We study the spectrum of scalar-isoscalar excitations in the color-flavor locked phase of dense quark matter. The sigma meson in this phase appears as a four-quark state (of diquark and antidiquark) with a well-defined mass and extremely small width, as a consequence of its small coupling to two pions. The quark particle/hole degrees of freedom also contribute significantly to the correlator just above the threshold $2\Delta$, where $\Delta$ is the superconducting gap.


I. INTRODUCTION

The generally accepted theory of strong interactions, Quantum Chromodynamics (QCD), has several remarkable and unique features, one of them being the spontaneous breakdown of chiral symmetry ($\chi_{SB}$) in the vacuum, which dictates the physics of the low energy hadronic world (light pseudoscalars). This is borne out by the predictive power of current algebra techniques, chiral effective Lagrangians and chiral perturbation theory to pion mediated processes in the threshold region [1–3]. More recently, a consistent calculational scheme extending to the resonance region has been developed [4]. Chiral symmetry breaking in the vacuum is characterized by the existence of an order parameter, the quark condensate $\langle \bar{q}q \rangle$. Long wavelength axial-like phase fluctuations of the condensate correspond to the familiar pions, while the quantum fluctuations of its norm represent the sigma meson $\sigma \sim \langle (\bar{q}q)^2 \rangle$. Empirically, the $\sigma$ meson is seen as a broad resonance in the $I = J = 0$ channel of $s$–wave $\pi\pi$ scattering, and it’s large width (600-1000 MeV), comparable to it’s mass (400-1200 MeV) [5] follows from its strong coupling to two pions. From a theoretical standpoint, the $\sigma$ meson is invoked in any linear realization of chiral symmetry breaking such as in the Gell-Mann Levy linear sigma model [6] or the Nambu Jona-Lasinio model [7]. Despite the difficulties in the experimental resolution of the sigma meson, there are strong theoretical arguments for it’s importance in vacuum hadronic physics [8]. It’s spectral function is expected to change when finite temperature and/or baryon density effects are included [9,10], these effects being linked to the evolution of the chiral condensate with temperature ($T$) and baryon density ($\mu_B$). Qualitatively similar effects can be reproduced with $p$–wave renormalization of the two pions to which the ‘bare sigma’ of mass $\sim 800$ MeV couples [11]. A particular and consistent feature of these studies, corroborated by experimental data from the CHAOS experiment on $A(\pi,2\pi)$ knock-out reactions, is the spectral enhancement at the $2m_\pi$ threshold, indicative of a softening of the $\sigma$ meson and perhaps a partial restoration of chiral symmetry at finite baryon density [9]. In this way, the sigma meson is more likely to be identified clearly in hot and dense matter.

Thus far, the spectral enhancement has been investigated at temperatures and densities around the chiral phase transition. At much higher densities corresponding to perhaps (5-10)$\rho_0$ or more, ($\rho_0$ being the saturation density of nuclear matter) the physics of the $\sigma$-meson is as yet unexplored, though certainly accessible in light of recent advances in theories of high density quark matter, which have shown that a BCS-like quark phase is energetically preferred. The idea of color superconducting quark matter is now a few decades old [12–14] but renewed interest in this field began only a few years ago [15–20]. Since single quarks are expected to pair with large gaps of a hundred MeV or more [15,16], it is the low energy excitations of the system that determine the response to small external perturbations. For the case of pairing among two light flavors, chiral symmetry remains unbroken whereas for three light flavors, a color-flavor locked (CFL) phase is favored which breaks chiral symmetry because left- and right-handed flavor rotations are locked to the color gauge field. In this letter, we study the response of CFL matter to an external scalar probe. We find a light $\sigma$ meson ($\sim 250$ MeV for large quark gaps) that is sharply peaked owing to it’s small coupling to a two-pion correlated state. Above a threshold energy of $2\Delta$, the quark particle/hole degrees of freedom can also couple to the external probe with appreciable strength. We consider the possibility of dilepton and neutrino production from the scalar mode in dense matter. These processes constitute the leptonic decay of the sigma meson in this dense phase.
The scalar probe will couple to a $q\bar{q}$ excitation. In this section, we therefore evaluate the contribution of the quark loop to the scalar correlator $\langle T\{\bar{\psi}(x)\psi(y)\}\rangle_{CFL}$. In momentum space, the quark loop is given by

$$G(Q) = -\frac{1}{2} \text{Tr}_{q,s,c,f,NG} \left[ \Gamma S(K) \Gamma S(P) \right]$$

(1)

Here, $Q = (Q_0, \mathbf{Q})$ is the external momentum, while $K = q + Q/2, P = q - Q/2$ with $q$ being the internal momentum of the loop. The trace is performed over the internal momentum $q$, spin $s$, color $c$, flavor $f$ and Nambu-Gorkov NG indices. The scalar vertex is simply $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where $1 = 1_c \times 1_f \times 1_s$. The identity matrix in flavor reads $\text{diag}(1,1,0)$ so that only $u,d$ quark fields are introduced. This is in accordance with the light quark nature of the sigma meson at zero density. At high density, the $s$-quark can however, appear inside the loop in between two gap insertions on the same quark line. The structure of the entries of the propagation matrix $S$ in the Nambu-Gorkov basis are detailed in the Appendix. As shown there, in the large $\mu$ limit, simplified expressions may be used for the quark propagator. As we are interested in generating a spectral function, we will study the energy dependence of the correlator and work at zero three-momentum $Q = (Q_0, \mathbf{0})$. The result of tracing over the internal symmetries yields (see Appendix A for details)

$$G(Q_0) = \int \frac{d^4 q}{(2\pi)^4} (T_1 + T_2 + T_3 + T_4);$$

(2)

$$T_1 = -\frac{1}{2} \text{Tr}_{s,c,f} \left[ 1S_{11}(K)1S_{11}(P) \right]$$

$$= \left\{ -6 \left( k_0 + k_{||} \right) (p_0 - \mu - |p|) \right\} - \frac{3(m_u^2 + m_d^2)}{4\mu^2} \left[ \left( \frac{p_0 + p_{||}}{p_0 - \epsilon^2_p} \right)^2 + \left( \frac{k_0 + k_{||}}{k_0 - \epsilon^2_k} \right)^2 \right]$$

$$+ \frac{m_u^2 + m_d^2 + 2m_s^2}{\mu^2} \left( \frac{|G(K)|^2}{k_0^2 - \epsilon^2_k} \right) + (K \leftrightarrow P).$$

(3)

$$T_3 = \frac{1}{2} \text{Tr}_{s,c,f} \left[ 1S_{12}(K)1S_{21}(P) \right]$$

$$= \frac{2m_u m_d}{\mu^2} \left( \frac{G(K)G^*(P)}{(k_0^2 - \epsilon^2_k)(p_0^2 - \epsilon^2_p)} \right).$$

(4)

$T_2$ and $T_4$ follow from $T_1$ and $T_3$ respectively under the substitutions $\pm \mu \leftrightarrow \mp \mu, \pm |q| \leftrightarrow \mp |q|, \pm G(q) \leftrightarrow \mp G^*(q)$. The terms in $T_1$ which come from the diagonal elements of the Nambu-Gorkov propagator have the following interpretation. The first term is the particle/hole-antiparticle/antihole contribution arising from the massless part of the quasiparticle propagator. The particle-hole contribution from a massless propagator is finite only for non-zero $\mathbf{Q}$, therefore it does not appear here. The reason is that $\psi\psi$ flips chirality whereas at $\mathbf{Q} = 0$, the particle and hole have the same helicity (same as chirality for massless particles). The second term owes its origin to mass insertions that can flip chirality (two mass insertions on only one of the two lines of the loop gives the first term in the square bracket while one mass insertion each in both lines gives the second). For quarks near the Fermi surface, the gap affects the propagation. The third term includes the possibility of having two gap insertions in between two mass insertions on the same quark line. $T_3$ originates from the off-diagonal parts of the Nambu-Gorkov propagator. As such, it involves gap insertions on each quark line as well as mass insertions to flip chirality. The delineation of these terms allows us to understand the associated contribution to the spectral function that relates to the imaginary part of the correlator. The imaginary parts are obtained after analytic continuation as $Q_0 = \omega + i\epsilon$, and are plotted in Fig.(1) for 2 sets of quark chemical potentials and gaps.

We begin by considering the first two terms of $T_1$ and $T_2$ which represent the quark-antiquark response from the ‘bulk’ of the matter. Assuming the antiquarks to be ungapped (since the anti-gap dependent pieces in the propagator appear only at $O(\mu^2)$), these terms are relevant to energies greater than $2\mu + \Delta$ (an antiquark needs energy $\approx 2\mu$ to be freed while the quark needs only an energy $\Delta$ (see Eqn.(B2)). The terms inside the square bracket from $T_1$ and $T_2$ which come from mass insertions (see Eqn.(B3)) can be reliably estimated only for typical energies greater than the light quark mass because of the small mass expansion of the quark propagators. This contribution arises at low energies and extends up until $\omega < 2\epsilon_q$. The remaining term from $T_1$ and $T_2$ involves two gap insertions in addition to two mass insertions on the same quark line. The resulting contribution to
the spectral function (see Eqn.(B6)) shows that the strange quark mass appears at this level even though the external current involves only light quarks. The reason is the color-flavor locked structure of the gap. Similar caveats as for the first two terms of $T_1, T_2$ apply here as well ($m_{u,d} < \omega < 2\epsilon_q$).

The contribution from the off-diagonal components of the quark propagator involve diquarks (Eqn.(B7)). It is proportional to $\Delta^2$ as well as the light quark masses and contributes significantly just above the threshold to break a pair, i.e., $2\Delta$. As we are interested in moderately large chemical potentials, we will rely on instanton-based non-perturbative estimates of the gap, which may be as large as a few hundred MeV at intermediate densities [18].

![Graph](image_url)

FIG. 1. Cut of the loop involving the quarks and diquarks. The response has the usual $\bar{q}q$ threshold as well as a pair-breaking threshold at $\omega = 2\Delta$.

### III. SCALAR CORRELATOR IN THE CFL PHASE II: SIGMA MESON

Based on weak coupling analyses to leading logarithm accuracy, effective Lagrangian approaches to QCD in the CFL phase have proven to be useful with applications to color-flavor anomalies, pseudo-scalar meson masses, hidden local symmetry and vector meson properties [21–25]. Here, we use the effective Lagrangian to obtain the coupling of the $\sigma$ meson in the CFL phase to a scalar probe, and we model its width in the dense phase via a Breit-Wigner resonance through its coupling to two generalized pions [21]. In this way, we show that the $\sigma$ meson appears as a sharp low mass excitation and makes an important contribution to the scalar correlator.

The low energy excitations of the CFL phase are two singlet modes associated with $U(1)_B$ and $U(1)_A$ symmetry breaking (corresponding to vector and axial baryon number fluctuations respectively) and an octet of Goldstone modes associated with chiral symmetry breaking. The octet is described by a low energy theory that bears strong resemblance to chiral perturbation theory [22,23]. The Goldstone modes are parametrized by a $3 \times 3$ unitary matrix $U$ which is a color singlet, transforming under $SU(3)_L \times SU(3)_R$ as $U \rightarrow g_L U g_R^\dagger$, where $U$ is related to axial-like fluctuations of the left and right handed diquark fields:

\[
L^{ai} \sim \epsilon^{abc} \epsilon^{ijk} \langle q^b_L q^c_L \rangle^*, \quad R^{ai} \sim \epsilon^{abc} \epsilon^{ijk} \langle q^b_R q^c_R \rangle^*, \quad \text{and} \quad U = LR^\dagger.
\]

The fields $L$ and $R$ carry color $(ijk)$ and flavor $(abc)$, and transform under $g_f \subset SU(3)_f$ and $g_c \subset SU(3)_C$, (where $f$ denotes left or right handed flavor) as

\[
L \rightarrow g_L L g_C^\dagger \quad \text{and} \quad R \rightarrow g_R R g_C^\dagger,
\]

respectively. The low energy effective theory is governed by the Lagrangian [24]
\[
\mathcal{L}_{\text{eff}} = \frac{f_{\pi}^2}{4} \text{Tr} \left[ \nabla_0 U \nabla_0 U^\dagger - v_{\pi}^2 \partial_i U \partial_i U^\dagger \right] \\
+ \left[ A_1 \text{Tr} (MU^\dagger) \text{Tr} (MU^\dagger) + A_2 \text{Tr} (MU^\dagger MU^\dagger) + A_3 \text{Tr} (MU^\dagger) \text{Tr} (M^\dagger U) + H.C. \right] + \ldots ,
\]\n
where \( \nabla_0 \) includes Bedaque-Schäfer effective mass terms in addition to the usual partial time derivative, and \( v_\pi^2 \) denotes the Goldstone boson velocities.\(^1\) In \([24,26]\), the coefficients \( A_{1,2,3} \) in the chiral theory have been matched to an effective high density theory at the Fermi surface, obtained by integrating out the high momentum modes \([27,28]\), such that the energy shifts from the mass terms in the two theories are in agreement. Their values are \([24,26]\)

\[
A_1 = -A_2 = \frac{3\Delta^2}{4\pi^2}, \quad A_3 = 0.
\]

As we are interested in the mass of the \( \sigma \) meson and it’s coupling to two pions, we show that they can be obtained from the mass terms in the following way. Although Eqn.(7) denotes a non-linear sigma model, we note that the sigma field represents fluctuations about the finite expectation value of \( \langle \langle q\bar{q} \rangle \rangle \). Therefore, fluctuations about the mass terms, which appear multiplied to the chiral field \( U = L^I R \), the phase field of \( \langle q\bar{q}\bar{q}\rangle \), can be understood as the sigma meson. This is analogous to fluctuating radially about the non-trivial minimum of the potential for an \( O(4) \) linear sigma model to obtain the sigma field. The leading mass terms in the chiral effective theory have a different form than those in vacuum chiral perturbation theory if instanton induced interactions are suppressed at high density \([16]\), leading to a restoration of \( U(1)_A \) symmetry. In that case, terms like \( \text{Tr} MU \) are forbidden. The mass terms appearing in Eqn.(7) instead respect the \( Z_2 \) symmetry of the theory. The restoration of \( U(1)_A \) symmetry is also the reason that we do not concern ourselves here with the physics of the \( \eta' \) meson (This may be included in the effective Lagrangian by considering the overall \( U(1) \) phases of the condensate as in \([24,26]\)). Thus, the parts of the effective Lagrangian that are relevant to the mass and coupling (to two pions) of the sigma are the terms proportional to \( A_1 \) and \( A_2 \). These terms contain the coefficients of the fields \( \sigma^2 \) and \( \sigma \pi \pi \). Expanding the chiral field \( U \) to \( O(\pi^2) \) via the parametrization \( U = \exp(2\pi^a \tau^a / f_\pi) \), (where the \( SU(3) \) generators \( \lambda^a = 2\tau^a \) are normalized as \( \text{tr}[\lambda^a \lambda^b] = 2\delta^{ab} \)), we find that (see Appendix C for details)

\[
m_{\sigma}^2 = \frac{6\Delta^2}{\pi^2}, \quad g_{\sigma \pi \pi} = \frac{m_{\sigma}^2(\delta m)}{f_{\pi}^2}
\]

where \( \delta m = m_d - m_u > 0 \). The non-linear sigma model is obtained by integrating out the sigma field, therefore, no kinetic term for it appears, which prevents from properly normalizing the \( \sigma \) mass term. However, we have in mind a linear realization of chiral symmetry breaking in order to assess the contribution of the \( \sigma \) meson to the scalar correlation function. Thus, the chiral field can as well be written as \( U = \sigma + i\gamma^5 \pi.\tau \), leading to the canonical form of the kinetic term \( \frac{1}{4}(\partial_{\mu} \sigma)^2 \). The spectral function of the \( \sigma \) meson is now modelled as a Breit-Wigner resonance with a mass \( m_{\sigma} \) and a width determined by it’s coupling to a two-pion state, as detailed in Appendix C and displayed in Fig.(2).

In order to assess the contribution of the \( \sigma \) meson to the scalar correlator, we need to compute the overlap of \( \bar{\psi}\psi \) with the \( \sigma \) field. This is not straightforward since the \( \sigma \) involves diquark fields. The procedure involves an extension of the method discussed in Ref. \([29]\) to couple fermion fields to the Goldstone modes, which also involve diquark fields. For the CFL realization, the invariant coupling of fermion fields to the Goldstones can be written as \([29]\)

\[
-\frac{\Delta}{2} \sum_{I=1,\ldots,3} \text{Tr}[\psi L^I]^T C \epsilon_I (\psi L^I) \epsilon_I ] = -\frac{\Delta}{2} \sum_{I,I'=1,\ldots,3} \text{Tr}[\psi^T C \epsilon_I L_{II'} \psi \epsilon_{I'}] .
\]

where \( \psi \) is to be considered as a \( 3 \times 3 \) matrix in color and flavor space, and \( (\epsilon_I)_{ab} = \epsilon_{Iab} \) with \( a, b \) denoting color indices. In a model where chiral symmetry breaking is realized linearly, the \( \sigma \) meson is introduced as the norm of the chiral field, with the pions being the fluctuations along the directions that are degenerate with respect to axial rotations. In that case, the chiral field \( U = \sigma + i\gamma_5 \pi.\tau \). Since \( L \) can be expanded as \( L = 1 + \frac{2\pi^a \tau^a}{f_{\pi}} + \ldots \) and \( U = L^I R = L^I \) (for a unitary choice of gauge), an analogous expression for the fermion coupling to \( \sigma \) follows

\(^1\)In the weak coupling limit, we have \( v_\pi^2 = 1/3 \) \([26]\). As a consequence of the breaking of Lorentz invariance in matter, there are two pion decay constants: the temporal decay constant \( f_T = f_S \) and the spatial decay constant \( f_S = v_\pi^2 f_S \). Indeed, axial vector current conservation demands that \( f_T w^2 - f_S q^2 = 0 \) for an on-shell pion.
FIG. 2. Spectral function of the $\sigma$ meson as modelled by a Breit-Wigner resonance, with mass and width determined from the chiral effective Lagrangian Eqn.(7).

$$\frac{-\gamma_5 \Delta}{2f_\pi} \sum_{I,I'=1,2,3} \text{Tr}[\psi^T C \epsilon_I I_2 \psi \epsilon_{I'}] \ , \tag{11}$$

where $I_2 = \text{diag}(1,1,0)$ since the $\sigma$ meson involves the $u,d$ flavors. Thus, only $I,I' = 1,2$ will give a finite result for the coupling. We can now proceed to evaluate the overlap between the coherent $\sigma$ excitation and the scalar operator $\bar{\psi}\psi$.

$$J_{\bar{\psi}\psi} = \Delta \sum_{I,I'=1,2} \text{Tr}[1 \ S(P) \ T \ S(K)]; \quad T = \left( \begin{array}{cc} 0 & \gamma_0^0 \epsilon_I (I_2) \epsilon_{I'} \gamma_0^0 \gamma_5^0 \\ \epsilon_I (I_2) \epsilon_{I'} \gamma_5^0 & 0 \end{array} \right) \ . \tag{12}$$

The traces are performed over Nambu-Gorkov indices and all other internal symmetries. Particulars of the loop momentum integration are explained in Appendix D and we quote the final result here (ignoring the energy dependence of the gap).

$$J_{\bar{\psi}\psi} = \Delta^2 \left( m_u + m_d + 2m_s \right) \left[ \Theta(2\Delta - \omega)2x_0 + \Theta(\omega - 2\Delta) \left( \int_{-\mu}^{\max(-\mu,-\sqrt{\omega^2/4-\Delta^2})} dq \right) + \int_{\sqrt{\omega^2/4-\Delta^2}}^{\Lambda_*} dq \right] \epsilon_q \ , \tag{13}$$

with $\Lambda_* = \left( \frac{4\Lambda_6}{\pi^2 m^*} \right)$ denoting an upper cutoff on the $dq$ integration that is set by the leading logarithm estimate of the gap with perturbatively screened gluons [30]. The $\sigma$ meson resonance contribution to the imaginary part of the scalar correlation function is given by $J_{\bar{\psi}\psi}\rho_{\sigma} J_{\bar{\psi}\psi}(\omega)$, which is shown included with the quark loop contribution in Fig(3). The spectral function derived from the full correlator is shown in Fig.(4).

**IV. $\sigma$ COUPLING TO ELECTROWEAK PROBES**

The relevance of scalar correlations to dilepton emission in a dense quark medium has been addressed previously in [31], where it was shown that resonant interactions in the $\bar{q}q$ scalar channel yield an emission rate that is much lower than even the perturbative $\bar{q}q$ Born rate by 2-3 orders of magnitude. Therein, the chosen densities and temperatures were centered in the vicinity of the chiral phase transition, and were motivated by fits to experimental hadron abundancies measured in ultrarelativistic S and Pb beam collisions at SPS energies. The dilepton rate from the vector-isovector ($\rho$) excitation in the ordered CFL phase, which may occur at somewhat higher densities and much
FIG. 3. Imaginary parts from the quark loop and the $\sigma$ meson contribution. The latter appears as a low mass sharp resonance.

FIG. 4. The full spectral function including contributions from the quark loop and the sigma meson.

lower temperatures than considered in [31] was assessed through the Hidden Local Symmetry approach in [25] and compared to standard hadronic approaches with in-medium effects. Here, we can comment on the possibility of dilepton emission from the scalar excitation along the lines of [31]. The CFL quarks couple to the in-medium photon $\tilde{A}_\mu$ with strength $\tilde{e} = e \cos \theta_{CFL}^\gamma$ via the vertex

$$ (\Gamma^\mu)_{ij}^a = (\bar{Q})_{ab}^i \hat{\gamma}^\mu = \frac{1}{\sqrt{3}} \left[ (\lambda_8)^{ij}_a \delta_{ab} - \delta_i^j (\lambda_8)_{ab} \right] \hat{\gamma}^\mu, \quad (14) $$

$$ ^2 \tan \theta_{CFL} = \frac{2e}{\sqrt{3}g}, \text{ and } e, \ g \text{ denote the gauge couplings of } A_\mu, \ G_\mu \text{ respectively.} $$
The spin and color-flavor (with conjugated quarks, respectively. According to Ref. [34], the general entries of the matrix read

\[ \mu_{M} \text{antiparticle-gap.} \]

\[ \psi_{f}(\text{emission via coupling to the effective coupling vanishes and the scalar mode does not contribute to dilepton emission. However, neutrino pair} \]

\[ \text{from this process is not expected to be compete with processes involving weak decays of the pseudoscalar Goldstone} \]

\[ \text{as far as the coupling to neutral currents is concerned.} \]

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\section*{APPENDIX A: NAMBU-GORKOV PROPAGATOR}

In the CFL phase, all quarks acquire a gap. It is convenient to describe their propagation in the Nambu-Gorkov formalism by the following matrix

\[ S = -i(\bar{\psi}\gamma) \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \]

in terms of the two-component Nambu-Gorkov field \( \psi=(\psi,\psi_{C}) \), where \( \psi \) refers to quarks and \( \psi_{C} = C\bar{\psi}^{T} \) to charge-conjugated quarks, respectively. According to Ref. [34], the general entries of the matrix read

\[ S_{11}(q) = -i(\psi(q)\bar{\psi}(q)) = \left\{ \left( G_{0}^{+}(q) \right)^{-1} - \gamma^{0}\Delta^{\dagger}(q)\gamma^{0}G_{0}^{-}(q)\Delta(q) \right\}^{-1} \]

\[ S_{12}(q) = -i(\psi(q)\bar{\psi}_{C}(q)) = -G_{0}^{+}(q)\gamma^{0}\Delta^{\dagger}(q)\gamma^{0}G_{0}(q)S_{11}(q) \]

\[ S_{21}(q) = -i(\psi_{C}(q)\bar{\psi}(q)) = -G_{0}^{-}(q)\Delta(q)S_{11}(q) \]

\[ S_{22}(q) = -i(\psi_{C}(q)\bar{\psi}_{C}(q)) = \left\{ \left( G_{0}^{-}(q) \right)^{-1} - \Delta(q)G_{0}^{+}(q)\gamma^{0}\Delta^{\dagger}(q)\gamma^{0} \right\}^{-1} \]

with \( (G_{0}^{\pm}(q))^{-1} = \frac{1}{\Delta} \pm \mu_{q}\gamma_{0} - m \), where \( \mu_{q} \) denotes the quark chemical potential. In general \( m = \text{diag}(m_{u}, m_{d}, m_{s}) \).

The spin and color-flavor \((c, f)\) structure of the superconducting gap \( \Delta \) is made evident as

\[ \Delta(q) = MG(q)\Lambda^{+}(q) + MG(q)\Lambda^{-}(q) \]

where \( \Lambda^{\pm}(q) = \frac{1}{2}(1 + \alpha.e_{q}) \) are the energy projectors and \( M = e_{q}^{a}e_{q}^{b}\gamma_{5} = M^{\dagger} \) with \((e_{a})^{bc} = e^{abc} \). \( G(q) \) denotes the antiparticle-gap.

Using the approximations \( M^{\dagger}M \approx 1_{c.f}, |q_{||} \approx |q| - \mu \), the quasiparticle energies read \( e_{q}^{2} = q_{||}^{2} + |G(q)|^{2}, e_{q}^{2} = (|q| + \mu)^{2} + |G(q)|^{2} \) and the entries in the propagator simplify as follows

\[ S_{11}(q) = \gamma^{0}\left( \frac{q_{0} + q_{0}}{q_{0}^{2} - e_{q}^{2}} \right)\Lambda^{-}(q) + \gamma^{0}\left( \frac{q_{0} - |q|}{q_{0}^{2} - e_{q}^{2}} \right)\Delta^{-}(q) \]

\[ S_{12}(q) = -MG^{*}(q)\frac{q_{0}^{2} - e_{q}^{2}}{2|q_{0}|^{2}}\Lambda^{+}(q) - MG(q)\frac{q_{0}^{2} - e_{q}^{2}}{2|q_{0}|^{2}}\left( \gamma^{0}M^{\dagger}m\Delta^{-}(q) + \gamma^{0}mM^{\dagger}\Lambda^{+}(q) \right) \]
The integral in Eq. (3) is most conveniently evaluated by contour integration in the complex $q_0$ plane after noting that $d^4q = 2\pi q_\perp dq_\perp dq_0$ where $q_\perp = |q_\perp|; q_\perp = q -(q, \hat{P})\hat{P}$ with $\hat{P}$ being a vector of magnitude $\mu$ directed normal to the Fermi surface. The $dq_\perp$ integration runs from 0 to $\Lambda_\perp = 2\mu$. The first terms from $T_1$ and $T_2$ which correspond to the particle-antiparticle piece then give

$$\text{Im}G_{q\bar{q}}(\omega) = \frac{6\mu^2}{\pi^2} \int_{-\infty}^{\infty} dq_\parallel \delta(\omega - 2\mu - \sqrt{q_\parallel^2 + \Delta^2}) \left( 1 + \frac{q_\parallel}{\sqrt{q_\parallel^2 + \Delta^2}} \right), \quad (B1)$$

where $dq_\parallel > 0$ pertains to particles while $dq_\parallel < 0$ to holes. The $dq_\parallel$ integration yields

$$\text{Im}G_{q\bar{q}}(\omega) = \frac{12\mu^2}{\pi} \int \frac{1}{\sqrt{1 - (\Delta/(\omega - 2\mu))^2}} \Theta(\omega - (2\mu + \Delta)) \Theta(\Lambda_r - \sqrt{\omega - 2\mu - \Delta^2}) \ . \quad (B2)$$

The branch cut at $\omega = 2\mu + \Delta$ signifies the threshold for particle-antiparticle excitations.

For the terms inside the square bracket of $T_1$ and $T_2$, we have

$$G(Q_0) = -\frac{3(m_3^2 + m_4^2)}{4\mu^2} \int \frac{d^4q}{(2\pi)^4} \left\{ \left( \frac{p_0 + p_\parallel}{\sqrt{p_0^2 - \epsilon_p^2}} \right)^2 + (p_\parallel \rightarrow -p_\parallel) \right. \right.$$  
$$\left. + \frac{(k_0 + k_\parallel)(p_0 + p_\parallel)}{(k_0^2 - \epsilon_k^2)(p_0^2 - \epsilon_p^2)} + (k_\parallel \rightarrow -k_\parallel) \right) + (K \rightarrow P) \right\} \ . \quad (B3)$$

The arrangement of poles in the complex $q_0$ plane depends on the external energy $\omega$. For the terms in the first line of Eqn.(B3), which arise from two mass insertions in the same line, the residues from the poles exactly cancel if $\omega > 2\epsilon_q$. A finite residue is obtained in the case $\omega < 2\epsilon_q$ only. The interpretation is that a particle disappears into the Dirac sea and reappears from it at a later time, corresponding to the two mass insertions. The terms on the second line which correspond to one mass insertion on each quark line, lack an imaginary part and do not contribute to the spectral function. Note that we have used the large $\mu$ approximation for antiparticles so that only particle lines can be cut (only particle propagators appear in Eqn.(B3)). Moreover, the external energy has to exceed the mass gap, so that the above expression is valid only for $\omega$ typically greater than the $u$ and $d$ quark mass. Neglecting the energy dependence of the gap to a first approximation (we will assume it to depend on the chemical potential only), we find

$$\text{Im}G_m(\omega) = -\frac{3}{16\pi^2} (m_u^2 + m_d^2)(w^2 - \Delta^2) \int_{-\infty}^{\infty} dq_\parallel \frac{\delta(2\epsilon_q - \omega)}{\epsilon_q^3} \ . \quad (B4)$$

If $\omega < 2\Delta, q_\parallel$ is unrestricted whereas if $\omega > 2\Delta, |q_\parallel| > \sqrt{\omega^2 - \Delta^2}$ must be satisfied (it should be noted that $q_\parallel$ can be at least $-\mu$). Hence

$$\text{Im}G_m(\omega) = \frac{3(m_u^2 + m_d^2)}{8\pi^2} (w^2 - \Delta^2) \left[ \Theta(2\Delta - \omega) \frac{2}{\Delta^2} + \Theta(\omega - 2\Delta) \left\{ \int_{-\mu}^{\max(-\mu, -\sqrt{\omega^2 - 4\Delta^2})} dq_\parallel + \int_{\sqrt{\omega^2 - 4\Delta^2}}^{\Lambda_r} \frac{dq_\parallel}{\epsilon_q^3} \right\} \right] \ . \quad (B5)$$

The third term from $T_1, T_2$ involving gap and mass insertions on the same quark line evaluates to

$$\frac{(m_u^2 + m_d^2 + 2m_q^2)\Delta^2}{2\pi^2} \left[ \Theta(2\Delta - \omega) \frac{2}{\Delta^2} + \Theta(\omega - 2\Delta) \left\{ \int_{-\mu}^{\max(-\mu, -\sqrt{\omega^2 - 4\Delta^2})} dq_\parallel + \int_{\sqrt{\omega^2 - 4\Delta^2}}^{\Lambda_r} \frac{dq_\parallel}{\epsilon_q^3} \right\} \right] \ . \quad (B6)$$

Finally, the diquark contribution reads
\[
\frac{m_u m_d \Delta}{\pi^2} \left[ \Theta(2\Delta - \omega) \frac{8}{\omega \sqrt{\Delta^2 - \omega^2}} \tan^{-1} \left( \frac{4\Delta^2}{\omega^2} - 1 \right) + \Theta(\omega - 2\Delta) \left\{ \frac{1}{\omega} \int_{\max(-\mu, -\sqrt{\omega^2/4 - \Delta^2})}^{\Delta} \frac{dq}{\sqrt{\omega^2/4 - \Delta^2}} \left( \omega^2/4 - \epsilon_q^2 \right) \right\} \right] + \left( \int_{-\mu}^{\max(-\mu, -\sqrt{\omega^2/4 - \Delta^2})} dq \right) + \int_{\Delta}^{-\mu} dq \right) \left( \omega^2/4 - \epsilon_q^2 \right) \right) \right] .
\]

where \( \omega \) is typically greater than the \( u, d \) quark masses.

**APPENDIX C: \( \sigma \) MESON MASS AND WIDTH**

The relevant terms from the effective Lagrangian Eqn.(7) are

\[
L_{eff} = A_1 \text{Tr}[(M + \sigma)U] \text{Tr}[(M + \sigma)U^\dagger] + A_2 \text{Tr}[(M + \sigma)U^\dagger(M + \sigma)U^\dagger]; \quad U = \exp(2i\pi^a \tau^a / f_\pi).
\]

Here, \( M = \text{diag}(m_u, m_d, m_s) \) while the \( \sigma \) field is decomposed as \( \sigma = \sigma \otimes 1_f^L \). The flavor structure involves only the \( u, d \) quarks. We expand the chiral field \( U \) to \( O(\pi^2) \) and compare the coefficients of the \( \sigma^2 \) and \( \sigma \pi \pi \) terms to the kinetic term and two-pion coupling term of the \( \sigma \) field.

\[
\frac{1}{2} m_\sigma^2 \sigma^2 = \frac{3\Delta^2}{\pi^2} \sigma^2, \quad g_{\sigma \pi \pi} i \epsilon^{a b i} n^a \sigma n^b \sigma = \frac{6 m \Delta^2}{f_\pi^2} \sigma n^{a b} \pi^a \pi^b \sigma,
\]

from which the mass of the sigma and it’s coupling to two pions follows(Eqn.(9)). The spectral function of the \( \sigma \) meson can be modelled as a Breit-Wigner resonance

\[
\rho_\sigma(\omega, \mathbf{q} = 0) = \frac{\text{Im} \Sigma(\omega)}{(\omega^2 - m_\sigma^2 - \text{Re} \Sigma(\omega))^2 + (\text{Im} \Sigma(\omega))^2}
\]

where \( \Sigma(\omega) \) denotes the self-energy of the \( \sigma \) meson dressed by the pion loop. It’s imaginary part relates to the decay width \( \Gamma_{\sigma \pi \pi} \) of the \( \sigma \) meson which is determined by \( g_{\sigma \pi \pi} \)

\[
\Gamma_{\sigma \pi \pi} = \frac{g_{\sigma \pi \pi}^2}{24 \pi m_\sigma^2} \sqrt{\frac{m_\sigma^2}{4} - m_\pi^2}
\]

The real part of the self-energy is determined by the mass-shell condition to be

\[
\text{Re} \Sigma_{\sigma \pi \pi}(\omega) = \frac{g_{\sigma \pi \pi}^2}{16 \pi^2} \left[ 1 - \frac{4m_\pi^2}{\omega^2} \right] \ln \left( \frac{1 + \frac{1 - 4m_\pi^2}{\omega^2}}{1 - \frac{1 - 4m_\pi^2}{\omega^2}} \right) - 2 \left( \frac{p_0}{\omega} \right) \ln \left( \frac{\omega_0 + p_0}{m_\pi} \right)
\]

\[
p_0 = \sqrt{\frac{m_\pi^2}{4} - m_\pi^2}; \quad \omega_0 = m_\sigma/2 = \sqrt{m_\pi^2 + p_0^2}
\]

**APPENDIX D: \( \sigma \) MESON CONTRIBUTION**

As the \( \sigma \) meson couples to a quark-conjugate quark pair, a gap insertion on a quark line is required for an overlap with a quark-antiquark source. Moreover, a chirality flip via a mass insertion is essential since left and right-handed quarks pair with their own chirality species only. This necessitates retaining the leading mass terms in the expansion of the propagators \( O(m^2) \) terms need not be retained however). Since there is an overall factor of \( \Delta/f_\pi \) in the \( \sigma \) coupling to the fermions, we expect the final result to be proportional to the product of \( \Delta^2/f_\pi^2 \) and some combination of the quark masses (including the strange quark mass which comes in because of the color flavor locked structure of the gap). Performing the trace over all internal symmetries, we obtain

\[
J_{\sigma \psi} = \frac{\Delta}{f_\pi} \int \frac{d^4q}{(2\pi)^4} \left\{ \alpha \left[ \frac{p_1|\Delta(K)}{(k_0^2 - c^2_k)(p_0^2 - c^2_p)} + \frac{k_1|\Delta(P)}{(k_0^2 - c^2_k)(p_0^2 - c^2_p)} \right] + \beta \left[ \frac{\Delta(K)}{k_0^2 - c^2_k} + \frac{\Delta(P)}{p_0^2 - c^2_p} \right] \right\}
\]

\[
\alpha = \frac{4(m_u + m_d + 2ms)}{\mu}; \quad \beta = \frac{4(m_u + m_d + 2ms)}{\mu^2}.
\]
To simplify the momentum integration, the gap is assumed to be energy-momentum independent, and can be pulled out of the integration. As before, $d^4q = 2\pi q_+ dq_+ dq_⊥ dq_0$ and the contour integration over complex $q_0$ is performed first. Then, the terms proportional to $q$ yield equal and opposite residues in both cases $\omega > 2\epsilon_q$ and $\omega < 2\epsilon_q$. Thus, if the mass and gap insertions are on different lines, the result is zero. A finite coupling is obtained only when these two insertions appear on the same quark line. In other words, only the mixed terms (the mass and gap insertions are on different lines, the result is zero. A finite coupling is obtained only when these two insertions appear on the same quark line. In other words, only the mixed terms ($\sim m_\Delta$) that appear in the expansion of the propagator contribute. These are precisely the terms proportional to $J$, which give

$$J_{\sigma \bar{\psi} \psi} = \frac{\beta^2 \Delta^2}{4\pi^2 f_\pi} \int_{-\infty}^\infty \frac{dq_+}{\epsilon_q} \Theta(\epsilon_q - \frac{\omega}{2})$$  \hspace{1cm} (D2)

For the case $\omega < 2\Delta$, the $q_\perp$ integration is unrestricted. Since typical momenta are of the order of the gap, we may run the integration from $\Delta$ to $\Lambda_\ast$. For the case $\omega > 2\Delta, |q_\perp| > \sqrt{\frac{\omega^2}{4} - \Delta^2}$ must be satisfied ($q_\perp$ can be at least $-\mu$). Hence

$$J_{\sigma \bar{\psi} \psi} = \frac{\Delta^2 (m_u + m_d + 2m_s)}{\pi^2 f_\pi} \left[ \Theta(2\Delta - \omega)2x_0 + \Theta(\omega - 2\Delta) \left( \int_{-\mu}^{\max(-\mu, -\sqrt{\omega^2/4 - \Delta^2})} dq_\perp \epsilon_q \right) \right]$$  \hspace{1cm} (D3)

where $x_0 = \ln(\Lambda_\ast/\Delta)$.