Instantons and chiral symmetry breaking in SU($N$) gauge theories.

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We address the question of whether the low modes of the Dirac operator are caused by topological objects such as instantons in SU($N$) gauge theories. We study the pseudo-scalar density of these modes, finding the size distributions of the instantons, and comparing it with the underlying gauge field. We find that, although the near-zero modes of the Dirac operator depend on topology for all $N$, their small instanton content decreases as $N$ increases.

1. INTRODUCTION

The mechanism behind spontaneous chiral symmetry breaking (S$\chi$SB) is an unsolved mystery in QCD. A popular hypothesis is that a weakly interacting dilute gas of instantons and anti-instantons could create a number of small non-zero eigenvalues of the Dirac matrix [?]. By the Banks-Casher relation [?], this leads to a non-zero chiral condensate, and therefore S$\chi$SB. Using chiral Dirac operators, such as the overlap operator [?], one can examine the scalar and pseudo-scalar densities of the low-lying eigenmodes of the Dirac operator numerically, searching for instantons. Recent studies of these modes in SU(3) have found a lumpy self-dual structure [?, ?, ?, ?, ?, ?]. Although this is predicted by the instanton liquid model (ILM), there is continuing discussion about whether these lumps are instantons.

Partly because of this, some people believe that the instanton liquid is destroyed by quantum effects [?, ?].

We are investigating what happens to the lumpy structure in the low modes of the Dirac operator when we change the number of colours [?]. To do so, we generated configurations on a $12^4$ lattice, for SU($N$) gauge theories ($N = 2, 3, 4, 5$), keeping the lattice spacing fixed at $a = 0.12$ fm by keeping a constant string tension [?].

2. TOPOLOGICAL CHARGE AND SMALL INSTANTONS

There are two different measures of topological charge which we used. The fermionic charge, $Q_f$, is defined as the difference between the number of negative and positive chirality zero-modes of the Dirac matrix [?]. Secondly, we can use a lattice equivalent of the continuum topological charge density, $FF(x)$, as a (“field theoretic”) topological charge density, $q(x)$. We have to cool the gauge field before calculating $q(x)$ to suppress UV fluctuations. A gauge charge, $Q_g$, is the sum of $q(x)$ over all lattice sites. In the continuum, $Q_g$ and $Q_f$ are equal. However, lattice artifacts can lead to $Q_g \neq Q_f$. We found considerably more discrepancies in SU(2) and SU(3) than in SU(4) and SU(5). In each of the configurations where there was a discrepancy, there was also a small instanton of size $\rho < 0.3$ fm (found by looking at the size of the peaks in $q(x)$) [?]. We concluded that for the lattice spacings we used, the overlap operator cannot resolve instantons below this...
Figure 1. Local chirality distributions for $\lambda^2 < 0.1$ (left) or $\lambda^2 < 0.03$ (right).

3. SIZE DISTRIBUTIONS

For each mode, we identified the peaks in the pseudo-scalar density. Using classical instanton formulae for the height and shape of the peaks [?], we calculated size distributions for the instantons contributing to the eigenmodes. We saw that as $N$ increases, the typical instanton size becomes slightly larger, and small instantons ($\rho \lesssim 0.4 \text{ fm}$) disappear, in agreement with theoretical predictions [?].

4. LOCAL CHIRALITY

The local chiral orientation angle, $X$ [?], is defined at each lattice site as

$$X = \frac{4}{\pi} \left( \arctan \left( \frac{\psi_+ \psi^\dagger_+}{\psi_+ \psi^\dagger_-} \right) \right) - 1.$$ 

(1)

$\psi_+$ and $\psi_-$ are the projections of the eigenmodes of the Dirac matrix onto the $\pm$ chiral sectors. An isolated (anti-)instanton, will have $X = \pm 1$, and for a classical ILM, this would dominate the local chirality distribution, a histogram of the values of $X$ on the lattice sites with the highest scalar density. If the modes are not caused by topology, then the distribution would be centred around $X = 0$. Overlapping instantons, instantons in a background of quantum fluctuations, or a background of large quantum fluctuations will be between the two extremes. Figure 1 shows a histogram of the values of $X$ on the 2% of lattice sites with the highest scalar density, averaged over two different ranges of eigenvalues $\lambda$.

As $N$ increases, or the eigenvalue increases, the peaks in the local chirality distributions move towards zero, behaving less like the expected distribution from classical instantons. If the shape of the distributions is determined by small instantons, then excluding those SU(3) configurations which contain narrow instantons would make the distribution behave more like SU(4), since there are fewer small instantons there. However we did not observe a significant effect when we tried this, and so we conclude that small instantons are not responsible for the different shape of the chirality distributions for the various gauge groups.

5. AUTO-CORRELATORS

To see whether a mode is influenced by topology, one can compare it with the underlying gauge field. We used a scale-invariant (dimensionless) auto-correlator which is (in the continuum) independent of the instanton size,

$$C_d^5(0) \equiv \int d^4x \left| \psi^\dagger(x) \gamma^5 \psi(x) \right|^d \text{sign} \left( \psi^\dagger(x) \gamma^5 \psi(x) \right) \left| q(x) \right|^{1-d} \text{sign} \left( q(x) \right),$$

(2)

where $d$ lies in the range $0 \leq d \leq 1$. If the pseudo-scalar density is correlated with topological charge (for example, if the eigenmode contains an instanton/anti-instanton pair), then $C_d^5(0) > 0$. If the eigenmode is not influenced by topology, then there will be no correlation between the pseudo-scalar and topological charge densities, giving $C_d^5(0) \sim 0$. We plot the auto-
correlator as a function of the number of cooling sweeps at which we calculate $q(x)$ (figure 2). The auto-correlator quickly rises to a maximum, and then decreases as the cooling distorts the gauge field. We can calculate, by modelling artificial instanton configurations, that for a classical instanton/anti-instanton pair, the auto-correlator should give a value $\sim 1.1$. Overlapping instantons, instantons in a background of quantum fluctuations, or non-instanton topological fluctuations would give a smaller value than this. The measured value, at 6 cooling sweeps (the maximum for the non-zero modes), is smaller than the expected classical value, and decreases as we increase the number of colours. Instantons seem to contribute less to the modes as we increase $N$. On the other hand, the zero modes, which we know are caused by topology, show a similar decrease. The ratio between the auto-correlators (at 6 cooling sweeps) for the non-zero modes and for the zero-modes is constant ($\sim 1.4$). This suggests that topology is important in chiral symmetry breaking $\forall$ SU($N$), but that classical instantons become less important as we increase $N$.

6. CONCLUSIONS

We found that discrepancies between the fermionic and gauge charges are caused by small instantons ($\rho < 0.3$ fm in this calculation). As we increase $N$, small instantons ($\rho \lesssim 0.4$ fm) are suppressed. Examination of the auto-correlator suggests that chiral symmetry breaking at large $N$ does have a topological origin. However, as we increase $N$, semi-classical instantons seem to become less important. Our results, from sections 4 and 5, imply that if the lumps in the low modes of the Dirac operator are instantons, then as we increase $N$, the instantons either overlap more heavily, and/or quantum fluctuations become more important.