Quantum feedback channels

In classical information theory, the capacity of a memoryless communication channel cannot be increased by the use of feedback. In quantum information theory the encoding channel means that feedback is possible, and feedback can increase the capacity of a noisy quantum channel [1]. Memory-assisted entanglement-assisted classical communication also shows that in various cases of non-zero capacity, feedback can increase the capacity of classical communication.

In the classical theory of information transmission through noisy channels, the expression for the classical information capacity $C$ is given by

$$C = \max_{P_X} \left[ H(X) - H(X|Y) \right],$$

where $P_X$ is the mutual information of the source and $H(X)$ is the entropy of the source.

In quantum information theory, the capacity of a quantum channel can be increased by the use of feedback. In this paper, it is shown that the entanglement-assisted classical capacity $C_E$ is precisely equal to the entanglement-assisted classical capacity without feedback $C_E$.

The entanglement-assisted classical capacity for a quantum channel $A$ is given by

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of the ensemble [9]. The fact that the quantity in Eq. (3) is additive is already known [10]. The additivity follows from the arbitrary nature of Alice and Bob’s initial ensemble. Any initial ensemble of states may be held by Bob, with part or all of the ensemble then sent through the feedback channel to Alice. Alice may then encode her information into her part of the ensemble and transmit this back through the noisy channel. As Bob’s initial states may consist of the states that have been generated in the previous steps of any optimal multiple use, or asymptotic, protocol, the maximum in Eq. (3) is a maximum for any step of such a protocol. The total change in mutual information of any n step protocol must necessarily be less than n times the maximum change in any single part of the protocol. The additivity thus ensures that the asymptotic quantity, \[ C_E^{QB} = \sup_n \frac{1}{n} \Delta \chi^{(n)}_A, \] (4)

where the second line follows due to the subadditivity of the first term in the first line and the linearity of the quantum operation \( \Lambda \). The rest of the proof follows the derivation of the upper bound on the entanglement assisted capacity given by Holevo [11]. By the monotonicity of the conditional entropy, \( S(Q|R) = S(\rho_{QR}) - S(\rho_R) \), each of the terms, \(-S(A_i'|B_i) = S(\rho_{A_i'}) - S((\Lambda_A \otimes \mathbb{I}_B)\rho_{A_i'B_iB_i'})\), is bound by,

\[ -S(A_i'|B_i) \leq -S(A_i'|B_i; E_i) = -S(A_i'|R_i), \] (8)

for, \( \rho_{A_i'B_i} = \rho_{A_i'B_iE_i} \), a purification of the state \( \rho_{A_i'R_i} \) with an environment \( E_i \). A pure state implies equality in the subentropies, \( S(\rho_{A_i'}) = S(\rho_{A_i'R_i}) \), and so from Eq. (8) we have,

\[ S(\rho_{A_i'}) - S((\Lambda_A \otimes \mathbb{I}_B)\rho_{A_i'B_i}) \leq S(\rho_{A_i'}) - S((\Lambda_A \otimes \mathbb{I}_R)\rho_{A_i'R_i}). \]

Given a purification \( \rho_{QR} \) of \( \rho_Q \) with reference system \( R \), the function, \( \rho_Q \rightarrow S(\rho_Q) - S((\Lambda_A \otimes \mathbb{I}_R)\rho_{QR}) \), is concave. Hence, the last two terms of Eq. (7) are bounded above by \( S(\rho_{A_i'}) - S((\Lambda_A \otimes \mathbb{I}_B)\rho_{A_i'B_iB_i'}) \), where \( \rho_{A_i'B_i} \) is a purification of the state, \( \rho_A = \sum_i p_i \rho_{A_i'B_i} \). Combining this bound with Eq. (7) and maximizing over the ensemble then gives the required inequality in Eq. (5).

The above result does not extend to the case of unassisted capacities with quantum feedback. As Bob can share an unlimited number of maximally entangled states with Alice through the feedback channel, the unassisted capacity with quantum feedback is equal to the entanglement assisted capacity. Any channel with entanglement assisted capacities higher than the corresponding unassisted capacities therefore has a higher capacity with quantum feedback, for example, the qubit erasure channel [12] or the qubit depolarizing channel with entanglement fidelity, \( 0.25 < F < 0.75 \) [13]. Whether or not classical feedback can increase the unassisted classical capacities of any noisy quantum channel is still not known. For the classical capacities we then have the inequalities,

\[ C \leq C^{FB} \leq C^{QB} = C_E, \] (9)

with at least one inequality in each set being strict for many types of channels. In the case of the unassisted quantum capacity an example exists whereby the capacity is increased by classical feedback, the qubit erasure channel. The qubit erasure channel has a known quantum capacity of, \( Q_{erasure} = 1 - 2\epsilon \), for erasure probability \( \epsilon \) [12, 14]. Alice begins by sending halves of maximally entangled states through the channel, which arrive intact with probability \( 1 - \epsilon \), and are rendered useless with probability \( \epsilon \). Bob informs Alice, via the feedback channel, whether or not the transmission has been successful. In this way, Alice and Bob can share maximally entangled states at a rate, \( R^{FB}_E = 1 - \epsilon \). On \( N \) uses of the channel
Alice and Bob need only utilize $M < N$ channels in order to share enough entanglement to use an entanglement-assisted code [15] for the remaining $N-M$ channels. Thus, $M R^{{	ext{FB}}} = (N - M) E_Q$, for $E_Q$ the minimum entanglement required for the entanglement-assisted code. The capacity for this protocol is then, $N Q^{{	ext{FB}}} = (N - M) Q_E$, and we find,

$$Q^{{	ext{FB}}} = \left(\frac{R^{{	ext{FB}}}}{R^{{	ext{FB}}} + E_Q}\right) Q_E . \quad (10)$$

For the qubit erasure channel, all the quantities on the right of Eq. (10) are known, with, $R^{{	ext{FB}}} = 1 - \epsilon$, $E_Q = \epsilon$, and, $Q_E = 1 - \epsilon$, and thus,

$$Q^{{	ext{FB}}} = 1 - 2\epsilon + \epsilon^2 > Q_{\text{erasure}} , \quad (11)$$

whenever, $0 < \epsilon < 1$. The qubit erasure channel is also an example of a channel for which $Q < Q_2$, where $Q_2$ is the quantum capacity with a two-way classical side channel. It may well be the case that, as well as forward classical communication providing no increase in capacity for the unassisted quantum capacity, $Q = Q_{1+2}$ [13, 14], the addition of forward classical communication may provide no increase to the quantum capacity assisted by classical feedback, that is, $Q^{{	ext{FB}}} = Q_2$.

In summary, it is known that the use of feedback in classical information theory cannot increase the capacity of a memoryless channel. In this paper, it was shown that an analogous result holds for quantum information through memoryless quantum channels supplemented with classical or quantum feedback, only in terms of the entanglement assisted capacities of the channel. For channels without entanglement assistance, the capacities with quantum feedback may be increased up to the corresponding entanglement assisted capacities, and an example is given of a channel for which classical feedback increases the quantum capacity.

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