. \tag{C.89}
\]

The "well-defined" anomaly can be deduced from Eq (3.49) and writes as [40]:
\[
\mathcal{S} \Gamma_S = \int \Delta_2^{(z, \bar{z})} = \int \frac{1}{2} \left[ \nu_x^2 \mathcal{L}_3 \mathcal{C}(z, \bar{z}) - \mathcal{C}(z, \bar{z}) L_3 \nu_x^2(z, \bar{z}) \right] dz \wedge d\bar{z}, \tag{C.90}
\]
where we indicate the third order Bol operator \([13, 9]\) \(L_3\) and the projective connection \(T(z, \overline{z})\)

\[
L_3 = \delta^3 + 2 T(z, \overline{z}) \partial + \partial T(z, \overline{z}),
\]

\[
T(z, \overline{z}) = \partial^2 \ln \partial Z(z, \overline{z}) - \frac{1}{2} (\partial \ln \partial Z(z, \overline{z}))^2 \equiv \left\{ Z(z, \overline{z}), z \right\}
\]

\[
= \frac{\delta^3 Z(z, \overline{z})}{\partial Z(z, \overline{z})} - \frac{3}{2} \left( \frac{\delta^2 Z(z, \overline{z})}{\partial Z(z, \overline{z})} \right)^2.
\]

If we introduce a \(W\)-invariant background connection \(\hat{A}_z (z, \overline{z})\), we can build a \(W\) invariant projective connection \(\hat{T}(z, \overline{z})\):

\[
\delta_W \hat{Z}(z, \overline{z}) = \delta_W \hat{T}(z, \overline{z}) = 0.
\]

So our Eq(3.53) as a trick yields a partial compensation of the anomaly by means of well defined counterterms:

\[
\delta_W \int dz \wedge d\overline{z} \mu(z, \overline{z}) \left( T(z, \overline{z}) - \hat{T}(z, \overline{z}) \right) = \int dz \wedge d\overline{z} \left[ \frac{1}{2} \left( \mu^2 \mu(z, \overline{z}) L_3 C(z, \overline{z}) - C(z, \overline{z}) L_3 \mu(z, \overline{z}) \right) + \left( \delta W \mu(z, \overline{z}) \right) \left( T(z, \overline{z}) - \hat{T}(z, \overline{z}) \right) \right].
\]

Now, if we define a new \(\Gamma_{Tree}\) action:

\[
\Gamma_{Tree} = \Gamma_S - 2 \hbar \int dz \wedge d\overline{z} \mu(z, \overline{z}) \left( T(z, \overline{z}) - \hat{T}(z, \overline{z}) \right),
\]

our procedure allows to define a new quantum action \(\Gamma_Q\), as in Eq(3.54), which gives a new anomalous Ward identity (at zero external Classical currents):

\[
\left( \frac{\partial Z(z, \overline{z})}{\delta Z(z, \overline{z})} + \frac{(\partial - \mu^2) \overline{\partial} X(z, \overline{z})}{1 - \mu^2(z, \overline{z})} \right) \Gamma_Q
\]

\[
\left. = - \hbar \left[ \overline{\partial} - \mu(z, \overline{z}) \partial \right] \left( T(z, \overline{z}) - \hat{T}(z, \overline{z}) \right). \right \} \quad \text{[C.96]}
\]

The \(Z(z, \overline{z})\) tadpole term is zero at zero string v.e.v. and at \(< T(z, \overline{z}) >= \hat{T}(z, \overline{z})\):

\[
\left. \frac{\delta \Gamma_Q}{\delta Z(z, \overline{z})} \right|_{\ T = \hat{T}} = \left. \frac{- \hbar}{\partial Z(z, \overline{z})} \left( \overline{\partial} - \mu(z, \overline{z}) \partial \right) \left( T(z, \overline{z}) - \hat{T}(z, \overline{z}) \right) \right|_{\ T = \hat{T}} = 0.
\]

But a non-trivial \(< Z(z, \overline{z}) Z(z', \overline{z}') >\) two-point function can be derived at the first order in \(\hbar\). The results shown in Eq(3.74) and(3.75) can be verified. Note that if \(\frac{\delta T(z, \overline{z})}{\delta Z(z, \overline{z})}\) is an arbitrary function of \(Z(z - z', \overline{z} - \overline{z}')\) then the propagator and the commutator are zero.

Higher order Green functions can be derived iterating the functional derivation process, they appear at the \(\hbar\) order and are non-zero.
The correspondence law between the ghosts $K_s(z, \overline{z})$ and $\mathcal{C}_s(z, \overline{z})$ for the $\mathcal{W}_3$-algebra can be found in several papers [33, 39]:

$$K_{1,1}(z, \overline{z}) = \frac{8}{3} \alpha_{2}^{(3)}(z, \overline{z}) C^{(2)}(z, \overline{z}) - \partial \mathcal{C}(z, \overline{z}) + \frac{1}{3} \partial^{2} \mathcal{C}^{(2)}(z, \overline{z}),$$

(C.98)

$$K_{1,2}(z, \overline{z}) = C(z, \overline{z}) - \partial \mathcal{C}^{(2)}(z, \overline{z}),$$

$$K_{1,3}(z, \overline{z}) = 2 \mathcal{C}^{(2)}(z, \overline{z}).$$

And the $Z^{(r)}(z, \overline{z}) = 1, 2$ coordinate transformations can be written [37]:

$$\delta_{\mathcal{W}} Z^{(1)}(z, \overline{z}) = C^{(1)}(z, \overline{z}) \partial Z^{(1)}(z, \overline{z}) + 2 \mathcal{C}^{(2)}(z, \overline{z}) \partial^{2} Z^{(1)}(z, \overline{z})$$

$$- \partial \mathcal{C}^{(2)}(z, \overline{z}) \partial Z^{(1)}(z, \overline{z}) - \frac{4}{3} \mathcal{C}^{(2)}(z, \overline{z}) \partial Z^{(1)}(z, \overline{z}) \partial \ln w(z, \overline{z}),$$

(C.99)

$$\delta_{\mathcal{W}} Z^{(2)}(z, \overline{z}) = C^{(1)}(z, \overline{z}) \partial Z^{(2)}(z, \overline{z}) + 2 \mathcal{C}^{(2)}(z, \overline{z}) \partial^{2} Z^{(2)}(z, \overline{z})$$

$$- \partial \mathcal{C}^{(2)}(z, \overline{z}) \partial Z^{(2)}(z, \overline{z}) - \frac{4}{3} \mathcal{C}^{(2)}(z, \overline{z}) \partial Z^{(2)}(z, \overline{z}) \partial \ln w(z, \overline{z}),$$

(C.100)

where we have indicated the determinants [6]

$$w(z, \overline{z}) = \begin{vmatrix} \partial Z^{(1)}(z, \overline{z}) & \partial Z^{(2)}(z, \overline{z}) \\ \partial^{2} Z^{(1)}(z, \overline{z}) & \partial^{2} Z^{(2)}(z, \overline{z}) \end{vmatrix},$$

$$v(z, \overline{z}) = \begin{vmatrix} \partial^{2} Z^{(1)}(z, \overline{z}) & \partial^{2} Z^{(2)}(z, \overline{z}) \\ \partial^{3} Z^{(1)}(z, \overline{z}) & \partial^{3} Z^{(2)}(z, \overline{z}) \end{vmatrix}. $$

The arguments already given in Eq(2.28) allow to write the $\overline{\mathcal{D}}$ derivatives of the coordinates as:

$$\overline{\mathcal{D}} Z^{(1)}(z, \overline{z}) = \mu_{z}^{(1)}(z, \overline{z}) \partial Z^{(1)}(z, \overline{z}) + 2 \mu_{z}^{(2)}(z, \overline{z}) \partial^{2} Z^{(1)}(z, \overline{z})$$

$$- \partial \mu_{z}^{(2)}(z, \overline{z}) \partial Z^{(1)}(z, \overline{z}) - \frac{4}{3} \mu_{z}^{(2)}(z, \overline{z}) \partial Z^{(1)}(z, \overline{z}) \partial \ln w(z, \overline{z}),$$

$$\overline{\mathcal{D}} Z^{(2)}(z, \overline{z}) = \mu_{z}^{(1)}(z, \overline{z}) \partial Z^{(2)}(z, \overline{z}) + 2 \mu_{z}^{(2)}(z, \overline{z}) \partial^{2} Z^{(2)}(z, \overline{z})$$

$$- \partial \mu_{z}^{(2)}(z, \overline{z}) \partial Z^{(2)}(z, \overline{z}) - \frac{4}{3} \mu_{z}^{(2)}(z, \overline{z}) \partial Z^{(2)}(z, \overline{z}) \partial \ln w(z, \overline{z}),$$

$$\mu_{z}^{(2)}(z, \overline{z}) = \frac{\partial \mathcal{C}^{(2)}(z, \overline{z})}{\partial c^{(0,1)}(z, \overline{z})},$$

$$\mu_{z}^{(1)}(z, \overline{z}) = \frac{\partial C^{(1)}(z, \overline{z})}{\partial c^{(0,1)}(z, \overline{z})},$$

(C.102)

so the Beltrami multipliers can be written as:

$$\mu_{z}^{(1)}(z, \overline{z}) = \overline{\mathcal{D}} Z^{(1)}(z, \overline{z}) = \mu_{z}^{(1)}(z, \overline{z}) + 2 \mu_{z}^{(2)}(z, \overline{z}) \frac{\partial^{2} Z^{(1)}(z, \overline{z})}{\partial Z^{(1)}(z, \overline{z})}$$

$$- \partial \mu_{z}^{(2)}(z, \overline{z}) - \frac{4}{3} \mu_{z}^{(2)}(z, \overline{z}) \partial \ln w(z, \overline{z}),$$

(C.103)

$$\mu_{z}^{(2)}(z, \overline{z}) = \overline{\mathcal{D}} Z^{(2)}(z, \overline{z}) = \mu_{z}^{(1)}(z, \overline{z}) + 2 \mu_{z}^{(2)}(z, \overline{z}) \frac{\partial^{2} Z^{(2)}(z, \overline{z})}{\partial Z^{(2)}(z, \overline{z})}$$

$$- \partial \mu_{z}^{(2)}(z, \overline{z}) - \frac{4}{3} \mu_{z}^{(2)}(z, \overline{z}) \partial \ln w(z, \overline{z}),$$

(C.104)
and the Bilal-Fock-Kogan coefficients can be represented in terms of the target Laguerre-Forsyth coordinates as:

\[
\mu_2(z, \overline{z}) = \frac{\left(\frac{\partial Z^2(z, \overline{z})}{\partial Z(z, \overline{z})} - \frac{\partial Z^1(z, \overline{z})}{\partial Z(z, \overline{z})}\right)}{2 \left(\frac{\partial^2 Z^2(z, \overline{z})}{\partial Z^2(z, \overline{z})} - \frac{\partial^2 Z^1(z, \overline{z})}{\partial Z^1(z, \overline{z})}\right)}
\]

(C.105)

\[
\mu_3(z, \overline{z}) = \frac{\partial Z^1(z, \overline{z})}{\partial Z(z, \overline{z})} - 2\mu_2^2(z, \overline{z}) \frac{\partial Z^1(z, \overline{z})}{\partial Z^1(z, \overline{z})} + \partial \mu_2(z, \overline{z}) + \frac{4}{3} \mu_2^2(z, \overline{z}) \partial \ln w(z, \overline{z})
\]

(C.106)

(we point out to the reader’s attention that on the real axis \( z = \overline{z} \) we have \( \mu_2(z, \overline{z}) = 0 \) and \( \mu_1(z, \overline{z}) = 1 \)). The anomaly Eq.(3.49) can be written in the \( \mathcal{W}(3) \) case as [40]:

\[
\delta \mathcal{W}\Gamma_s = \int \Delta_2(z, \overline{z}) = \int \left\{ \left( C(z, \overline{z}) L_3 \mu_2(z, \overline{z}) - \mu_3(z, \overline{z}) L^3 C(z, \overline{z}) \right) - \frac{1}{3} \left( C^2(z, \overline{z}) L_5 \mu_2(z, \overline{z}) - \mu_3(z, \overline{z}) L^5 C^2(z, \overline{z}) \right) \right. \\
- 8 \left( C^1(z, \overline{z}) \mu_2(z, \overline{z}) - \mu_3(z, \overline{z}) C(z, \overline{z}) \right) \partial \mathcal{W}(z, \overline{z}) \\
- 24 \mathcal{W}(z, \overline{z}) \left( C^1(z, \overline{z}) \partial \mu_2(z, \overline{z}) - \mu_3(z, \overline{z}) \partial C(z, \overline{z}) \right) \\
\left. + C^2(z, \overline{z}) \partial \mu_2(z, \overline{z}) - \mu_3(z, \overline{z}) \partial C^2(z, \overline{z}) \right\} dz \wedge d\overline{z},
\]

(C.107)

where the fifth order \( L_5 \) Bol operator reads:

\[
L_5 = \partial^5 + 10 T(z, \overline{z}) \partial^3 + 15 (\partial T(z, \overline{z})) \partial^2 + \left[ 9 (\partial T(z, \overline{z})) + 16 T^2(z, \overline{z}) \right] \partial \\
+ 2 \left[ (\partial^3 T(z, \overline{z})) + 8 T(z, \overline{z}) (\partial T(z, \overline{z})) \right].
\]

(C.108)

Anyhow in this case the projective connection \( T(z, \overline{z}) \) is written in terms of the coordinates as:

\[
T(z, \overline{z}) = \frac{1}{2} (\partial^2 \ln w(z, \overline{z})) - \frac{1}{3} (\partial \ln w(z, \overline{z}))^2 + \frac{\nu(z, \overline{z})}{w(z, \overline{z})},
\]

(C.109)

while the spin 3 tensor, \( \mathcal{W}(z, \overline{z}) \), takes the expression:

\[
\mathcal{W}(z, \overline{z}) = \frac{1}{24} \left( \frac{1}{2} (\partial^3 \ln w(z, \overline{z})) - \partial^2 \ln w(z, \overline{z}) \partial \ln w(z, \overline{z}) - \frac{2}{9} (\partial \ln w(z, \overline{z}))^3 \right) \\
+ \frac{1}{16} \left( \frac{\partial \nu(z, \overline{z})}{w(z, \overline{z})} - \frac{5 \nu(z, \overline{z})}{3 w(z, \overline{z})} \partial \ln w(z, \overline{z}) \right).
\]

(C.110)

The partial counterterm compensation trick of Eq.(3.53) can be repeated:

\[
\delta \mathcal{W} \int dz \wedge d\overline{z} \left[ \left( \mu_2^2(z, \overline{z}) - \partial \mu_2(z, \overline{z}) \right) \left( T(z, \overline{z}) - \frac{\partial}{\partial T} T(z, \overline{z}) \right) \right]
\]
\[ + 8 \mu^{(2)}_\tau(z, \tau) \left( W(z, \tau) - \mathcal{W}(z, \tau) \right) = \]
\[ \int \Delta^1_2(z, \tau) dz \wedge d\tau + \int dz \wedge d\tau \left[ \delta W \left( \mu^{\tau}_\tau(z, \tau) - \partial \mu^{(2)}_\tau(z, \tau) \right) \left( T(z, \tau) - \tilde{T}(z, \tau) \right) \right. \]
\[ + 8 \left( \delta W \mu^{(2)}_\tau(z, \tau) \right) \left( W(z, \tau) - \mathcal{W}(z, \tau) \right) \]  
(C.111)

(Where \( \tilde{T}(z, \tau) \) and \( \mathcal{W}(z, \tau) \) are the components of the background matrix \( \mathcal{A}_\tau(z, \tau) \) twined to the currents \( T(z, \tau) \) and \( W(z, \tau) \) respectively).

So we define a new quantum Action \( \Gamma_Q \) as in Eq(3.56) by
\[
\Gamma_Q = \Gamma_S - \hbar \int dz \wedge d\tau \left[ \left( \mu^{\tau}_\tau(z, \tau) - \partial \mu^{(2)}_\tau(z, \tau) \right) \left( T(z, \tau) - \tilde{T}(z, \tau) \right) \right. \]
\[ + 8 \mu^{(2)}(z, \tau) \left( W(z, \tau) - \mathcal{W}(z, \tau) \right) \]  
(C.112)

with anomaly:
\[
\delta W \Gamma_Q = \int dz \wedge d\tau \left[ \delta W \left( \mu^{\tau}_\tau(z, \tau) - \partial \mu^{(2)}_\tau(z, \tau) \right) \left( T(z, \tau) - \tilde{T}(z, \tau) \right) \right. \]
\[ + 8 \left( \delta W \mu^{(2)}(z, \tau) \right) \left( W(z, \tau) - \mathcal{W}(z, \tau) \right) \].  
(C.113)

The Ward identities can be obtained if we differentiate with respect \( C^{(1)}(z, \tau) \), and \( C^{(2)}(z, \tau) \); so if we put \( < \mathcal{X} > = 0 \) we get respectively:
\[
\left. \left( \partial Z^{(1)}(z, \tau) \frac{\delta \Gamma}{\delta Z^{(1)}(z, \tau)} + \partial Z^{(2)}(z, \tau) \frac{\delta \Gamma}{\delta Z^{(2)}(z, \tau)} \right) \right|_{< \mathcal{X} > = 0}
= \frac{\delta}{\delta C^{(1)}(z, \tau)} \left\{ \int d\tau' \wedge d\tau'' \left[ \delta W \left( \mu^{\tau'}(\tau', \tau'') - \partial \mu^{(2)}(\tau', \tau'') \right) \left( T(\tau', \tau'') - \mathcal{W}(\tau', \tau'') \right) \right. \right. \]
\[ + 8 \left( \delta W \mu^{(2)}(\tau', \tau'') \right) \left( W(\tau', \tau'') - \mathcal{W}(\tau', \tau'') \right) \left. \left. \right|_{< \mathcal{X} > = 0} \right\} \]  
(C.114)

and
\[
\left. \left[ \left( 3 \partial^2 Z^{(1)}(z, \tau) + \partial Z^{(1)}(z, \tau) \partial - \frac{4}{3} \partial Z^{(1)}(z, \tau) \partial \ln w(z, \tau) \right) \frac{\delta \Gamma}{\delta Z^{(1)}(z, \tau)} \right. \right. \]
\[ + \left. \left. \left( 3 \partial^2 Z^{(2)}(z, \tau) + \partial Z^{(2)}(z, \tau) \partial - \frac{4}{3} \partial Z^{(2)}(z, \tau) \partial \ln w(z, \tau) \right) \frac{\delta \Gamma}{\delta Z^{(2)}(z, \tau)} \right|_{< \mathcal{X} > = 0} \right. \]
= \frac{\delta}{\delta C^{(2)}(z, \tau)} \left\{ \int d\tau' \wedge d\tau'' \left[ \delta W \left( \mu^{\tau'}(\tau', \tau'') - \partial \mu^{(2)}(\tau', \tau'') \right) \left( T(\tau', \tau'') - \mathcal{W}(\tau', \tau'') \right) \right. \right. \]
\[ + 8 \left( \delta W \mu^{(2)}(\tau', \tau'') \right) \left( W(\tau', \tau'') - \mathcal{W}(\tau', \tau'') \right) \left. \left. \right|_{< \mathcal{X} > = 0} \right\} \]  
27
Substituting the functional derivative \( \frac{\delta \Gamma}{\delta Z^{(1)}(z, \bar{z})} \) in Eq (C.115) from Eq (C.114), surprisingly gives rise to after some algebra, an algebraic (instead of differential) equation for \( \frac{\delta \Gamma}{\delta Z^{(1)}(z, \bar{z})} \), namely

\[
\frac{\delta \Gamma}{\delta Z^{(1)}(z, \bar{z})}_{<\mathcal{X}> = 0} = \frac{h}{3 \left( \frac{\delta^2 Z^{(1)}(z, \bar{z})}{\delta Z^{(1)}(z, \bar{z})} - \frac{\partial Z^{(1)}(z, \bar{z})}{\partial Z^{(2)}(z, \bar{z})} \right) \partial^2 Z^{(2)}(z, \bar{z})} - \partial Z^{(2)}(z, \bar{z}) \partial \left( \frac{\partial Z^{(1)}(z, \bar{z})}{\partial Z^{(2)}(z, \bar{z})} \right)
\]

\[
\int d\zeta' \wedge d\bar{\zeta}' \left\{ \frac{\delta}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \left( \frac{\delta W^{(2)}(\zeta', \bar{\zeta}')}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \right) \right\} - \frac{4}{3} \partial Z^{(2)}(z, \bar{z}) \partial \ln w(z, \bar{z}) \left\{ \frac{1}{\partial Z^{(2)}(z, \bar{z})} \frac{\delta}{\delta \mathcal{C}^{(1)}(z, \bar{z})} \left( \frac{\delta W^{(2)}(\zeta', \bar{\zeta}')}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \right) \right\}
\]

\[
\left( \mathcal{T}(\zeta', \bar{\zeta}') - \mathcal{\tilde{T}}(\zeta', \bar{\zeta}') \right) + 8 \frac{\delta}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \left( \frac{\delta W^{(2)}(\zeta', \bar{\zeta}')}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \right) \left( \mathcal{W}(\zeta', \bar{\zeta}') - \mathcal{\tilde{W}}(\zeta', \bar{\zeta}') \right)
\]

(C.116)

and from Eq (C.114) it is possible to find a complicated expression for \( \frac{\delta \Gamma}{\delta Z^{(1)}(z, \bar{z})} \).

These tadpoles are zero for \( \mathcal{T}(z, \bar{z}) = \mathcal{\tilde{T}}(z, \bar{z}) \), and \( \mathcal{W}(z, \bar{z}) = \mathcal{\tilde{W}}(z, \bar{z}) \), but give rise to non zero Green functions when further functional derivatives are performed. One has

\[
\frac{\delta \Gamma}{\delta Z^{(1)}(z', \bar{z}'') \delta Z^{(1)}(z, \bar{z})} \left\{ \begin{array}{c}
\mathcal{T} = \mathcal{\tilde{T}} \\
\mathcal{W} = \mathcal{\tilde{W}} \\
<\mathcal{X}> = 0
\end{array} \right. = \frac{h}{3 \left( \frac{\delta^2 Z^{(1)}(z, \bar{z})}{\delta Z^{(1)}(z, \bar{z})} - \frac{\partial Z^{(1)}(z, \bar{z})}{\partial Z^{(2)}(z, \bar{z})} \right) \partial^2 Z^{(2)}(z, \bar{z})} - \partial Z^{(2)}(z, \bar{z}) \partial \left( \frac{\partial Z^{(1)}(z, \bar{z})}{\partial Z^{(2)}(z, \bar{z})} \right)
\]

\[
\int d\zeta' \wedge d\bar{\zeta}' \left\{ \frac{\delta}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \left( \frac{\delta W^{(2)}(\zeta', \bar{\zeta}')}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \right) \right\} - \frac{4}{3} \partial Z^{(2)}(z, \bar{z}) \partial \ln w(z, \bar{z}) \left\{ \frac{1}{\partial Z^{(2)}(z, \bar{z})} \frac{\delta}{\delta \mathcal{C}^{(1)}(z, \bar{z})} \left( \frac{\delta W^{(2)}(\zeta', \bar{\zeta}')}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \right) \right\}
\]

\[
\times \left( \mathcal{T}(\zeta', \bar{\zeta}') - \mathcal{\tilde{T}}(\zeta', \bar{\zeta}') \right) + 8 \frac{\delta}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \left( \frac{\delta W^{(2)}(\zeta', \bar{\zeta}')}{\delta \mathcal{C}^{(2)}(z, \bar{z})} \right) \left( \mathcal{W}(\zeta', \bar{\zeta}') - \mathcal{\tilde{W}}(\zeta', \bar{\zeta}') \right)
\]

(C.116)
Now if we exchange the functional derivative order sequence \((z, \bar{z}) \rightarrow (z'', \bar{z}'')\) we can differentiate, with easy derivable (but rather complicated) expressions, the noncommutative character of the complex coordinates considered as quantum fields.

References


