Confinement without a center: the exceptional group $G(2)$

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We discuss theories with the exceptional centerless gauge group $G(2)$, paying attention to confinement and the pattern of chiral symmetry breaking. Exploiting the Higgs mechanism to break the symmetry down to $SU(3)$, we also present how the familiar features of confinement and chiral symmetry breaking of $SU(3)$ gauge theories reemerge. $G(2)$ gauge theories show up as an unusual theoretical framework to study $SU(3)$ gauge theories without the “luxury” of a center.

1. Introduction and motivations

In the last few years, accumulating numerical evidence of the relevance of center vortices in the effective mechanism of confinement in non-Abelian gauge theories has been collected. Center vortices and twist sectors [1] are present in a pure gauge theory with symmetry group $G$ if $\Pi_1(G/\text{center}(G)) \neq 0$, $\Pi_1$ being the first homotopy group. The exceptional group $G(2)$ is the simplest group without center vortices and twist sectors. Thus, it is interesting to investigate how confinement can show up in a theory with such a gauge group. Moreover, a property making $G(2)$ particularly interesting is that it has $SU(3)$ as a subgroup. In our study we have focused our attention on the way confinement and the pattern of chiral symmetry breaking show up in $G(2)$ gauge theories and how they change into the more familiar $SU(3)$ case as the symmetry gets broken.

2. $G(2)$: basic generalities

$G(2)$ is a subgroup of $SO(7)$. Its fundamental representation (rep) $\{7\}$ is 7 dimensional and a matrix $\Omega$ satisfies the following constraints:

\begin{align}
\det \Omega &= 1; \quad \delta_{ab} = \delta_{a'b'} \Omega_{aa'} \Omega_{bb'} \\
T_{abc} &= T_{a'b'c'} \Omega_{aa'} \Omega_{bb'} \Omega_{cc'}; \quad T = \text{antisym}. \quad (2)
\end{align}

In addition to the two defining $SO(7)$ properties (1), $G(2)$ leaves invariant a completely antisymmetric three-index tensor $T$ and is generated by 14 of the 21 $SO(7)$ generators. $G(2)$ has rank 2 and so its reps can be drawn on a plane. For instance, this is the diagram of the fundamental one $\{7\}$:

![Figure 1. The weight diagram of the 7-dimensional fundamental representation of $G(2)$.

$G(2)$ has $SU(3)$ as a subgroup in the real reducible rep $\{3\} \oplus \{\bar{3}\} \oplus \{1\}$. In a suitable basis, 8 of the 14 $G(2)$ generators can be written in the

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following way

$$\Lambda_a = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} \lambda_a & 0 & 0 \\ 0 & -\lambda_a^* & 0 \\ 0 & 0 & 0 \end{array} \right).$$

(3)

where $\lambda_a$, $a = 1, \ldots, 8$ are the usual $3 \times 3$ Gell–Mann matrices. The group $G(2)$ inherits from $SO(7)$ the property of having real reps and so $G(2)$ “quarks” are equivalent to $G(2)$ “anti-quarks”. $G(2)$ has a trivial center and there is no feature similar to “$N$–ality”, holding for $SU(N)$ groups. All $G(2)$ reps mix together and, consequently, a heavy $G(2)$ “quark” can be screened by $G(2)$ “gluons”. Thus, the string breaks already in the pure glue sector without the need for dynamical $G(2)$ “quarks”. Finally, similarly to $SU(3)$, in $G(2)$ Yang–Mills theory there are instantons — $\Pi_3(G(2)) = \mathbb{Z}$ — and two kinds of monopoles — $\Pi_2(G(2)/U(1)^2) = \mathbb{Z}^2$.

3. $G(2)$ Yang–Mills

Let us first consider $G(2)$ Yang–Mills (YM) theory. There are 14 $G(2)$ “gluons” which transform according to the adjoint rep $\{14\}$. Restricting to the $SU(3)$ subgroup, this rep is reducible and splits up into the sum $\{8\} \oplus \{3\} \oplus \{\bar{3}\}$. Thus, 8 of the 14 $G(2)$ “gluons” are related among themselves as gluons and the remaining 6 $G(2)$ “gluons” form a triplet and an antitriplet. These last 6 $G(2)$ “gluons” behave like quarks and antiquarks w.r.t. the $SU(3)$ color degrees of freedom. The $G(2)$ YM theory is asymptotically free and we expect it to be confining in the low energy regime but with a vanishing string tension (defined as the slope of the heavy quark potential when the distance goes to infinity), since the string breaks by pair production of dynamical $G(2)$ “gluons”. So, contrary to $SU(3)$ YM theory, in $G(2)$ YM theory we expect confinement to resemble more closely that in QCD but without the complications related to dynamical fermions. Due to the screening of $G(2)$ “quarks” by $G(2)$ “gluons”, the Wilson loop is not a good order parameter for confinement. At $T = 0$, one can consider the Fredenhagen–Marcu operator $[2]$ as an order parameter for confinement. By strong coupling computation, one obtains confining behaviour in this regime. $SU(N)$ YM theory has a deconfinement phase transition at finite temperature. Confined and deconfined phases differ by the way the center symmetry is realized. In $G(2)$ YM, since the center is trivial, it is unclear how to define an order parameter to investigate the issue of a finite temperature phase transition.

4. $G(2)$ Yang–Mills + Higgs $\{7\}$

Let us now break the $G(2)$ gauge symmetry to $SU(3)$. This can be accomplished by adding to the $G(2)$ YM theory a Higgs field in the fundamental rep $\{7\}$ of $G(2)$. The 8 $G(2)$ “gluons” related among themselves as gluons stay massless while the remaining 6 get a mass proportional to the v.e.v. of the Higgs field. If the mass of these $G(2)$ “gluons” is not too high, they participate in the dynamics but, as the v.e.v. of the Higgs field increases, they progressively decouple and, in the end, we are left with an $SU(3)$ gauge theory. Thus, a Higgs field in the rep $\{7\}$ gives us a handle to smoothly interpolate between $G(2)$ and $SU(3)$. The breaking of the string between two heavy $G(2)$ “quarks” happens for the pair production of these 6 massive $G(2)$ “gluons” and so the breaking scale is related to their mass. As this mass increases, the breaking scale gets larger as well. When the 6 massive $G(2)$ “gluons” completely decouple, it is sent to infinity and we recover the familiar picture of the unbreakable $SU(3)$ string.

5. $G(2)$ QCD

Let us now add to the $G(2)$ YM theory $N_f$ flavours of Majorana fermions in the fundamental rep $\{7\}$. In this way, we obtain a theory like QCD but with gauge group $G(2)$. As above, we will exploit the Higgs mechanism to smoothly interpolate between $G(2)$ and $SU(3)$. When we break the $G(2)$ symmetry to $SU(3)$, the rep $\{7\}$ reduces to the sum of $\{3\}$, $\{\bar{3}\}$ and a color singlet. Thus, reexpressing the Majorana degrees of freedom in
the following way
\[
\begin{pmatrix}
\psi_M^{(1)} \\
\psi_M^{(2)} \\
\psi_M^{(3)} \\
\psi_M^{(4)} \\
\psi_M^{(5)} \\
\psi_M^{(6)} \\
\psi_M^{(7)}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\psi_M^{(1)} = \psi_M^{(1)} + i\psi_M^{(4)} \\
\psi_M^{(2)} = \psi_M^{(2)} + i\psi_M^{(5)} \\
\psi_M^{(3)} = \psi_M^{(3)} + i\psi_M^{(6)} \\
c\psi_M^{(1)} = \psi_M^{(1)} - i\psi_M^{(4)} \\
c\psi_M^{(2)} = \psi_M^{(2)} - i\psi_M^{(5)} \\
c\psi_M^{(3)} = \psi_M^{(3)} - i\psi_M^{(6)} \\
\psi_M^{(7)} = \psi_M^{(7)}
\end{pmatrix}
\] (4)

we see that, by the Higgs mechanism, we are interpolating between a $G(2)$ QCD-like theory with $N_f$ Majorana fermions and QCD with $N_f$ Dirac quark flavours plus one Majorana fermion. However, this last particle is an $SU(3)$ colour singlet and so does not feel the $SU(3)$ strong interactions. We now consider the issue of the pattern of chiral symmetry breaking in $G(2)$ QCD and how this pattern interpolates to that of QCD as the v.e.v. of the Higgs field becomes large. Let us discuss separately the cases $N_f = 1$ and $N_f \geq 2$.

• $N_f = 1$. Consider first the case of one flavour. In QCD the baryon number symmetry is $U(1)_{L=R}$. It stays unbroken and there is no Goldstone particle. In $G(2)$ QCD, due to the reality of the $G(2)$ reps, $G(2)$ “quarks” and $G(2)$ “anti-quarks” are indistinguishable. Left ($L$) and right ($R$) components do not transform independently but $L = R^*$. So the baryonic $U(1)_{L=R}$ symmetry of QCD becomes $U(1)_{L=R=R^*} = Z_B(2)$ in $G(2)$ QCD and the number of $G(2)$ “quarks” is conserved only modulo two. Thus we have two kinds of states: those with an odd (uuGGG, uu, ...) and those with an even (uu, ...) number of $G(2)$ “quarks” u, bound or not with $G(2)$ “gluons” G. If we now add the Higgs field to the dynamics, 6 $G(2)$ “gluons” become massive and one can show that the states uGGG start to become heavy. The mixing between $G(2)$ “quarks” and $G(2)$ “anti-quarks” – which is mediated by the massive $G(2)$ “gluons” – becomes weaker and baryon number violating processes are rare. Then the $U(1)_{L=R} = U(1)_B$ symmetry of QCD reemerges as an approximate symmetry, becoming exact when the 6 massive $G(2)$ “gluons” decouple from the dynamics.

• $N_f \geq 2$. Let us now consider the case of two or more flavours. The Abelian part of the chi-ral symmetry is that discussed in the $N_f = 1$ case, so in the following we will only take into account the non–Abelian part. In QCD the pattern of chiral symmetry breaking is $SU(N_f)_L \otimes SU(N_f)_R \rightarrow SU(N_f)_{L=R}$ with $(N_f^2 - 1)$ Goldstone bosons. Again, in $G(2)$ QCD, left and right components are not independent and are related by $L = R^*$. So the unbroken chiral symmetry is $SU(N_f)_{L=R^*}$ which breaks down to the vector subgroup $SU(N_f)_{L=R=R^*} = SO(N_f)$. As a consequence, there are $N_f(N_f + 1)/2 - 1$ Goldstone bosons, one for every broken generator. As before, we add a Higgs field in the fundamental rep $\{7\}$ in order to smoothly interpolate between $G(2)$ QCD and QCD. It is now better to start from the QCD case – i.e. we assume a very large v.e.v. for the Higgs field – and move to $G(2)$ QCD by decreasing this value. $(N_f - 1)$ of the $(N_f^2 - 1)$ Goldstone bosons of QCD are self-conjugate (e.g. $\pi^0$ for $N_f = 2$) and the remaining $2[N_f(N_f + 1)/2 - 1]$ ones are pairwise conjugate (e.g. $\pi^\pm$ for $N_f = 2$). One can consider linear combinations of these pairwise conjugated particles to build up states which are even (e.g. $\pi^+ + \pi^-$) and odd (e.g. $\pi^+ - \pi^-$) under charge conjugation. The odd states are invariant under $SO(N_f)$ and so acquire a mass as the v.e.v. of the Higgs field becomes smaller and smaller, that is as the broken symmetry reduces from $SU(N_f)_{L=R} \rightarrow SU(N_f)_{L=R=R^*}$. The other $(N_f - 1) + N_f(N_f - 1)/2 = N_f(N_f + 1)/2 - 1$ ones stay instead massless.

6. Conclusions

We have studied $G(2)$ gauge field theories with and without fermions. We have focused our attention on confinement and the pattern of chiral symmetry breaking. Adding a Higgs field in the fundamental rep $\{7\}$, we can smoothly interpolate between $G(2)$ and $SU(3)$. In particular, we have discussed how the familiar confinement and pattern of chiral symmetry breaking of $SU(3)$ gauge theories reemerge as the $G(2)$ gauge symmetry gets broken. In conclusion, we have considered $G(2)$ theories as a theoretical laboratory to study $SU(3)$ gauge theories in an unusual context and without the “luxury” of a center. In a forthcoming paper, we will report on our study with more
details. In that paper, we will also discuss the $G(2)$ YM theory with 1 Majorana fermion flavour in the adjoint rep $\{14\}$ (SUSY–$G(2)$) and present analytic results in the strong coupling approximation to support our heuristic investigations.

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