We introduce the concept of the Einstein–Podolsky–Rosen (EPR) paradox and consider the structure of the entanglement in the bipartite case. We consider the expression of the entanglement between two modes in the context of the EPR paradox. We explore the entanglement between two modes and discuss the implications of this expression. We conclude that the entanglement between two modes is not the same as the entanglement between the modes in the EPR paradox. We show that the entanglement between two modes is a different expression of the entanglement between two modes. We further explore the entanglement between two modes and discuss the implications of this expression. We conclude that the entanglement between two modes is not the same as the entanglement between the modes in the EPR paradox.

\[ (\rho_A \otimes \rho_B) = \text{proj}_H \]
where $a_A^\dagger, a_B^\dagger$ denote the single particle creation operators in the two modes ($A$ and $B$) respectively, and $\hat{N}_A = a_A^\dagger a_A, \hat{N}_B = a_B^\dagger a_B$ are the corresponding boson number operators. The parameter $E_d$ is the single-atom tunneling amplitude, $\Delta_B$ is the difference in the chemical potential between the wells and $K$ corresponds to the atom-atom interaction. Here we only consider $K > 0$, corresponding to a repulsive interaction between atoms. The total particle number, $\hat{N}_A + \hat{N}_B$ is a conserved quantity and is set to the constant value $N$. If we add the constant term
\[
\frac{K}{8} \left( \hat{N}_A + \hat{N}_B \right)^2
\]
the first term in the Hamiltonian (1) becomes
\[
\frac{K}{4} \left( \hat{N}_A^2 + \hat{N}_B^2 \right)
\]
as we expect for repulsive $s$-wave scattering.

This Hamiltonian (1) is in fact a two-site version of the Bose-Hubbard model which describes bosonic particles with repulsive interactions, hopping through a potential lattice [28, 29]. In the Bose-Hubbard model, instead of two modes, there is an infinite lattice of potential wells (or modes) with coherent single-atom tunneling between nearest neighbor modes.

A similar condensed matter system where there is the coupling of two BEC modes is that of atom-molecule Bose-Einstein condensate. In such a situation there exists coherent coupling between atomic and molecular BEC’s respectively which constitute the two modes of the system. The simplest Hamiltonian, recently studied by Vardi et al. [30], which describes the atom-molecule BEC takes the form
\[
H_{AM} = \frac{\delta}{2} a^\dagger a + \frac{\Omega}{2} \left( a^\dagger a^\dagger b + b^\dagger a a \right)
\]
where $a^\dagger$ and $b^\dagger$ denote the creation operators for the atomic and molecular modes, respectively. $\Omega$ is a measure of the strength of the matrix elements for creation and destruction of molecules and $\delta$ is the molecular binding energy in the absence of coupling. The total atom number $\hat{N}_{atm} = n_a + 2n_b$, where $n_a = a^\dagger a$, $n_b = b^\dagger b$, commutes with the Hamiltonian, so is a constant of the motion. Both Hamiltonians (1) and (4) have recently been shown by Zhou et al. [31, 32] to be exactly solvable in the context of the algebraic Bethe ansatz.

III. MANY-PARTICLE ENTANGLEMENT

For qubits - two-dimensional systems represented by the states $|0\rangle$ and $|1\rangle$ - the canonical maximally entangled state is the Einstein-Podolsky-Rosen-Bohm (EPR) pair,
\[
\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)
\]
also known as a Bell state, in reference to the inequalities established by Bell [33]. The tripartite analogue of this state is the Greenberger-Horne-Zeilinger-Mermin (GHZ) state
\[
\frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)
\]
while the corresponding $m$-partite state is given by
\[
\frac{1}{\sqrt{2}} (|0^\otimes m\rangle + |1^\otimes m\rangle)
\]
These states are also known as $m$-particle Cat (m-Cat) states, in honour of Schrödinger’s cat. For systems with three or more subsystems, while most definitely entangled, we cannot say whether the Cat states are maximally entangled. Since there is no definite measure for arbitrary multiparticle entanglement, there is no clear notion of the structure of the maximally entangled states in such systems.

The $d$-dimensional analog of the 2-dimensional qubit is referred to as the qudit. For qudits, represented by the set of states $\{|i\rangle\}$ where $i = 0, \ldots, d-1$, a Cat state would be
\[
\frac{1}{\sqrt{2}} (|00\rangle + |(d-1)(d-1)\rangle).
\]

By the standard measure of entanglement for bipartite systems (the entropy of entanglement which is discussed in section IV) this is not the maximally entangled state. While state (8) is entangled, a maximally entangled state is of the form
\[
\frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle.
\]
While Cat states are the canonical maximally entangled states for systems consisting of two qubits, for higher dimensions and number of subsystems, the maximally entangled states correspond to uniform distributions over the tensor product basis.

In [16], to determine the structure of the canonical entangled states, the system described by the Hamiltonian (1) was decomposed into $N$ subsystems consisting of the individual bosons, each with an internal degree of freedom described by a two dimensional Hilbert space spanned by the two states $|A\rangle$ and $|B\rangle$. In this description, the system is viewed as a collection of $N$ single-qubit subsystems. These internal degrees of freedom can be used to define a two mode description just as the polarization degree of freedom of the electromagnetic field defines individual modes. In this case the annihilation and creation operators, appearing in equation (1), refer to single particle states distinguished by an internal degree of freedom rather than spatially localized single particle states discussed in this paper. However this does not change our point of view regarding the lack of physical significance of entanglement at the level of single atoms.
In [16] it was argued that the maximally entangled state in this case is the $N$-Cat state, which is a coherent superposition state of all particles in mode $A$ and all particles in mode $B$, i.e.
\[
\frac{1}{\sqrt{2}} \left( |A^\otimes N \rangle + |B^\otimes N \rangle \right).
\] (10)

While it cannot be said that this is the maximally entangled state, it does indeed have some entanglement. However, there is a problem with this choice of subsystem partitioning. By the nature of Bose-Einstein condensation, bosons within a condensate are indistinguishable. At no point can one make a physical measurement of the state of an individual particle in the condensate. For entanglement to exist between two systems the individual systems have to be distinguishable. While it is easy to imagine quantum measurements sensitive to individual particles, such operations could not be realized in the laboratory [17]. While one can first remove individual particles from the condensate in order to measure them, the resulting state of the condensate is thereby changed, and it is unclear how the results of such measurements would reveal the multi-atom entangled state of the condensate prior to the removal of the measured particles. This implies that the decomposition into individual boson subsystems is not physically realizable and while one can still write the Hilbert space of the system as a tensor product of the Hilbert spaces of individual bosons, this is not an appropriate description for realizable measurements upon the condensate. In other words, the system of coupled BEC's is best viewed as a bipartite entangled system rather than as a collection of $N$ single-particle subsystems.

Of course there is nothing to stop us from calculating the entanglement between indistinguishable particles according to some measure. However entanglement is a physical resource that enables useful tasks in quantum communication and computation. In all such tasks it is necessary that the entangled subsystems be distinguishable at some point in the protocols. For systems described by the Josephson Hamiltonian (1) it cannot be said that there is physically useful entanglement between individual bosons when they exist in condensate.

### IV. ENTANGLEMENT BETWEEN THE TWO MODES

Since the individual bosons are not physically accessible, distinguishable subsystems of the pair of tunnel-coupled BEC’s described by (1), we need to consider other possible decompositions into subsystems if we are to investigate entanglement characteristics in this system. While we cannot measure which mode of the coupled BEC’s a specific particle is in, the occupation number of a given mode is a physical observable. The two modes, be they spatially separated, or differing in some internal quantum number, are clearly distinguishable subsystems. We can thus view the pair of coupled BEC’s as a bipartite system of the two modes. It is relatively simple to investigate the entanglement between the modes since there is a unique measure of entanglement for two-component systems. Since the modes are distinguishable the entanglement between them is accessible and thus potentially useful for some quantum information or communication protocol. This had been demonstrated by Dunningham et al. [34] who have proposed a scheme for entanglement swapping involving two pairs of tunnel-coupled BEC’s. This is used to concentrate the entanglement between two modes.

In this interpretation, while the entanglement involves many particles it is actually between the modes of the system. To illustrate this, consider the situation where we have just one particle in the system. In this scenario the modes can have occupation numbers of zero and one, so are spanned by the states $|0\rangle$ and $|1\rangle$. Consider the state
\[
\frac{1}{\sqrt{2}} \left( |0\rangle|1\rangle + |1\rangle|0\rangle \right).
\]
Clearly, with respect to the partition into modes, this single-particle state is entangled which implies we have entanglement with only a single particle. An analogous single-particle entanglement has been generated optically and used in a quantum teleportation protocol [36].

The state of each mode is characterized by its occupation number. Because $N$ is constant, a general state of the system $|\psi\rangle$ can be written in term of the Fock states by
\[
|\psi\rangle = \sum_{n=0}^{N} \alpha_n |n\rangle |N-n\rangle
\]
(11)
where $\alpha_n$ are complex coefficients, i.e., $n$ bosons in mode $A$ implies there are $N-n$ bosons in mode $B$.

The standard measure of entanglement of pure states of bipartite systems is the entropy of entanglement, which is the von Neumann entropy of the reduced density operator of either of the subsystems [2, 35]. The reduced density operator of a subsystem is found by tracing out the other subsystem via the partial trace. If $\rho$ is the density operator describing some state of a bipartite system, the reduced density operator for subsystem $A$ is defined by
\[
\rho_A = \text{Tr}_B (\rho)
\] (12)
where $\text{Tr}_B$ is the partial trace over subsystem B. The entropy of entanglement is then given by
\[
E(\rho_A) = -\text{Tr} (\rho_A \log (\rho_A))
\] (13)
\[
= -\sum_k \lambda_k \log (\lambda_k)
\] (14)
where the logarithm is taken in base 2, and $\{ \lambda_k \}$ are the set of eigenvalues of the reduced density operator, $\rho_A$. 

The value of \( E \) varies between 0, for separable product states, to a maximum of \( \log d \) (where \( d \) is the dimension of the Hilbert space of the subsystem) for maximally entangled states corresponding to a completely mixed density operator.

The entropy of entanglement can be calculated from the reduced density operator of either of the subsystems without loss of generality. This follows from the Schmidt decomposition of pure states, which demonstrates that the eigenvalues of the reduced density operators of the two subsystems are identical (pg. 109 of [2]).

**Schmidt decomposition:** For any pure state \( |\psi\rangle \) of a bipartite composite system there exist orthonormal states \( |i_A\rangle \) for subsystem \( A \) and orthonormal states \( |i_B\rangle \) for subsystem \( B \) such that

\[
|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle,
\]

where \( \lambda_i \) are non-negative, real numbers known as Schmidt coefficients, satisfying \( \sum_i \lambda_i^2 = 1 \). It is easy to see from the Schmidt decomposition that the reduced density operators for the two subsystems are, respectively, \( \rho_A = \sum_i \lambda_i^2 |i_A\rangle \langle i_A| \) and \( \rho_B = \sum_i \lambda_i^2 |i_B\rangle \langle i_B| \) which have identical eigenvalues.

Using the Fock basis, from equation (11) the density operator describing a general state of the system is given by

\[
\rho = |\psi\rangle \langle \psi| = \sum_{m,n=0}^N c_m^* c_n |m\rangle \langle N-m| \langle n| \langle N-n|.
\]

Taking the partial trace with respect to mode \( B \) yields the reduced density operator for mode \( A \),

\[
\rho_A = \text{tr}_B (\rho) = \sum_{m,n=0}^N c_m^* c_n |m\rangle \langle N-m| |N-n\rangle \langle n|.
\]

From expression (17) we can see that the reduced density operator in this case is diagonal in the Fock basis and the eigenvalues are simply \( \lambda_i = |c_i|^2 \). Thus the entropy of entanglement between the two modes of the coupled BEC’s is given by

\[
E (\rho_A) = - \sum_{n=0}^N |c_n|^2 \log |c_n|^2.
\]

To determine the maximally entangled state, expression (18) can be optimized with respect to \( x_n = |c_n|^2 \) by imposing the normalization condition \( \sum_{n=0}^N |c_n|^2 = 1 \) with a Lagrange multiplier, \( \mu \) i.e. we maximize

\[
L = - \sum_{n=0}^N (x_n \log x_n - \mu x_n) + \mu.
\]

Differentiating with respect to \( x_n \) gives

\[
\frac{\partial L}{\partial x_n} = \mu - \log x_n - \frac{1}{\ln 2} = 0
\]

which implies

\[
x_n = 2^{\mu - \frac{1}{\ln 2}}
\]

for all \( n \). From the normalization condition

\[
x_n = \frac{1}{\sqrt{N+1}}, \quad \forall n.
\]

So a state with maximum entanglement will have coefficients

\[
c_n = \frac{e^{i\theta_n}}{\sqrt{N+1}}
\]

where \( \theta_n \) is some phase angle. This corresponds to a completely mixed density operator, as expected for a state with maximal entanglement. Thus we can express the canonical maximally entangled state, \( |MES\rangle \) for the system described by the Josephson Hamiltonian (1) as

\[
|MES\rangle = \frac{1}{\sqrt{N+1}} \sum_{n=0}^N |n\rangle |N-n\rangle.
\]

From equation (18), the maximal entanglement is

\[
E_{\text{max}} = - \sum_{n=0}^N \frac{1}{N+1} \log \left( \frac{1}{N+1} \right)
\]

\[
= - \log \left( \frac{1}{N+1} \right)
\]

\[
= \log (N+1)
\]

As mentioned previously, this is what is expected for the maximum entanglement, since the dimension of the Hilbert space of the individual modes is \( N+1 \) (see page 510 of [2]).

### A. Entanglement of the Ground State

As mentioned in Section II, the systems consisting of a pair of tunnel-coupled BEC’s is the simplest system described by the Bose-Hubbard model and corresponds to a lattice potential with just two sites. For the Bose-Hubbard model, in the limit of an infinite lattice, there is a quantum phase transition where the ground state changes from superfluid phase to the Mott insulator phase [28]. Such a transition from the Mott insulator to superfluid phase was recently experimentally observed by Greiner et al. [37].

In the Mott insulator state, particles tend to be localized at the individual lattice sites with no phase coherence across the lattice, whereas in the superfluid
state, each atom is spread over the entire lattice and there exists long-range phase coherence across the lattice. This transition from the Mott insulator to the superfluid state occurs as the ratio of the coupling between lattice sites to the interaction strength increases. As long-range coherences in quantum systems are intrinsically linked to entanglement, it is of interest to quantify the entanglement in the system in relation to this transition. Since there is no measure for the entanglement in systems consisting of three or more subsystems, the two mode system here is the only Bose-Hubbard model for which we can currently give a complete description of the entanglement.

Making the two modes identical (by setting the bias, $\Delta \mu$ to 0), the Hamiltonian (1) was diagonalised numerically for increasing values coupling to interaction ratio, $\frac{\epsilon J}{K}$, and the entanglement of the ground state was determined via equation (17).

Figure 1 shows the results of this analysis for different values of the total particle number, $N$. Since $\epsilon J$ is the tunneling parameter, the larger its value the stronger the interaction between the modes of the system. As such, it is intuitive that for no coupling, the entanglement between the modes is zero and as the coupling increases, the entanglement between the modes in the ground state increases.

The entanglement $\sqrt{\epsilon J}$ asymptotically approaches a constant value as the ratio $\frac{\epsilon J}{K} \to \infty$, which is illustrated more clearly in figure 2, which shows the results for $N = 100$. Now we consider the two extreme parameter values. Firstly for $\epsilon J = 0$, the ground state will have an equal, fixed number atoms in each of the two modes and is therefore the localized state,

$$|\psi_{\text{loc}}\rangle = \left(\frac{N}{2}\right) \left(\frac{N}{2}\right).$$  \hspace{1cm} (25)

Clearly this state has zero entanglement. For $K = 0$, the Hamiltonian consists of just the tunneling term

$$H_{\text{tun}} = -\frac{\epsilon J}{2}(a_A^\dagger a_B + a_B^\dagger a_A)$$  \hspace{1cm} (26)

and it is easy to show that the ground state of such a Hamiltonian for a single particle is the bonding state

$$|+\rangle = \frac{1}{\sqrt{2}}(a_A^\dagger + a_B^\dagger)|0\rangle|0\rangle$$  \hspace{1cm} (27)

where $|0\rangle|0\rangle$ is the vacuum state. So in the ground state for $K = 0$ each individual particle is in the bonding state and the state of the system is the $N$-particle analogue of the bonding state

$$|\psi_{+}\rangle = \frac{1}{\sqrt{2^N N!}}(a_A^\dagger + a_B^\dagger)^N|0\rangle|0\rangle.$$  \hspace{1cm} (28)

This state is the two-site analogue of the superfluid phase, with each atom being spread over the two modes. From equation (18), the corresponding entanglement for this state is

$$E(\rho_+) = -\sum_{n=0}^{N} \frac{1}{2^N N!} \left(\begin{array}{c}N \\ n\end{array}\right) \log \left(\frac{1}{2^N N!} \left(\begin{array}{c}N \\ n\end{array}\right)\right).$$  \hspace{1cm} (29)

Thus for zero coupling, the ground state is the localized state (25) which has no entanglement. As soon as the coupling begins to increase, the entanglement between the modes increases rapidly, as the occupation number of the modes is no longer exact and fluctuations in the phase decrease. As the tunneling amplitude continues to increase, the entanglement asymptotically approaches a maximum value, given by expression (29), which corresponds to the bonding state. In this state, the occupation
In this case, there is no indication of
the amount of entanglement between the modes.
However, for $\theta < \theta_0$ or $\theta > \theta_1$ the
entanglement is lost. From the figure above,
the conclusion is depicted that the
entanglement between the modes in the
entanglement of the system does not increase
in the right direction, in the interaction case.
Moreover, the system does not change the
entanglement of the system, that does not increase
in the right direction, in the interaction case.

\[ x = x + \frac{\varepsilon}{N} = x' \]

\[ \left( \begin{array}{c} p_1 \\ p_2 \\ \end{array} \right) = \left( \begin{array}{c} p_1 \\ p_2 \\ \end{array} \right) + \frac{\varepsilon}{N} \]

C. Dynamics of Entanglement

Dynamics of Entanglement

\[ \begin{align*}
    \langle \hat{r}_x \rangle &= \langle \hat{r}_y \rangle \\
    \langle \hat{r}_z \rangle &= \langle \hat{r}_z \rangle
\end{align*} \]
the original peak in the probability distribution flattens
over the evolution, approaching a completely even distribu-
tion, corresponding to the maximally entangled state
(23). As the evolution continues, the distribution begins
to peak at the extremes and the entanglement decreases.

In terms of modal entanglement, the dynamical scheme
proposed in [16] can still be used to create a close to
maximally entangled state, over a shorter time period
than for the creation of the inaccessible many-particle
entangled state.

It is of interest to note that the critical parameter ratio
(39) is the same as that found by Milburn et al. [13] in
regards to a transition in the dynamics from self-trapping
to delocalization. For an initial condensate localized in
one mode, when \( \frac{2\Omega}{\chi N} > 1 \), the condensate distribution
will remain localized within the mode as it evolves. For
\( \frac{2\Omega}{\chi N} < 1 \), the evolution results in a delocalization of
the condensate distribution between the two modes.

V. THE ATOM-MOLECULE BEC

The atom-molecule BEC described by Hamiltonian (4)
is a similar system to that of the tunnel-coupled pair of
BEC’s. In neither system can we consider the individual
particles (the individual atoms and molecules) as separate,
distinguishable subsystems but both consist of the
coherent coupling of two distinct BEC’s. In the
atom-molecule BEC, the two modes of the system do not dif-
fer spatially or by some internal quantum number but are
rather two chemically distinct components. Nonetheless,
the determination of the entanglement between the atomic
and molecular modes is analogous to the calcula-
tions above for the tunnel-coupled BEC’s.

As before, the state of each mode is characterized by it’s
occupation number, however in the case of the atom-
molecule BEC the set of Fock states spanning the Hilbert
space of the system depends upon whether the total num-
bre of atoms, \( N_{\text{atm}} \), is even or odd. In the case of an even
\( N_{\text{atm}} \), a general state, \( |\chi\rangle \) of the system can be expanded as

\[
|\chi\rangle = \sum_{n=0}^{M} d_n |2n\rangle |M-n\rangle
\]

where \( M = N_{\text{atm}}/2 \), while for \( N_{\text{atm}} \) odd, the general
state \( |\phi\rangle \) can be expressed as

\[
|\phi\rangle = \sum_{n=0}^{M} d_n |2n+1\rangle |M-n\rangle
\]

where in this case, \( M = (N_{\text{atm}} - 1)/2 \) and the \( \{d_n\} \) are
complex coefficients defining the state. In analogy with
expression (17) for the reduced density operator for the
tunnel-coupled BEC’s, the reduced density operator for a
general state of the atom-molecule BEC is given by

\[
\rho_0 = \sum_{n=0}^{M} |d_n|^2 |M-n\rangle \langle M-n|
\]

where the partial trace has been taken with respect to the
atomic mode and \( M \) is defined as above for even and odd
total atom number, \( N_{\text{atm}} \). Thus the entropy of entangle-
ment between the atomic and molecular modes is given
by equation (18), the same expression as for the tunnel-
coupled BEC’s, where \( N \), the total particle number, is
replaced by \( M \) as defined above i.e.

\[
E(\rho_0) = -\sum_{n=0}^{M} |d_n|^2 \log |d_n|^2.
\]

Since the dimension of the subspace of the modes is \( M \),
the maximally entangled states, analogous to (23), are

\[
|M ES_{\text{even}}\rangle = \frac{1}{\sqrt{M+1}} \sum_{n=0}^{M} |2n\rangle |M-n\rangle,
\]

for \( N_{\text{atm}} \) even, and

\[
|M ES_{\text{odd}}\rangle = \frac{1}{\sqrt{M+1}} \sum_{n=0}^{M} |2n+1\rangle |M-n\rangle,
\]

for \( N_{\text{atm}} \) odd and will have entanglement \( \log (M+1) \).

Following the same numerical analysis as in section,
IV A figure 6 shows the results for the variation in the
entanglement of the ground state of the atom-molecule
BEC for differing values of the ratio of the parameters,
\( \frac{\Omega}{\chi} \) and total number of atoms, \( N_{\text{atm}} \). To relate the en-
tanglement structure of the ground state shown in figure
6 to some physical properties of the system we need to
consider other properties of the ground state for increasing
total atom number and parameter ratio \( \frac{\Omega}{\chi} \). Zhou et
al. [32] considered the two zero temperature correlations
\( \langle n_a \rangle \), the average atomic occupation number and the
coherence correlator, \( \theta = \langle a^\dagger a b^\dagger b \rangle \).

Figure 7 shows

\[
\text{FIG. 6: The entropy of entanglement of the ground state of the atom-molecule BEC for increasing values of the ratio } \frac{\Omega}{\chi} \text{ and atom number, } N_{\text{atm}}.
\]
the results for the average atomic occupation number and the coherence correlator for the same parameter ranges used for the entanglement calculations.

![Graph](image)

**FIG. 7.** The average atomic occupation number (a) and the coherence correlator (b) for the ground state of the atom-molecule BEC. Note that both the average atomic occupation number and the coherence correlator have been scaled using the total atom number, \(N\).

We should note that the results here using direct numerical diagonalization of the Hamiltonian concur with the results found by Zhou et al. [32] utilizing the exact solution.

From figure 7a it can be seen that the maximum entanglement in the ground state occurs where the average atomic occupation is comparable to the average molecular occupation. As indicated in [32], the threshold coupling for the formation of a predominantly molecular BEC is \(g^2 \approx 1.4 \sqrt{N_{\text{atm}}N_{\text{m}}/N_{\text{m}}}\). In the limit of large \(N_{\text{atm}}\), the threshold for the molecular BEC is in fact a quantum phase transition. Figure 8 shows the comparative results for the average atomic occupation number, the coherence correlator and the entanglement for \(N_{\text{atm}} = 100\). That the entanglement is not maximally quantum critical point is quite different to the behavior of the transverse Ising model, studied in references [3, 38]. The entanglement characteristics of the transverse Ising model are, of course, much more complicated, since it consists of many distinguishable subsystems. In [3] it was conjectured that, in the sense of entanglement sharing - how much two-party entanglement can be distributed amongst a given number of parties - the ground state was maximally entangled at the critical point. At the critical point the ground state saturates the bounds of entanglement sharing. The ground state entanglement is not maximal at the critical point, the state is still strongly entangled. This would make intuitive sense, given that the property responsible for the long-range correlations in quantum phase transitions is entanglement.

![Graph](image)

**FIG. 8:** The average atomic occupation number, coherence correlator and the entanglement for the ground state of the atom-molecule BEC, for \(N_{\text{atm}} = 100\). All three properties have been scaled with respect to their maximum value so as to compare the characteristics of these properties.

The plots of the results for the entanglement and the coherence correlator share a common structure, however the maximum values occur for different parameter values. This could mean that there is possibly another correlation that is more closely related to the entanglement between the atomic and molecular modes.

**VI. CONCLUSION**

We have argued here that in a system consisting of a pair of tunnel-coupled BEC’s, the individual bosons within the condensates cannot be viewed as distinguishable subsystems. Subsequently entanglement in this system should not be viewed as between the individual bosons. A more physically relevant description of this system is as a bipartite system where the subsystems are the two modes. Using this description we have analyzed quantitatively the entanglement between the two modes in the ground state of the coupled BEC’s and its relation to the Mott insulator to superfluid phase transition. This idea was extended to consider the entanglement between the atomic and molecular modes in an atom-molecule BEC.

On top of this, we have demonstrated that the dynamical scheme of [16], argued to be viable with current state of the art technology, can be used to create a highly entangled state between the modes of the BEC system, over a smaller time-scale.

As mentioned earlier, the amount of entanglement depends upon how the system is decomposed. In section IV C, it was shown that the tunnel-coupled two-mode system can be viewed as a pseudo-angular momentum system - a single "qudit." In this description - viewing the
solely as a single qudit - it appears that there is no entanglement present in the system. Entanglement is only seen when the system is viewed in terms of subsystems, in this case, the two modes. In other words, the entanglement cannot be characterized when we neglect information about the underlying subsystems and only consider properties of the system as a whole.

A possible way to create entanglement between individual bosons in the tunnel-coupled system would be to engineer some state within the condensate traps, then free the particles (see [39, 40] for examples of this procedure applied to other BEC systems). Once the bosons are free from the condensate they become distinct allowing entanglement to form between them. However, whilst the bosons remain in condensate they are indistinguishable and cannot become entangled with each other.

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