ON THE MASS OF A UNIFORM DENSITY STAR IN HIGHER DIMENSIONS

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Abstract

In the framework of higher dimensions the mass of a uniform density star is evaluated. The four dimensional upper bound for mass to radius ratio obtained by Schwarzschild is generalised in the framework of higher dimensional spacetime. It is found that the analog upper bound for mass to radius ratio in higher dimensions tends to increase at first as the number of dimensions of space-time increases, it attains a maximum at nine dimensions thereafter decreases. It is found that $D = 4$ is the lowest space-time dimensions for which mass to radius ratio of a uniform density star can be derived.

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The idea that space-time dimensions should be extended from four to higher dimensions came from the work of Kaluza and Klein [1] who first tried to unify gravity with electromagnetism. The approach has been revived and considerably generalized after realizing that many interesting theories of particle interactions need spacetime dimensions more than four for their formulation. For example, Superstring theory is considered as one of the promising candidate which may unify gravity with the other fundamental forces in nature which requires 10 dimensions for its consistent formulation. The present ideas in dimensional reduction suggest that our cosmos may be a 3-brane evolves in a D-dimensional space-time. Cadeau and Woolgar [2] recently addressed this issue in the context of black holes which led to a homogeneous but non-FRW-braneworld cosmologies. There has been a growing interest in recent years to obtain a higher dimensional analog of a 4 dimensional general relativistic result. Several work in the literature appeared including the higher dimensional generalisation of the spherically symmetric Schwarzschild and Reisner Nordstrom black holes [3,4], Kerr black holes [5], Vaidya solution [6] etc.. Mandelbrot [7] studied the problem on the variability of dimensions in which he describes how a ball of thin thread is seen as an observer changes scale. An object which seems to be a point object from a far point becomes a 3-dimensional ball at a closer distance. Thus as an observer moves down through various scales the ball appears to change shape. While the embedding dimensions for the ball has not changed, the effective dimension of the contents does change. It is possible that there are compact [8] or non-compact [9] dimensions present at a certain point. At this scale, the $(3 + 1)$ metric is simply not true although one obtains a valid description with general relativity. Liu *et al* [10]

reported solar system tests based on a 5-dimensional extension of the Schwarzschild metric and Cassisi *et al* [11] have examined the effects of higher dimensions on stellar evolution. Yu and Ford [8] reported that observable effects of higher dimensions may be found from light cone fluctuations. Günther and Zhuk [12] investigated the observable consequences of spontaneous compactification hypothesis for the extra dimension. At present dimensional physics has become an active area of investigation with some promise of future experimental insights [13].

In this paper I intend to obtain namely a higher dimensional generalisation of the Schwarzschild mass formula in 4 dimensional space-time for a uniform density static star. We consider a star in higher dimensional universe which is in hydrostatic equilibrium to obtain the mass and radius relation for a stable configuration. The star under study is a D -dimensional ball of uniform density.

The Einstein's field equation in higher dimensions is given by

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_D T_{ab} \quad (1)$$

where D is the total number of dimensions, $G_D = G.V_{D-4}$ is the gravitational constant in D dimensions, G denotes the 4 dimensional Newton's constant and V_{D-4} is the volume of the extra dimensions, R_{ab} is Ricci tensor and T_{ab} is the energy momentum tensor. We consider a higher dimensional spherically symmetric static space-time in the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\mu(r)}dr^2 + f^2(r)d\Omega_n^2 \quad (2)$$

where $n = D - 2$ and $d\Omega_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_2(d\theta_3^2 + \dots + \sin^2\theta_{n-1}d\theta_n^2)$ represents the metric on the n -sphere in polar coordinates. The energy momentum

tensor is $T_{ab} = (\rho, -p, -p, -p, -p, \dots, -p)$. The field eq.(1) then reduces to the study of the following three equations using $f(r) = r$:

$$\frac{ne^{-2\mu}\mu'}{r} + \frac{n(n-1)}{2} \left(\frac{1 - e^{-2\mu}}{r^2} \right) = 8\pi G_D \rho, \quad (3)$$

$$\frac{ne^{-2\mu}\nu'}{r} - \frac{n(n-1)}{2} \left(\frac{1 - e^{-2\mu}}{r^2} \right) = 8\pi G_D p, \quad (4)$$

$$e^{-2\mu} \left(\nu'' + \nu'^2 - \mu'\nu' - \frac{(n-1)(\mu' - \nu')}{r} \right) - \frac{(n-1)(n-2)}{2} \left(\frac{1 - e^{-2\mu}}{r^2} \right) = 8\pi G_D p, \quad (5)$$

we use $c = 1$ and prime denotes derivative w.r.t. r . The exterior of such a star should match with the D -dimensional generalisation of Schwarzschild solution obtained by Myers and Perry [3]. Here we consider an ansatz

$$e^{-2\mu} = 1 - \frac{C}{r^{n-1}} \quad (6)$$

where C is a constant related to the mass of a higher dimensional star. Using field eq.(4) we get

$$\nu' = \frac{\frac{r^{n+1}}{n} 8\pi G_D p + \frac{n-1}{2} C}{r(r^{n-1} - C)} \quad (7)$$

For hydrostatic equilibrium, the Tolman-Openheimer-Volkoff (TOV) equation is derived using field eqs.(3)-(5) which is given by

$$\frac{dp}{dr} = -(\rho + p)\nu'. \quad (8)$$

We now eliminate ν' using eq.(7) and obtain

$$\frac{dp}{dr} = -\frac{(\rho + p) \left[\frac{r^{n+1}}{n} 8\pi G_D p + \frac{n-1}{2} C \right]}{r(r^{n-1} - C)} \quad (9)$$

For a star having uniform density we have

$$\begin{aligned}\rho &= \rho_o, \quad r \leq R \\ \rho &= 0, \quad r > R\end{aligned}\tag{10}$$

where R is the radial size of the star. The mass of a higher dimensional star is given by

$$M = \frac{n A_n C}{16\pi G_D}\tag{11}$$

where $n = D - 2$, $A_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$. To determine the constant C we find the volume of $(n + 1)$ dimensional ball of radius r which is obtained below

$$V = \int_0^r V_n dr = \frac{2\pi^{\frac{n+1}{2}} r^{n+1}}{(n+1)\Gamma(\frac{n+1}{2})}.\tag{12}$$

where the volume of the n -dimensional surface is $V_n = 2\pi^{\frac{n+1}{2}} r^n / \Gamma(\frac{n+1}{2})$. Thus using equations (11) and (12) with the energy density condition we get $C = 16\pi G_D \rho_o r^{n+1} / n(n+1)$. The TOV equation can be written as

$$\frac{dp}{dr} = -\frac{(\rho_o + p)[(n+1)p + (n-1)\rho_o] r}{2\rho_o(\alpha^2 - r^2)}\tag{13}$$

with $\alpha^2 = n(n+1)/16\pi G_D \rho_o$. On integrating the differential equation (13) we get

$$\frac{(n+1)p + (n-1)\rho_o}{p + \rho_o} = K\sqrt{\alpha^2 - r^2}\tag{14}$$

where K is an integration constant. It is now easy to express pressure in terms of ρ_o . Using the boundary condition at $r = R$ i.e., pressure at the surface of the star should vanish we have $p(R) = 0$, one can determine the integration constant, $K = (n-1)/\sqrt{\alpha^2 - R^2}$. Thus the pressure in terms of radial coordinate can be expressed as

$$p = \rho_o \frac{(n-1)(\sqrt{\alpha^2 - r^2} - \sqrt{\alpha^2 - R^2})}{(n+1)\sqrt{\alpha^2 - R^2} - (n-1)\sqrt{\alpha^2 - r^2}}.\tag{15}$$

The pressure at the centre $r = 0$ of the star becomes

$$p_c = \rho_o \frac{(n-1)(\alpha^2 - \sqrt{\alpha^2 - R^2})}{(n+1)\sqrt{\alpha^2 - R^2} - (n-1)\alpha}. \quad (16)$$

The important feature of the above solution is that it necessarily imposes a constraint connecting the mass of a given star (M) and radius (R). To derive this constraint, we simply consider the case that at $r = r_o$ the pressure becomes infinite where

$$r_o^2 = \alpha^2 - \left(\frac{n+1}{n-1}\right)^2 (\alpha^2 - R^2) \quad (17)$$

which depends on the dimensions of the universe in addition to the density of a star. Since the pressure is a scalar, this infinity will persist all over. To avoid this one gets a constraint on the mass to radius ratio of a star which is

$$\frac{M}{R^{D-3}} < \frac{(D-2)^2 \pi^{\frac{D-3}{2}}}{2G_D(D-1)^2 \Gamma(\frac{D-1}{2})}. \quad (18)$$

Thus one gets a stable configuration for a star satisfying the above limit. It is important to state here that no star of uniform density can have a mass and radius exceeding the above limit. In 4-dimensions ($D = 4$) the Schwarzschild limit $\frac{2M}{R} < \frac{8}{9}$ is obtained taking ($G_4 = 1$) [14]. A table showing the variation of $\frac{G_D M}{R^{D-3}}$ with the dimensions D are given below :

<i>Variation of $\frac{G_DM}{R^{D-3}}$ with space – time dimensions</i>		
$D = n + 2$	n	<i>upper limit on $\frac{G_DM}{R^{n-1}}$</i>
3	1	0.1250
4	2	0.4444
5	3	0.8836
6	4	1.3404
7	5	1.7135
8	6	1.93364
9	7	1.9783
10	8	1.8666
11	9	1.6438
12	10	1.3630
13	11	1.0714
14	12	0.8027
15	13	0.5757
26	24	$1.7560 \cdot 10^{-4}$

Table : 1

We note the following :

- The upper bound on $\frac{G_DM}{R^{D-3}}$ first increases with the increase of dimensions of the space-time, it attains a maximum value. The maximum value for an integer dimensional space-time is found at $D = 9$. Thereafter, it decreases to a very small value as one goes on increasing the dimensions of the universe. Thus compared to the superstring dimensions $D = 10$ the above ratio is found to be least in $D = 4$ dimensions which is the observable dimension. However, the ratio $\frac{C}{R^{D-3}}$ always remains less than one.

- The mass to radius ratio in D dimensions is $\frac{M}{R} < f(D) R^{D-4}$, where $f(D)$ is a function of dimensions of the spacetime in which the star is embedded. Thus the

mass to radius ratio is a number only for $D = 4$ otherwise it depends on the radius also as R^{D-4} . When $D = 2 + 1$ dimensions, one obtains an upper limit on the mass of the star which is independent of the radius, an unphysical result. However, when $D = 3 + 1$ dimensions, one derives a consistent mass to radius ratio for a uniform density star. Thus $D = 4$ is the lowest dimensions in this case to formulate the stellar study.

- It may be important to mention here that the four dimensional analog of the Mass to radius ratio in higher dimensional spacetime gives an interesting result if one vary dimensions along the real axis (i.e., considering fractional space-time dimensions). The same mass to radius ratio obtained in four dimensions is obtained for a fractional higher dimensions also which supports the work of Kobelev [15] . This issue will be discussed elsewhere.

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