Exact solutions to the Dirac equation
for a Coulomb potential in $D + 1$ dimensions

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The Dirac equation is generalized to $D + 1$ space-time. The conserved angular momentum operators and their quantum numbers are discussed. The eigenfunctions of the total angular momenta are calculated for both odd $D$ and even $D$ cases. The radial equations for a spherically symmetric system are derived. The exact solutions for the system with a Coulomb potential are obtained analytically. The energy levels and the corresponding fine structure are also presented.

Key words: Dirac equation, $D + 1$ dimensions, Exact solutions, SO(D) group.

I. INTRODUCTION

The exact solutions of the Schrödinger equation in the real three-dimensional space for a hydrogen atom and for a harmonic oscillator were important technical achievements in quantum mechanics [1,2]. During the past half century, the mathematical tools for the orbital angular momentum operators and their eigenfunctions in an arbitrary $D$-dimensional space have been presented [3–8]. The nonrelativistic $D$-dimensional Coulombic and the harmonic oscillator problems have been studied in some detail by many authors [9–15]. The solutions of the Dirac equation, however, have been studied in the usual three- [17–21], two- [22] and one-dimensional [23] space. Motivated by the recent interest of higher-dimensional field theory, we generalize the Dirac equation to $D + 1$ space-time. The conserved total angular momentum operators and their quantum numbers are dis-
discussed. The eigenfunctions of the total angular momenta are calculated for both odd $D$ and even $D$ cases. From the viewpoint of mathematics, this problem is a typical application of group theory to physics. In terms of the eigenfunctions, we obtain the radial equations for the spherically symmetric system, and analytically solve the radial equations for the quantum Coulombic system.

This paper is organized as follows. Section 2 is devoted to the generalization of the Dirac equation to $D + 1$ space-time. In Sec. 3, the conserved angular momentum operators and their quantum numbers are discussed. The eigenfunctions of the total angular momentums are calculated for both odd $D$ and even $D$ cases in terms of the method of group theory. The radial equations for the system with a spherically symmetric potential are derived. In Sec. 4, the wave functions of bound states for a Coulombic system, which are expressed by the confluent hypergeometric functions, are presented. The energy levels and the corresponding fine structure are also discussed. Some conclusions are given in Sec. 5.

II. DIRAC EQUATION IN $D + 1$ DIMENSIONS

The Dirac equation in $D + 1$ dimensions can be expressed as [24]

$$i \sum_{\mu=0}^{D} \gamma^\mu (\partial_\mu + ieA_\mu) \Psi(x, t) = M \Psi(x, t),$$

(1)

where $M$ is the mass of the particle, and $(D+1)$ matrices $\gamma_\mu$ satisfy the anti-commutative relations:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu} 1,$$

(2)

with the metric tensor $\eta^{\mu\nu}$ satisfying

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \left\{ \begin{array} { l l } 
\delta_{\mu\nu} & \text{ when } \mu = 0 \\
-\delta_{\mu\nu} & \text{ when } \mu \neq 0.
\end{array} \right.$$  

(3)

For simplicity, the natural units $\hbar = c = 1$ are employed throughout this paper. Consider the special case where only the zero component of $A_\mu$ is non-vanishing and spherically symmetric:

$$eA_0 = V(r), \quad A_a = 0, \quad \text{when } a \neq 0,$$

(4)

The Hamiltonian $H(x)$ of the system is expressed as

$$i \partial_0 \Psi(x, t) = H(x) \Psi(x, t), \quad H(x) = \sum_{a=1}^{D} \gamma^0 \gamma^a p_a + V(r) + \gamma^0 M,$$

$$p_a = -i \partial_a = -i \frac{\partial}{\partial x^a}, \quad 1 \leq a \leq D.$$  

(5)
The orbital angular momentum operators \( L_{ab} \), the spinor operators \( S_{ab} \), and the total angular momentum operators \( J_{ab} \) are defined as follows
\[
L_{ab} = -L_{ba} = i x_a \partial_b - i x_b \partial_a, \quad S_{ab} = -S_{ba} = i \gamma_a \gamma_b / 2, \quad 1 \leq a < b \leq D
\]
\[
J_{ab} = L_{ab} + S_{ab}, \quad J^2 = \sum_{a<b=2} J_{ab}^2, \quad L^2 = \sum_{a<b=2} L_{ab}^2, \quad S^2 = \sum_{a<b=2} S_{ab}^2.
\]

The eigenvalue of \( J^2 \) (\( L^2 \) or \( S^2 \)) is denoted by the Casimir \( C_2(M) \), where \( M \) is the highest weight of the representation to which the total (orbital or spinor) wave function belongs. We will discuss the Casimir in the next section. It is easy to show by the standard method [24] that \( J_{ab} \) and \( \kappa \) are commutant with the Hamiltonian \( H(x) \),
\[
\kappa = \gamma^0 \left\{ \sum_{a<b} i \gamma^a \gamma^b L_{ab} + (D - 1)/2 \right\} = \gamma^0 \left\{ J^2 - L^2 - S^2 + (D - 1)/2 \right\}.
\]

### III. THE RADIAL EQUATIONS

Since the potential \( V(r) \) is spherically symmetric, the symmetry group of the system is \( \text{SO}(D) \) group. Erdelyi [3] and Louck [5,7] introduced the hyperspherical coordinates in the real \( D \)-dimensional space
\[
\begin{align*}
x^1 &= r \cos \theta_1 \sin \theta_2 \ldots \sin \theta_{D-1}, \\
x^2 &= r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{D-1}, \\
x^b &= r \cos \theta_{b-1} \sin \theta_k \ldots \sin \theta_{D-1}, \quad 3 \leq b \leq D - 1, \\
x^D &= r \cos \theta_{D-1} \\
\sum_{a=1}^{D} (x^a)^2 &= r^2.
\end{align*}
\]

The unit vector along \( x \) is usually denoted by \( \hat{x} = x/r \). The volume element of the configuration space is
\[
\prod_{a=1}^{D} dx^a = r^{D-1} dr d\Omega, \quad d\Omega = \prod_{a=1}^{D-1} (\sin \theta_a)^{a-1} d\theta_a,
\]
\[
0 \leq r \leq \infty, \quad -\pi \leq \theta_1 \leq \pi, \quad 0 \leq \theta_2 \leq \pi, \quad 2 \leq b \leq D - 1.
\]

Now, let us sketch some necessary knowledge of the \( \text{SO}(D) \) group. From the representation theory of Lie groups [25–27], the Lie algebras of the \( \text{SO}(2N+1) \) group and the \( \text{SO}(2N) \) group are \( B_N \) and \( D_N \), respectively. Their Chevalley bases with the subscript \( \mu \), \( 1 \leq \mu \leq N - 1 \), are same:
\[
\begin{align*}
H_{\mu}(J) &= J_{(2\mu-1)(2\mu)} - J_{(2\mu+1)(2\mu+2)}, \\
E_{\mu}(J) &= \left( J_{(2\mu)(2\mu+1)} - i J_{(2\mu-1)(2\mu+1)} - i J_{(2\mu)(2\mu+2)} - J_{(2\mu-1)(2\mu+2)} \right) / 2, \\
F_{\mu}(J) &= \left( J_{(2\mu)(2\mu+1)} + i J_{(2\mu-1)(2\mu+1)} + i J_{(2\mu)(2\mu+2)} - J_{(2\mu-1)(2\mu+2)} \right) / 2.
\end{align*}
\]

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But, the bases with the subscript \( N \) are different:
\[
H_N(J) = 2J_{(2N-1)(2N)},
\]
\[
E_N(J) = J_{(2N)(2N+1)} - iJ_{(2N-1)(2N+1)},
\]
\[
F_N(J) = J_{(2N)(2N+1)} + iJ_{(2N-1)(2N+1)},
\] (10b)

for \( \text{SO}(2N+1) \), and
\[
H_N(J) = J_{(2N-3)(2N-2)} + J_{(2N-1)(2N)},
\]
\[
E_N(J) = \left( J_{(2N-2)(2N-1)} - iJ_{(2N-3)(2N-1)} + iJ_{(2N-2)(2N)} + J_{(2N-3)(2N)} \right) / 2,
\]
\[
F_N(J) = \left( J_{(2N-2)(2N-1)} + iJ_{(2N-3)(2N-1)} - iJ_{(2N-2)(2N)} + J_{(2N-3)(2N)} \right) / 2,
\] (10c)

for \( \text{SO}(2N) \). The operator \( J_{ab} \) can be replaced with \( L_{ab} \) or \( S_{ab} \) depending on the wave functions one is discussing. \( H_\mu(J) \) span the Cartan subalgebra, and their eigenvalues for an eigenstate \( |m\rangle \) in a given irreducible representation are the components of a weight vector \( m = (m_1, \ldots, m_N) \):
\[
H_\mu(J)|m\rangle = m_\mu|m\rangle, \quad 1 \leq \mu \leq N.
\] (11)

If the eigenstates \( |m\rangle \) for a given weight \( m \) are degenerate, this weight is called a multiple weight, otherwise, a simple one. \( E_\mu \) are called the raising operators and \( F_\mu \) the lowering ones. For an irreducible representation, there is a highest weight \( M \), which is a simple weight and can be used to describe the irreducible representation. Usually, the irreducible representation is also called the highest weight representation and directly denoted by \( M \). The Casimir \( C_2(M) \) can be calculated by the formula (e.g. see (1.131) in [27])
\[
C_2(M) = M \cdot (M+2\rho) = \sum_{\mu,\nu=1}^{N} M_\mu d_\mu (A^{-1})_{\mu\nu} (M_\nu + 2),
\] (12)

where \( \rho \) is the half sum of the positive roots in the Lie algebra, \( A^{-1} \) is the inverse of the Cartan matrix, and \( d_\mu \) are the half square lengths of the simple roots.

The orbital wave function in \( D \)-dimensional space is usually expressed by the spherical harmonic \( Y_{m}^{(l)}(\hat{x}) \) [5,7], which belongs to the weight \( m \) of the highest weight representation \((l) \equiv (l, 0, \ldots, 0)\). For the highest weight state, \( m = (l) \), we have
\[
Y_{(l)}^{(l)}(\hat{x}) = N_{D,l} r^{-l} (x_1 + ix_2)^l,
\]
\[
N_{D,l} = \begin{cases} 2^{-N-l} \left( \frac{(2l + 2N - 1)!}{\pi^N l!(l + N - 1)!} \right)^{1/2} & \text{when } D = 2N + 1 \\ \left( \frac{(l + N - 1)!}{2\pi^N l!} \right)^{1/2} & \text{when } D = 2N, \end{cases}
\] (13)

where \( N_{D,l} \) is the normalization factor. Its partners \( Y_{m}^{(l)}(\hat{x}) \) can be calculated from \( Y_{(l)}^{(l)}(\hat{x}) \) by lowering operators \( F_\mu(L) \). The Casimir for the spherical harmonic \( Y_{m}^{(l)}(\hat{x}) \) is calculated by Eq. (12):
\[
L^2 Y_{m}^{(l)}(\hat{x}) = C_2[(l)] Y_{m}^{(l)}(\hat{x}), \quad C_2[(l)] = l(l + D - 2).
\] (14)
The spinor wave functions as well as those for the total angular momentum are different for \( D = 2N + 1 \) and \( D = 2N \), and will be discussed separately in the following subsections.

**A. The SO\((2N+1)\) case**

For \( D = 2N + 1 \) we define
\[
\gamma^0 = \sigma_3 \times 1, \quad \gamma^a = (i\sigma_2) \times \beta_a, \quad 1 \leq a \leq 2N + 1, \tag{15}
\]
where \( \sigma_a \) is the Pauli matrix, \( 1 \) denotes the \( 2^N \)-dimensional unit matrix, and \((2N + 1)\) matrices \( \beta_a \) satisfy the anticommutative relations
\[
\beta_a \beta_b + \beta_b \beta_a = 2\delta_{ab} 1, \quad a, b = 1, 2, \ldots, (2N + 1). \tag{16}
\]
The dimension of \( \beta_a \) matrices is \( 2^N \). Thus, the spinor operator \( S_{ab} \) becomes a block matrix
\[
S_{ab} = 1 \times \overline{S}_{ab}, \quad \overline{S}_{ab} = -i \beta_a \beta_b / 2. \tag{17}
\]
The relation between \( S_{ab} \) and \( \overline{S}_{ab} \) is similar to the relation between the spinor operators for the Dirac spinors and for the Pauli spinors. The operator \( \kappa \) becomes
\[
\kappa = \sigma_3 \times \kappa, \quad \kappa = -i \sum_{a < b} \beta_a \beta_b L_{ab} + (D - 1)/2. \tag{18}
\]
The fundamental spinor \( \chi(\mathbf{m}) \) belong to the fundamental spinor representation \( (s) \equiv (0, \ldots, 0, 1) \). From Eq. \( (12) \) the Casimir for the representation \( (s) \) is calculated to be \( C_2[(s)] = (2N^2 + N)/4 \).

The product of \( Y_\mathbf{m}^{(l)}(\hat{x}) \) and \( \chi(\mathbf{m}') \) belong to the direct product of two representative \( (l) \) and \( (s) \), which is a reducible representation:
\[
(l) \times (s) \simeq (l, 0, \ldots, 0, 1) \oplus (l - 1, 0, \ldots, 0, 1). \tag{19}
\]

In other words, in order to construct a wave function belonging to the representation \( (j) \equiv (l, 0, \ldots, 0, 1) \) there are two different ways: the combination of \( Y_\mathbf{m}^{(l)}(\hat{x}) \chi(\mathbf{m}') \) and that of \( Y_\mathbf{m}^{(l+1)}(\hat{x}) \chi(\mathbf{m}') \). They have different eigenvalues of \( \kappa \). Since the system is spherically symmetric, we only need to calculate the highest weight state for the representation \( (j) \) in terms of the Clebsch-Gordan coefficients
\[
\phi_{[K],(j)}(\hat{x}) = \sum_{\mathbf{m}} Y_\mathbf{m}^{(l)}(\hat{x}) \chi[(j) - \mathbf{m}][(l + 1), \mathbf{m}, (s), (j) - \mathbf{m}](j), (j)) = N_{D,l} r^{-l}(x^1 + ix^2)^l \chi[(s)],
|K| = C_2[(j)] - C_2[(l)] - C_2[(s)] + N = l + N. \tag{20}
\]

\[
\phi_{-[K],(j)}(\hat{x}) = \sum_{\mathbf{m}} Y_\mathbf{m}^{(l+1)}(\hat{x}) \chi[(j) - \mathbf{m}][(l) + 1, \mathbf{m}, (s), (j) - \mathbf{m}](j), (j)) = N_{D,l} r^{-l-1}(x^1 + ix^2)^l \chi[(s)] + (x^{2N-1} + ix^{2N}) \chi[(0, \ldots, 0, 1, \overline{T})]
+ (x^{2N-3} + ix^{2N-2}) \chi[(0, \ldots, 0, 1, \overline{T}, 1)] + \ldots
+ (x^3 + ix) \chi[(1, \overline{T}, 0, \ldots, 0, 1)] + (x^1 + ix^2) \chi[(\overline{T}, 0, \ldots, 0, 1)] \chi[(0, \ldots, 0, 1)]
\]
\[
-|K| = C_2[(j)] - C_2[(l + 1)] - C_2[(s)] + N = -l - N. \tag{21}
\]
The wave function $\Psi_{K,(j)}(\mathbf{x})$ of the total angular momentum belonging to the irreducible representation $(j)$ can be expressed as

$$\Psi_{K,(j)}(\mathbf{x},t) = r^{-N}e^{-iEt} \begin{pmatrix} F(r)\phi_{K,(j)}(\mathbf{x}) \\ iG(r)\phi_{-K,(j)}(\mathbf{x}) \end{pmatrix},$$

$$H_1(J)\Psi_{K,(j)}(\mathbf{x}) = l\Psi_{K,(j)}(\mathbf{x}), \quad H_N(J)\Psi_{K,(j)}(\mathbf{x}) = \Psi_{K,(j)}(\mathbf{x}), \quad 2 \leq \mu \leq N - 1,$$

$$\kappa \Psi_{K,(j)}(\mathbf{x}) = K\Psi_{K,(j)}(\mathbf{x}), \quad K = \pm (l + N).$$

Its partners can be calculated from it by the lowering operators $F_\mu(J)$.

The radial equation will depend upon the explicit forms of $\beta_a$ matrices. We express $\beta_a$ matrices by direct products of $N$ Pauli matrices $\sigma_a$ [28]:

$$\beta_{2m-1} = \prod_{a=1}^{m-1} \sigma_1 \times \sigma_3 \times \ldots \times \sigma_3,$$

$$\beta_{2m} = \prod_{a=1}^{m-1} \sigma_1 \times \sigma_2 \times \sigma_3 \times \ldots \times \sigma_3,$$

$$\beta_{2N+1} = \sigma_3 \times \sigma_3 \times \ldots \times \sigma_3.$$ (23)

In terms of the explicit forms of $\beta_a$, we obtain

$$\left(\bar{\beta} \cdot \mathbf{x}\right) \phi_{K,(j)}(\mathbf{x}) = r^{-1} \sum_{a=1}^{2N+1} \beta_a x^a \phi_{K,(j)}(\mathbf{x}) = \phi_{-K,(j)}(\mathbf{x}),$$

$$\left(\bar{\beta} \cdot \mathbf{p}\right) r^{-N} \phi_{K,(j)}(\mathbf{x}) = \sum_{a=1}^{2N+1} \beta_a p_a r^{-N} \phi_{K,(j)}(\mathbf{x}) = iKr^{-N-1} \phi_{-K,(j)}(\mathbf{x}).$$ (24)

Substituting $\Psi_{K(j)}(\mathbf{x})$ into the Dirac equation (5) we obtain the radial equation

$$\frac{dG(r)}{dr} + \frac{K}{r}G(r) = [E - V(r) - M]F(r),$$

$$-\frac{dF(r)}{dr} + \frac{K}{r}F(r) = [E - V(r) + M]G(r).$$ (25)

### B. The SO(2N) case

As is well known, the spinor representation of SO(2N) group is reducible and can be reduced to two inequivalent fundamental spinor representations $(+s) \equiv (0, 0, \ldots, 0, 1)$ and $(-s) \equiv (0, 0, \ldots, 0, 1, 0)$. From Eq. (12) the Casimir for both spinor representations are calculated to be $C_2([\pm s]) = (2N^2 - N)/4$. In terms of $\beta_a$ matrices given in Eq. (23), we define $\gamma^\mu$ matrices for $D = 2N$:

$$\gamma^0 = \beta_{2N+1}, \quad \gamma^a = \beta_{2N+1}\beta_a, \quad 1 \leq a \leq 2N.$$ (26)

$\gamma^0$ is a diagonal matrix where half of the diagonal elements are equal to +1 and the remaining to −1. Because the spinor operator $S_{ab}$ and the operator $\kappa$ are commutant
with $\gamma^0$, each of them becomes a direct sum of two matrices, referring to the rows with the eigenvalues $+1$ and $-1$ of $\gamma^0$, respectively. The fundamental spinors $\chi_{\pm}(m)$ belong to the fundamental spinor representations $(+s)$ and $(-s)$, respectively, and satisfy

$$\gamma^0 \chi_{\pm}(m) = \pm \chi_{\pm}(m).$$  

(27)

The product of $Y^{(l)}(\hat{x})$ and $\chi_{\pm}(m')$ belong to the direct product of two representation $(l)$ and $(\pm s)$, which is a reducible representation:

$$(l) \times (+s) \simeq (l, 0, \ldots, 0, 1) \oplus (l - 1, 0, \ldots, 0, 1, 0),$$

$$(l) \times (-s) \simeq (l, 0, \ldots, 0, 1, 0) \oplus (l - 1, 0, \ldots, 0, 1).$$

(28)

There are two kinds of representations for the total angular momentum: the representation $(j_1) \equiv (l, 0, \ldots, 0, 1)$ and the representation $(j_2) \equiv (l, 0, \ldots, 0, 1, 0)$. Their Casimir are the same:

$$C_2[(j_1)] = C_2[(j_2)] = l(l + 2N - 1) + (2N^2 - N)/4.$$  

(29)

There are two different ways to construct a wave function belonging to the representation $(j_1)$: the combination of $Y^{(l)}(\hat{x})\chi_{+}(m')$ and that of $Y^{(l+1)}(\hat{x})\chi_{-}(m')$. Due to the spherical symmetry, we only calculate the highest weight state for the representation $(j_1)$ by the Clebsch-Gordan coefficients:

$$\phi_{K, (j_1)}(\hat{x}) = Y^{(l)}(\hat{x})\chi_{+}[(+s)] = N_{D,l}r^{-l}(x^1 + ix^2)^l\chi_{+}[(+s)],$$

$$\phi_{-K, (j_1)}(\hat{x}) = \sum_m Y^{(l+1)}_m(\hat{x})\chi_{-}[(j_1) - m]\langle(l + 1), m, (-s), (j_1) - m|(j_1), (j_1)\rangle$$

$$= N_{D,l}r^{-l-1}(x^1 + ix^2)^l \{ (x^{2N-1} + ix^{2N})\chi_{-}[-(s)]$$

$$+ (x^{2N-3} + ix^{2N-2})\chi_{-}[(0, \ldots, 0, 1, \bar{T}, 0)] +$$

$$+ (x^{2N-5} + ix^{2N-4})\chi_{-}[(0, \ldots, 0, 1, \bar{T}, 0, 1)] + \ldots$$

$$+ (x^3 + ix^4)\chi_{-}[(1, \bar{T}, 0, \ldots, 0, 1)] + (x^1 + ix^2)\chi_{-}[(\bar{T}, 0, \ldots, 0, 1)] \},$$

$$K = C_2[(j_1)] - C_2[(l)] - C_2[(+s)] + N - 1/2 = l + N - 1/2.$$  

(30)

For the representation $(j_2) \equiv (l, 0, \ldots, 0, 1, 0)$ we have

$$\phi_{K, (j_2)}(\hat{x}) = \sum_m Y^{(l+1)}_m(\hat{x})\chi_{+}[(j_2) - m]\langle(l + 1), m, (+s), (j_2) - m|(j_2), (j_2)\rangle$$

$$= N_{D,l}r^{-l-1}(x^1 + ix^2)^l \{ (x^{2N-1} - ix^{2N})\chi_{+}[(+s)]$$

$$+ (x^{2N-3} + ix^{2N-2})\chi_{+}[(0, \ldots, 0, 1, \bar{T}, 0)] +$$

$$+ (x^{2N-5} + ix^{2N-4})\chi_{+}[(0, \ldots, 0, 1, \bar{T}, 1, 0)] + \ldots$$

$$+ (x^3 + ix^4)\chi_{+}[(1, \bar{T}, 0, \ldots, 0, 1, 0)] + (x^1 + ix^2)\chi_{+}[(\bar{T}, 0, \ldots, 0, 1, 0)] \},$$

$$\phi_{-K, (j_2)}(\hat{x}) = Y^{(l)}(\hat{x})\chi_{-}[-(s)] = N_{D,l}r^{-l}(x^1 + ix^2)^l\chi_{-}[-(s)],$$

$$K = C_2[(j_2)] - C_2[(l + 1)] - C_2[(+s)] + N - 1/2 = -l - N + 1/2.$$  

(31)
In terms of the explicit forms of $\beta_a$ we obtain

$$
(\vec{\beta} \cdot \hat{x}) \phi_{K,(j\omega)}(\hat{x}) = r^{-1} \sum_{a=1}^{2N} \beta_a a^a \phi_{K,(j\omega)}(\hat{x}) = \phi_{-K,(j\omega)}(\hat{x}),
$$

$$
(\vec{\beta} \cdot \vec{p}) r^{-N+1/2} \phi_{K,(j\omega)}(\hat{x}) = \sum_{a=1}^{2N} \beta_a p_a r^{-N+1/2} \phi_{K,(j\omega)}(\hat{x}) = iKr^{-N-1/2} \phi_{-K,(j\omega)}(\hat{x})
$$

(32)

$\omega = 1$ or 2.

The wave function $\Psi_{K,(j\omega)}(x)$ of the total angular momentum belonging to the irreducible representation $(j\omega)$ can be expressed as

$$
\Psi_{|K\rangle,(j\omega)}(x,t) = r^{-N+1/2}e^{-iEt} \left\{ F(r)\phi_{|K\rangle,(j\omega)}(\hat{x}) + iG(r)\phi_{-|K\rangle,(j\omega)}(\hat{x}) \right\},
$$

$$
\Psi_{-|K\rangle,(j\omega)}(x,t) = r^{-N+1/2}e^{-iEt} \left\{ F(r)\phi_{-|K\rangle,(j\omega)}(\hat{x}) + iG(r)\phi_{|K\rangle,(j\omega)}(\hat{x}) \right\},
$$

$$
k\Psi_{K,(j\omega)}(x) = K\Psi_{K,(j\omega)}(x), \quad K = \begin{cases} 
  l + N - 1/2, & \text{when } \omega = 1 \\
  -l - N + 1/2, & \text{when } \omega = 2,
\end{cases}
$$

(33)

$$
H_1(J)\Psi_{K,(j\omega)}(x) = |\Psi_{K,(j\omega)}(x)|,
$$

$$
H_{N-1}(J)\Psi_{K,(j\omega)}(x) = 0, \quad H_N(J)\Psi_{K,(j\omega)}(x) = \Psi_{K,(j\omega)}(x),
$$

$$
H_{N-1}(J)\Psi_{K,(j\omega)}(x) = \Psi_{K,(j\omega)}(x), \quad H_N(J)\Psi_{K,(j\omega)}(x) = 0,
$$

$$
H_\mu(J)\Psi_{K,(j\omega)}(x) = 0, \quad 2 \leq \mu \leq N - 2.
$$

IV. SOLUTIONS TO THE RADIAL EQUATION IN D+1 DIMENSIONS

Although the wave functions and the eigenvalues $K$ are different for the $D = 2N + 1$ case and the $D = 2N$ case, the forms of the radial equations are unified

$$
\frac{dG(r)}{dr} + \frac{K}{r}G(r) = [E - V(r) - M]F(r),
$$

$$
-\frac{dF(r)}{dr} + \frac{K}{r}F(r) = [E - V(r) + M]G(r).
$$

(34)

For definiteness we discuss the attractive Coulomb potential

$$
V(r) = -\frac{\xi}{r}, \quad \xi = Z\alpha > 0,
$$

(36)
where $\alpha = 1/137$ is the fine structure constant. It is easy to see that the solution for the repulsive potential can be obtained from that for the attractive potential by interchanging
\[ F_{KE} \leftrightarrow G_{-K-E}, \quad V(r) \leftrightarrow -V(r). \] (37)

From the Sturm-Liouville theorem [29], there are bound states with the energy less than and near $M$ for the attractive Coulomb potential and with the energy larger than and near $-M$ for the repulsive potential, if the interaction is not too strong. It is convenient to introduce a dimensionless variable $\rho$ in Eq. (35) for bound states:
\[ \rho = 2r\sqrt{M^2 - E^2}, \quad 0 < E < M. \] (38)

Solving $F(\rho)$ from Eq. (35),
\[ F_{KE}(\rho) = \left( -\frac{1}{2} \sqrt{\frac{M - E}{M + E} + \frac{\xi}{\rho}} \right)^{-1} \left[ \frac{dG_{KE}(\rho)}{d\rho} + \frac{K}{\rho} G_{KE}(\rho) \right], \] (39)
we obtain a second-order differential equation of $G_{KE}(\rho)$:
\[ \frac{d^2 G_{KE}(\rho)}{d\rho^2} + \left[ -\frac{1}{4} - \frac{K^2 - \xi^2 + K}{\rho^2} + \frac{E\xi}{\rho\sqrt{M^2 - E^2}} \right] G_{KE}(\rho) + \left[ \rho - \frac{\rho^2}{2\xi} \sqrt{\frac{M - E}{M + E}} \right]^{-1} \left[ \frac{dG_{KE}(\rho)}{d\rho} + \frac{K}{\rho} G_{KE}(\rho) \right] = 0. \] (40)

From the behavior of $G_{KE}(\rho)$ at the origin and at the infinity, we define
\[ G_{KE}(\rho) = \rho^\lambda e^{-\rho/2} R(\rho), \quad \lambda = \sqrt{K^2 - \xi^2} > 0, \]
\[ \omega = \frac{1}{2\xi} \sqrt{\frac{M - E}{M + E}}, \quad \tau = \frac{E\xi}{\sqrt{M^2 - E^2}}, \] (40)
and obtain
\[ (\rho - \omega \rho^2) \frac{d^2 R(\rho)}{d\rho^2} + \left[ \omega \rho^2 - (2\lambda + 1) \rho + 2\lambda + 1 \right] \frac{dR(\rho)}{d\rho} + [\omega(\lambda - \tau) \rho + \omega(K + \lambda) + \tau - \lambda - 1/2] R(\rho) = 0. \] (41)

Eq. (41) can be solved by the power series expansion method for for (3+1)-dimensions [21] and (2+1)-dimensions [22]. The results in $D + 1$ dimensions are as follows
\begin{align*}
F_{KE}(\rho) & = \frac{(M^2 - E^2)^{1/4}}{\Gamma(2\lambda + 1)} \left[ (M \pm E) E \Gamma(n' + 2\lambda + 1) \right]^{1/4} \rho^{\lambda - \rho/2} \\
& \times \left[ (K + \tau M/E) \, _1F_1(-n', 2\lambda + 1, \rho) \mp n' \, _1F_1(1 - n', 2\lambda + 1, \rho) \right], \quad (42)
\end{align*}
\[ \int_0^\infty (|F_{KE}(\rho)|^2 + |G_{KE}(\rho)|^2) \, dr = 1, \] (43)
\[ n' = \tau - \lambda = 0, 1, 2, \ldots \]  

(44)

where \( \mathbf{1}_F(\alpha, \beta, \rho) \) is the confluent hypergeometric function. When \( n' = 0 \), \( K \) has to be positive. Introduce the principal quantum number

\[ n = |K| - (D - 3)/2 + n' = |K| - (D - 3)/2 + \tau - \lambda = 1, 2, \ldots \]  

(45)

The principal quantum number \( n \) can be equal to 1 only for \( K = (D - 1)/2 \) and equal to other positive integers for both signs of \( K \). The energy \( E \) can be calculated from Eqs. (40), (44) and (45)

\[ E = M \left[ 1 + \frac{\xi^2}{(\sqrt{K^2 - \xi^2} + n - |K| + (D - 3)/2)^2} \right]^{-1/2}. \]  

(46)

Expanding Eq. (46) in powers of \( \xi^2 \), we have

\[ E \simeq M \left\{ 1 - \frac{\xi^2}{2[n + (D - 3)/2]^2} - \frac{\xi^4}{2[n + (D - 3)/2]^4} \left( \frac{n + (D - 3)/2}{|K|} - \frac{3}{4} \right) \right\}, \]  

(47)

where the first term on the right hand side is the rest energy \( M (c^2 = 1 \) in our conventions), the second one coincides with the energy from the solutions to the Schrödinger equation, and the third one is the fine structure energy, which removes the degeneracy between the states of the same \( n \).

V. CONCLUSIONS

In this paper we generalized the Dirac equation to (D+1)-dimensional space-time. The conserved angular momentum operators and their quantum numbers are discussed. The eigenfunctions of the total angular momentums are calculated for both odd \( D \) and even \( D \) cases, respectively. The unified radial equations for a spherically symmetric system are obtained. The radial equations with a Coulomb potential are solved by the power series expansion approach. The exact solutions are expressed by the confluent hypergeometric functions. The eigenvalues as well as their fine structure energy are also studied. Our solutions coincide with those in 3+1 dimensional [21] and 2+1 dimensional [22] space-time.

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[22] Shi-Hai Dong and Zhong-Qi Ma, Exact solutions to the Dirac equation with a Coulomb potential in $2+1$ dimensions, preprint.


