Lorentz-Invariant Non-Commutative QED

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Abstract

Lorentz-invariant non-commutative QED (NCQED) is constructed such that it should be a part of Lorentz-invariant non-commutative standard model (NCSM), a subject to be treated in later publications. Our NCSM is based on Connes’ observation that the total fermion field in the standard model may be regarded as a bi-module over a flavor-color algebra. In this paper, it is shown that there exist two massless gauge fields in NCQED which are interchanged by $C'$ transformation. Since $C'$ is reduced to the conventional charge conjugation $C$ in the commutative limit, the two gauge fields become identical to the photon field in the same limit, which couples to only four spinors with charges $\pm 2, \pm 1$. Following Carlson-Carone-Zobin, our NCQED respects Lorentz invariance employing Doplicher-Fredenhagen-Roberts algebra instead of the usual algebra with constant $\theta^{\mu \nu}$. In the new version $\theta^{\mu \nu}$ becomes an integration variable. We show using a simple NC scalar model that the $\theta$ integration gives an invariant damping factor instead of the oscillating one to the nonplanar self-energy diagram in the one-loop approximation. Seiberg-Witten map shows that the $\theta$ expansion of NCQED generates exotic but well-motivated derivative interactions beyond QED with allowed charges being only $0, \pm 1, \pm 2$. 

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1 Introduction

This is the first in a series of papers which are devoted to discuss the standard model on non-commutative space-time. The non-commutative standard model (NCSM) was already taken up by several authors\cite{1,2,3}. Our approach is different from theirs in that their model building is affected in no way by Connes’ reformulation\cite{4,5} of the standard model, while we are motivated by the assumption\cite{6} that an (associative) algebra underlies the gauge symmetry, and their non-commutative space-time with constant $\theta^{\mu\nu}$ breaks Lorentz symmetry, while we maintain Lorentz invariance following Carlson, Carone and Zobin\cite{7}.

Let us first recall the main reorganization introduced by Connes’ intrusion. Consider $U(1)$ gauge theory. Connes realized it by the algebra $A = C^\infty(M_4) \otimes C$ whose unitaries constitute $U(1)$. From the linearity of the algebra representation $\rho(a + b) = \rho(a) + \rho(b)$ the allowed representation of the algebra is restricted to be of the form $\rho(b) = \text{diag} (b, \cdots, b^*, \cdots)$ for $b = b(x) \in A$, meaning the Abelian charge to be $\pm 1$ in Connes’ realization of $U(1)$ gauge theory\cite{8}, since the gauge group is given by the unitary group of the algebra, $U(C^\infty(M_4) \otimes C)) = \text{Map}(M_4, U(1))$.

Next consider leptons. Since they have flavor and the Abelian charges $Y(e_R) = -2, Y(l_L) = -1, Y(\nu_R) = 0$ (assuming the right-handed neutrino), we represent them as a bi-module. In general, in an $A$-$B$ bi-module $M$ the two commuting operations are defined,

$$a(bx) = b(ax), \quad a \in A, \quad b \in B, \quad x \in M.$$  

It is common to write one operation as a right action,

$$a(xb) = (ax)b, \quad a \in A, \quad b \in B, \quad x \in M.$$  

Similarly, we write the standard gauge transformation in the lepton sector in the form,

$$g\psi = (g\psi)u^* = g(\psi u^*) \equiv g\psi u^*,$$

$$g = \left( \begin{array}{cc} g_L & 0 \\ 0 & g_R \end{array} \right) \otimes 1_{N_g}, \quad g = \left( \begin{array}{cc} u & 0 \\ 0 & u^* \end{array} \right),$$  

where $g_L \in SU(2)_L, u = e^{i\alpha}$ and $\psi = \left( \begin{array}{c} \psi_L \\ \psi_R \end{array} \right)$ with $\psi_L = \left( \begin{array}{c} \nu \\ e \end{array} \right)_L$ and $\psi_R = \left( \begin{array}{c} \nu_R \\ e_R \end{array} \right)$ in $N_g$ generations. We distinguish between the left-handed doublets and the right-handed singlets by the subscripts, $L$ and $R$, outside and inside the vector notation, ( ), respectively. The form $u = e^{i\alpha}$ is due to the normalization $Y(\phi) = 1$. (See below.) Since $g$ and $u^*$ commute, the leptonic $\psi$ is a bi-module over a flavor algebra, $C^\infty(M_4) \otimes (H \oplus C)$, $H$ being real quaternions. The unitary group
of the algebra, \( U(C^\infty(M_4) \otimes (H \oplus C)) = \text{Map}(M_4, SU(2) \times U(1)) \) is the flavor group.

Let us now introduce quarks into the scheme. The total fermion field

\[
\psi = \begin{pmatrix}
  l_L & q^L & d^L & \phi_0^* \\
  l_R & q^R & d^R & \phi_0^* \\
  u_R & \phi_+ & \phi_0 & \phi_0^* \\
  d_R & \phi_- & \phi_0 & \phi_0^*
\end{pmatrix},
\]

receives the standard gauge transformation,

\[
\psi \rightarrow g \psi = (g \psi) G = g(\psi G) \equiv g \psi G,
\]

where the left action by \( g \) is the same as in (1), while the color gauge transformation is written as

\[
G = \begin{pmatrix}
  u^* & 0 \\
  0 & v^T
\end{pmatrix}.
\]

Here \( v \) belongs to the unitary group of the color algebra \( C^\infty(M_4) \otimes M_3(C) \). Since no algebra exists whose unitary group is color \( SU(3) \), we need Connes’ unimodularity condition to reproduce the correct hypercharge of quarks,

\[
det G = 1.
\]

Putting \( v = e^{i\beta} v' \) with \( \det v' = 1 \) the unimodularity condition (5) implies

\[-\alpha + 3\beta = 0.\]

This correctly reproduces the fractional hypercharge of quarks, \( Y(q_L) = 0 + 1/3 = 1/3, Y(u_R) = 1 + 1/3 = 4/3, Y(d_R) = -1 + 1/3 = -2/3 \). (3) defines the total fermion field as a bi-module \(^2\). On the other hand, the gauge transformation of Higgs \( h = \begin{pmatrix}
  \phi_0^* & \phi_+ \\
  -\phi_- & \phi_0
\end{pmatrix} \) looks like

\[
h \rightarrow g h = (g_L h) g_R^\dagger = g_L (h g_R^\dagger) \equiv g_L h g_R^\dagger.
\]

Consequently, matrix-valued Higgs is also regarded as a (single) bi-module. The spontaneous breakdown of symmetry is triggered by the finite vacuum expectation value \( \langle h \rangle = (v/\sqrt{2}) 1_2 \) so that it is given by

\[
g_L \rightarrow g_R.
\]

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\(^1\)This name was suggested by I. S. Sogami\(^9\) to represent chiral leptons and quarks in a unified way.

\(^2\)One can easily reassemble \( \psi \) so as to move the right action to the left with commutativity from semi-simple group structure becoming apparent.

\(^3\)The hypercharge of Higgs \( \phi \) is normalized to be +1, leading to the choice \( u = e^{i\alpha} \).
In a series of papers we interpret the bi-module structure of the total fermion field in the framework of non-commutative gauge theory (NCGT), defining the total fermion field as a non-commutative (NC) bi-module so that the round brackets in (1) and (3) mean only the associativity. We are motivated to study NCSM for two reasons. Firstly, it was shown in Ref. 10) that, in non-commutative QED (NCQED), Abelian charge is restricted to be ±1 and 0. This is similar to the restriction in Connes’ realization of $U(1)$ gauge theory. If we consider NCQED as only a part of a larger theory, NCSM, it is necessary to incorporate the value ±2 in NCQED to account for the Abelian charge $Y(e_{R}) = -2$. This is accomplished by considering NC bi-module in exactly the same way as we explained $Y(e_{R}) = -2$ by considering the bi-module. We are thus led to the second motivation, namely, if the field quantities are defined on non-commutative space-time, the left and right actions are distinguished, interpreting the bi-module structure (3) as a two-sided gauge transformation in NCGT.

In this paper, we restrict ourselves to the lepton sector and consider only in the broken phase, (7), that is, NCQED of leptons. Hence we encounter only $u$ but both $u$ and $u^*$ appear in the gauge transformations. In later communications, we will consider NCSM in the lepton and quark sectors. Although NCQED has been the subject of intensive study in its own interests, it is based on the Lorentz-non-covariant algebra $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ with constant antisymmetric matrix $\theta^{\mu\nu}$. Hence it violates Lorentz symmetry. Quite recently, Carlson, Carone and Zobin[7] constructed a Lorentz-invariant NCGT by employing DFR(Doplicher-Fredenhagen-Roberts) algebra of non-commutative space-time, which replaces $\theta^{\mu\nu}$ with an anti-symmetric tensor operator $\hat{\theta}^{\mu\nu}$. According to their formulation the old $\theta^{\mu\nu}$ plays a role as an argument of 4-dimensional covariant fields as extra 6-dimensional coordinates. Consequently, the action contains an integration over the extra dimensions, too, with unknown weighting function $W(\theta)$. The $\theta$ in the old version becomes an integration variable. The function $W(\theta)$ is not yet determined from the first principle except for normalization and its evenness. However, the authors in Ref. 7) performed detailed perturbation calculation for $2\gamma \rightarrow 2\gamma$ scattering to compare with the standard model prediction. We shall show using a simple NC scalar model that the $\theta$ integration with a model choice of $W(\theta)$ gives an invariant damping factor instead of the oscillating one to the nonplanar self-energy diagram in the one-loop approximation. The IR-singularity which can not be discriminated from the $\theta \rightarrow 0$ singularity, may be avoided if we take UV limit at the same time as the commutative limit. The $\theta$ expansion[14] using Seiberg-Witten map[15]

\footnote{For instance, we can no longer write (1) as if both actions operate on the spinor only from the left. In particular, this means that $e_{R}$ and $\nu_{R}$ remain acted from both sides, $e_{R} \rightarrow g e_{R} = u^* e_{R} u^*$ and $\nu_{R} \rightarrow g \nu_{R} = u \nu_{R} u^*$, which would read $g e_{R} = u^* u^* e_{R} = (u^*)^2 e_{R}$ and $g \nu_{R} = u u^* \nu_{R} = \nu_{R}$ in the standard notation, respectively. If we start with this notational convention, the left-handed doublet and the right-handed singlet behave `differently' on non-commutative space-time as assumed in Ref.1). (In Refs. 1) and 2) $\nu_{R}$ is not considered.)}
defines QED in the *smooth* commutative limit in the Lagrangian level, while higher order terms in $\theta$ involve exotic but well-organized derivative interactions\[14\].

To reveal another aspect of NCQED relevant to NC bi-module we recall that the minimal interaction in QED can be written in two different but equivalent ways,

$$e\bar{\psi}\gamma^\mu A_\mu \psi = e\bar{\psi}\gamma^\mu \psi A_\mu.$$ 

Since one cannot freely move the operators on non-commutative space-time, the above two ways of writing force us to naturally consider the different gauge fields in NCQED, one corresponding to $A_\mu$ sandwiched between the spinors and the other to $A_\mu$ outside the spinors\[5\]. They are destined to fuse into the single photon field in the commutative limit\[6\].

By the same token we are motivated to introduce\[11\] two different charge conjugation transformations in NCQED since, in QED, we have two different but equivalent ways of writing the charge conjugation,

$$(e\bar{\psi}\gamma^\mu A_\mu \psi)^c = \begin{cases} e\bar{\psi}^c\gamma^\mu A_\mu^c \psi^c, \\ -e\psi^c T \gamma^\mu T A_\mu^c \bar{\psi}^c T. \end{cases}$$

The latter line is the proper generalization of the usual charge conjugation $C$, while the former defines another charge conjugation transformation, called $C'$.

The plan of this paper goes as follows. We define fields on DFR algebra in the next section, which largely owes Ref. 7). In section 3, we construct Lorentz-invariant NCQED to accommodate fermions with the Abelian charges $\pm 2, \pm 1, 0$. As shown in Ref. 7), technically, there is only a slight modification from the Lorentz-non-invariant NCQED. Our NCQED possesses two gauge fields which are each other’s $C'$ conjugates. In the section 4, we shall demonstrate that the $\theta$ integration actually gives an invariant damping factor in loop-integration using a simple NC scalar model in Euclidean metric and suggest a way to avoid IR-singularity\[13\]. We discuss Seiberg-Witten map in our NCQED characterized by the two gauge fields to be identified with the single photon field in the commutative limit in the section 5. Conclusions are given in the last section. Appendix A discusses a non-smoothness in the commutative limit of the derivative operator for constant $\theta$ algebra. In the Appendix B a proof will be given on the correspondence of operator product to Moyal product.

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5Since we take trace on operator space, whether we move $A_\mu$ to the left or to the right of the spinors is irrelevant.

6Two-sided gauge transformation is familiar in NCGT. Our assertion is that it happens to make two independent gauge fields fuse into a single one in the commutative limit.
2 Fields defined on DFR algebra

A field on a non-commutative space-time is an operator in a classical sense. It must have a definite transformation property under the Lorentz group acting on the operator coordinates \(\hat{x}^\mu\). If it defines a Lorentz-invariant theory, the algebra of the operator coordinates must be Lorentz-covariant. Namely, \(\hat{x}^\mu = \Lambda_{\mu\nu} \hat{x}^\nu\) must satisfy the same algebra as the algebra obeyed by \(\hat{x}^\mu\) as viewed in the primed reference frame connected with the unprimed reference frame by a Lorentz transformation \((\Lambda^\mu_\nu)\). In conformity with this requirement we employ as in Re. 7) DFR (Doplicher-Fredenhagen-Roberts) algebra\(^{[12]}\) spanned by the hermitian operators \(\hat{x}^\mu, \hat{\theta}^\mu\) with \(\hat{\theta}^\mu = -\hat{\theta}^{\mu'} (\mu, \nu = 0, 1, 2, 3)\) which satisfy the commutation relations

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad [\hat{\theta}^\mu, \hat{x}^\rho] = [\hat{\theta}^{\mu\nu}, \hat{\theta}^{\rho\sigma}] = 0. \tag{8}
\]

This algebra is Lorentz-covariant, allowing us to define operator scalar, spinor, vector and tensor fields \(\varphi(\hat{x}, \hat{\theta})\). For instance, if \(\varphi(\hat{x}, \hat{\theta}) = \hat{A}_\mu(\hat{x}, \hat{\theta})\) is operator vector field, it transforms as \(\hat{A}_\mu'(\hat{x}', \hat{\theta}') = \Lambda_{\mu\nu} \hat{A}_\nu(\hat{x}, \hat{\theta})\) where \(\hat{x}'^\mu = \Lambda_{\mu\nu} \hat{x}^\nu\) and \(\hat{\theta}_\mu' = \Lambda_{\mu\nu} \hat{\theta}_\nu\). If replace \(\hat{\theta}^{\mu\nu}\) with \(\theta^{\mu\nu}\), a real constant antisymmetric matrix, we write \(\hat{\varphi}(\hat{x})\) instead of \(\hat{\varphi}(\hat{x}, \hat{\theta})\). Although we may impose the condition \(\varphi'(x') = \varphi(x)\) to define operator scalar field, the algebra spanned by \(\hat{x}^\mu\) with constant \(\theta\) is no longer Lorentz covariant. Hence Lorentz symmetry is lost in any theory based on the algebra \([\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\) for constant \(\theta\). Only in the limit \(\theta^{\mu\nu} = 0\), the simultaneous eigenvalues of \(\hat{x}^\mu\) can be regarded as a label of a point in \(M_4\) where we define a scalar field by \(\varphi'(x') = \varphi(x)\) with \(x' = \Lambda x\) and \(x'^2 = x^2\). However, the commutative limit is discontinuous in this case. A symptom concerns with the derivative operator, which will be discussed in the Appendix A. This suggests that the Lorentz invariance should be maintained from the outset. In the series of papers we employ Lorentz invariant formulation by Carlson-Carone-Zobin\(^{[7]}\).

It is well-known that the field \(\hat{\varphi}(\hat{x}, \hat{\theta})\) is in one to one correspondence with \(c\)-number field \(\varphi(x, \theta)\). The field \(\varphi(x, \theta)\) is obtained by replacing the operators \(\hat{x}^\mu, \hat{\theta}^{\mu\nu}\) in the Weyl representation

\[
\hat{\varphi}(\hat{x}, \hat{\theta}) = \frac{1}{(2\pi)^4} \int d^4k d^6\sigma \varphi(k, \sigma) e^{ikx + i\sigma \hat{\theta}} \equiv \frac{1}{(2\pi)^4} \int d^4k d^6\sigma \varphi(k, \sigma) e^{ikx + i\sigma \hat{\theta}}. \tag{9}
\]

with \(c\)-numbers \(x^\mu, \theta^{\mu\nu}\), respectively,

\[
\varphi(x, \theta) = \frac{1}{(2\pi)^4} \int d^4k d^6\sigma \varphi(k, \sigma) e^{ikx + i\sigma \theta}. \tag{10}
\]

It goes without saying that, if \(\hat{\varphi}(\hat{x}, \hat{\theta})\) is operator scalar, spinor, vector, and tensor fields, then \(\varphi(x, \theta)\) is also scalar, spinor, vector, and tensor fields, respectively. Although it has ten-dimensional

\(^{[7]}\)This invariance is replaced by the Lorentz covariance of the algebra spanned by the operator coordinates.
arguments, its transformation property is defined with respect to the 4-dimensional Lorentz group. If we put $\theta^\mu\nu = 0$ and define $\tilde{\varphi}(k) = \int d^6\sigma \varphi(k, \sigma)$, (10) gives the usual Fourier transform of a 4-dimensional field $\varphi(x) \equiv \varphi(x, 0)$ with Fourier component $\tilde{\varphi}(k)$. Consequently, the limit $\theta^\mu\nu \to 0$ corresponds to the commutative limit. The inverse Fourier transform is given by

$$\tilde{\varphi}(k, \sigma) = \frac{1}{(2\pi)^6} \int d^4x d^6\sigma \varphi(x, \theta)e^{-ikx - i\sigma \theta}. \quad (11)$$

Since the translation $\hat{x}^\mu \to \hat{x}^\mu + a^\mu \hat{1}$ for any $c$-number $a^\mu$ with $\hat{1}$ being the unit operator is an automorphism of the DFR algebra, we can define the operator

$$\hat{\varphi}(\hat{x} + a\hat{1}, \hat{\theta}) = \frac{1}{(2\pi)^4} \int d^4x d^6\sigma \varphi(k, \sigma)e^{ik\mu(\hat{x}^\mu + a^\mu \hat{1}) + i\sigma_{\mu\nu}\hat{\theta}^\mu\nu}, \quad (12)$$

from which the derivative of an operator is defined by

$$\partial_\mu \hat{\varphi}(\hat{x}, \hat{\theta}) = \frac{\partial}{\partial a^\mu} \hat{\varphi}(\hat{x} + a\hat{1}, \hat{\theta})|_{a=0} = \frac{1}{(2\pi)^4} \int d^4k d^6\sigma ik^\mu \tilde{\varphi}(k, \sigma)e^{ik\hat{x} + i\sigma \hat{\theta}}. \quad (13)$$

We also define the trace [7] 9

$$\text{tr} \, \hat{\varphi}(\hat{x}, \hat{\theta}) = \frac{1}{(2\pi)^6} \int d^6\sigma \varphi(0, \sigma)\tilde{W}(\sigma) = \int d^4x d^6\theta \varphi(x, \theta)W(\theta), \quad (14)$$

where

$$W(\theta) = \frac{1}{(2\pi)^6} \int d^6\sigma \tilde{W}(\sigma)e^{-i\sigma \theta}, \quad (15)$$

with the normalization

$$\int d^6\theta W(\theta) = 1. \quad (16)$$

It is clear that the function $W(\theta)$ has dimensions, $[L^{-12}]$. Taking the trace of the first equation of (8) we get $\text{tr} \, \hat{\theta}^\mu\nu = 0$ so that

$$\int d^6\theta W(\theta) \theta^\mu\nu = 0, \quad (17)$$

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8 We never take the limit $[\hat{x}^\mu, \hat{x}^\nu] \to 0$. (This helps avoid an embarrassment encountered in defining the derivative operator in the case, $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^\mu\nu$, at $\theta^\mu\nu \to 0$. We shall discuss this point in the Appendix A.) Conversely, we can build 4-dimensional field theory solely based on DFR algebra. Define field $\hat{\varphi}(\hat{x}, \hat{\theta})$ on DFR algebra and take the limit $\theta^\mu\nu \to 0$. Quantization is to be performed on the $c$-number field $\varphi(x) = \varphi(x, \theta = 0)$. The operator coordinates remain operators even in this limit. In this sense, the commutative limit is smooth.

9 If $\varphi$ is a matrix, matrix trace is also implied.
since Weyl symbol corresponding to the operator $\hat{\theta}^{\mu\nu}$ is $\theta^{\mu\nu}$. Thus $W(\theta)$ is an even function \(^\text{10}\). The commutative limit corresponds to

$$W(\theta) = W^{(0)}(\theta) \equiv \delta^6(\theta) \equiv \delta(\theta^{01})\delta(\theta^{02})\delta(\theta^{03})\delta(\theta^{12})\delta(\theta^{23})\delta(\theta^{31}).$$

(18)

This has a correct dimension, $[L^{-12}]$, since $\theta$ has dimensions of length squared.

The Weyl symbol of the operator product is given by the Moyal product. To see this put

$$\hat{\varphi}_1(\hat{x}, \hat{\theta})\hat{\varphi}_2(\hat{x}, \hat{\theta}) = \frac{1}{(2\pi)^4} \int d^4kd^6\sigma \tilde{\varphi}_{12}(k, \sigma)e^{ik\hat{x} + i\sigma\hat{\theta}}.$$  

(19)

Then we can show that

$$\varphi_{12}(x, \theta) = \frac{1}{(2\pi)^4} \int d^4kd^6\sigma \tilde{\varphi}_{12}(k, \sigma)e^{i(kx + \sigma\theta)}$$

$$= e^{\frac{i}{2}g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \theta^\nu}} \varphi_1(x, \theta)\varphi_2(y, \theta)|_{x=y}$$

$$\equiv \varphi_1(x, \theta) * \varphi_2(x, \theta).$$  

(20)

The proof will be given in the Appendix B. Namely, if $\hat{\theta}^{\mu\nu}$ belongs to the center of the algebra, the product of operators corresponds to the Moyal product as if $\hat{\theta}^{\mu\nu}$ is a $c$-number $\theta^{\mu\nu}$. Only difference lies in the additional dependence of the Weyl symbol on $\theta$, $\varphi(x, \theta)$. This is the most important observation by Carlson-Carone-Zobin\(^7\).

By definition (20) we have

$$\int d^4x\varphi_1(x, \theta) * \varphi_2(x, \theta) = \int d^4x\varphi_1(x, \theta)\varphi_2(x, \theta) = \int d^4x\varphi_2(x, \theta) * \varphi_1(x, \theta),$$

(21)

so that it follows

$$\text{tr} \hat{\varphi}_1(\hat{x}, \hat{\theta})\hat{\varphi}_2(\hat{x}, \hat{\theta}) = \int d^4xd^6\theta W(\theta)\varphi_1(x, \theta)\varphi_2(x, \theta) = \int d^4xd^6\theta W(\theta)\varphi_2(x, \theta)\varphi_1(x, \theta)$$

$$= \text{tr} \hat{\varphi}_2(\hat{x}, \hat{\theta})\hat{\varphi}_1(\hat{x}, \hat{\theta}).$$  

(22)

Moyal product of three Weyl symbols is similarly defined.

$$\hat{\varphi}_1(\hat{x}, \hat{\theta})\hat{\varphi}_2(\hat{x}, \hat{\theta})\hat{\varphi}_3(\hat{x}, \hat{\theta}) = \frac{1}{(2\pi)^4} \int d^4kd^6\sigma \tilde{\varphi}_{123}(k, \sigma)e^{ik\hat{x} + i\sigma\hat{\theta}}$$

$$\varphi_{123}(x, \theta) = \frac{1}{(2\pi)^4} \int d^4kd^6\sigma \tilde{\varphi}_{123}(k, \sigma)e^{i(kx + \sigma\theta)}$$

$$= \varphi_1(x, \theta) * \varphi_2(x, \theta) * \varphi_3(x, \theta).$$  

(23)

\(^{10}\)The authors in Ref. 7 are led to the condition $W(\theta) = W(-\theta)$ from Lorentz invariance.
The associativity is proven from that of operators. Using the cyclic property of the trace we can show the cyclic property of Moyal products under integration

\[ \int d^4 x \varphi_1(x, \theta) \ast \varphi_2(x, \theta) \ast \varphi_3(x, \theta) = \int d^4 x \varphi_2(x, \theta) \ast \varphi_3(x, \theta) \ast \varphi_1(x, \theta) \]

\[ = \int d^4 x \varphi_3(x, \theta) \ast \varphi_1(x, \theta) \ast \varphi_2(x, \theta), \]

(24)

where we have omitted integration \( \int d^6 \theta W(\theta) \).

### 3 Lorentz-invariant NCQED

Let us now consider Lorentz-invariant NCQED for fermion. Since fields on DFR algebra (8) only demands additional dependence on the variable \( \theta \) as compared with those defined on the Lorentz-non-covariant algebra \([\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\), Lorentz-invariant NCQED closely follows from the Lorentz-non-invariant NCQED. For the reason explained in the Introduction, we employ NCQED as detailed in Ref.11). Thus we consider the following gauge transformations for 8 spinors

\[ \chi_1(x, \theta) \rightarrow \hat{g}\chi_1(x, \theta) = U(x, \theta) \ast \chi_1(x, \theta) \ast U^\dagger(x, \theta), \]

\[ \chi_2(x, \theta) \rightarrow \hat{g}\chi_2(x, \theta) = U^\dagger(x, -\theta) \ast \chi_2(x, \theta) \ast U(x, -\theta), \]

\[ \psi_1(x, \theta) \rightarrow \hat{g}\psi_1(x, \theta) = U(x, \theta) \ast \psi_1(x, \theta), \]

\[ \psi_2(x, \theta) \rightarrow \hat{g}\psi_2(x, \theta) = \psi_2(x, \theta) \ast U(x, -\theta), \]

\[ \psi_3(x, \theta) \rightarrow \hat{g}\psi_3(x, \theta) = U^\dagger(x, -\theta) \ast \psi_3(x, \theta), \]

\[ \psi_4(x, \theta) \rightarrow \hat{g}\psi_4(x, \theta) = \psi_4(x, \theta) \ast U^\dagger(x, \theta), \]

\[ \psi_5(x, \theta) \rightarrow \hat{g}\psi_5(x, \theta) = U(x, \theta) \ast \psi_5(x) \ast U(x, -\theta), \]

\[ \psi_6(x, \theta) \rightarrow \hat{g}\psi_6(x, \theta) = U^\dagger(x, -\theta) \ast \psi_6(x, \theta) \ast U^\dagger(x, \theta), \]

(25)

where the gauge parameter is assumed to be \( \ast \) unitary,

\[ U(x, \theta) \ast U^\dagger(x, \theta) = U^\dagger(x, \theta) \ast U(x, \theta) = 1, \]

\[ U(x, \theta) = (e^{i\alpha(x, \theta)})_\ast \]

\[ \equiv 1 + i\alpha(x, \theta) + \frac{1}{2!} (i\alpha(x, \theta)) \ast (i\alpha(x, \theta)) + \cdots. \]

(26)
The product of $U(x,-\theta)$ is defined by

$$U_1(x,-\theta) \ast U_2(x,-\theta) \equiv U_2(x,-\theta) \ast U_1(x,-\theta),$$

(27)

so that the group property with respect to $\ast$ product is retained,

$$U_1^\dagger(x,-\theta) \ast U_2^\dagger(x,-\theta) \equiv U_2^\dagger(x,-\theta) \ast U_1^\dagger(x,-\theta) \equiv (U_1(x,-\theta) \ast U_2(x,-\theta))^\dagger. \tag{28}$$

In Ref. 11) we did not consider the $\theta$ dependence of the gauge parameter, although we implicitly assumed the $\theta$ dependence of the fields including the gauge fields. (See the section 5.)

In the commutative limit, we have only 8 spinors all of which receive $U(1)$ gauge transformation

$$\psi(x) \rightarrow ^g\psi(x) = U(x)\psi(x), \quad U(x) = e^{iQ\alpha(x)}, \tag{29}$$

where $\alpha(x) = \alpha(x,0)$ and the charge $Q$ is determined as follows.

1. The set $\{\chi_1, \psi_1, \psi_4\}$ couples to the gauge field transforming like

$$A_{\mu}(x,\theta) \rightarrow ^gA_{\mu}(x,\theta) = U(x,\theta) \ast A_{\mu}(x,\theta) \ast U^\dagger(x,\theta) + \frac{2i}{e}U(x,\theta) \ast \partial_{\mu}U^\dagger(x,\theta). \tag{30}$$

As a consequence, $Q(\chi_1) = 0$, $Q(\psi_1) = +1$ and $Q(\psi_4) = -1$ in units of $e/2$. This is the charge quantization in NCQED\textsuperscript{[10]}.

2. On the other hand, the set $\{\chi_2, \psi_2, \psi_3\}$ interacts with another gauge field with different transformation property

$$A_{\mu}'(x,\theta) \rightarrow ^gA_{\mu}'(x,\theta) = U^\dagger(x,-\theta) \ast A_{\mu}'(x,\theta) \ast U(x,-\theta) + \frac{2i}{e}U^\dagger(x-\theta) \ast \partial_{\mu}U(x-\theta). \tag{31}$$

Hence, $Q(\chi_2) = 0$, $Q(\psi_2) = +1$ and $Q(\psi_3) = -1$ in units of $e/2$. This is also the charge quantization in NCQED.

3. Apparently, $\{\psi_5, \psi_6\}$ with $Q(\psi_5) = +1$ and $Q(\psi_6) = -1$ in units of $e$ couple to both gauge fields. In the commutative limit, we may put $\psi_1 = \psi_2$, $\psi_3 = \psi_4$ so that the two gauge fields should become identical up to sign. That is, the photon field $A_{\mu}(x)$ is given by

$$A_{\mu}(x) = A_{\mu}(x,0) = -A_{\mu}'(x,0). \tag{32}$$

In fact, inverting the sign of $\theta$ in (30) and comparing the result with (31), we can put

$$A_{\mu}'(x,\theta) = -A_{\mu}(x,-\theta), \tag{33}$$
since $\Delta_\mu \equiv A'_\mu(x, \theta) + A_\mu(x, -\theta)$ is subject to homogeneous transformation and can be put zero in a gauge-invariant way \(^{11}\). In the commutative limit, we have the single photon field which couples to four spinors, $\psi_1 = \psi_2, \psi_3 = \psi_4, \psi_5$ and $\psi_6$, only. In other words, if we define QED as a limiting theory of Lorentz-invariant NCQED, we are allowed to have only four spinors with charges $\pm 2, \pm 1$ in units of $e/2$ and one neutral spinor. We shall give very plausible argument in favor of the spinors $(\chi_1, \psi_6)$ to represent leptons in Nature in later publications where we obtain our NCQED from a spontaneously broken gauge theory of Lorentz-invariant non-commutative Weinberg-Salam model (NCWS). ((1) implies two kinds of spinors $\chi_1$ and $\psi_6$ to describe leptons. See the footnote on p.4.) This is why we employed NCQED in Ref.11), which considers all possible spinors. In other references, such spinors like $\psi_5$ or $\psi_6$, which will be responsible for the charged leptons considering the bi-module structure (1) literally, are not introduced.

We now list all gauge couplings for 8 spinors

\[
\begin{align*}
\frac{e}{2} \bar{\chi}_1(x, \theta) * \gamma^\mu (A_\mu(x, \theta) * \chi_1(x, \theta) - \chi_1(x, \theta) * A_\mu(x, \theta)), \\
\frac{e}{2} \bar{\chi}_2(x, \theta) * \gamma^\mu (A'_\mu(x, \theta) * \chi_2(x, \theta) - \chi_2(x, \theta) * A'_\mu(x, \theta)), \\
\frac{e}{2} \bar{\psi}_1(x, \theta) * \gamma^\mu A_\mu(x, \theta) * \psi_1(x, \theta), \\
\frac{e}{2} \bar{\psi}_2(x, \theta) * \gamma^\mu \psi_2(x, \theta) * A'_\mu(x, \theta), \\
\frac{e}{2} \bar{\psi}_3(x, \theta) * \gamma^\mu A'_\mu(x, \theta) * \psi_3(x, \theta), \\
\frac{e}{2} \bar{\psi}_4(x, \theta) * \gamma^\mu \psi_4(x, \theta) * A_\mu(x, \theta), \\
\frac{e}{2} \bar{\psi}_5(x, \theta) * \gamma^\mu A_\mu(x, \theta) * \psi_5(x, \theta) - \psi_5(x, \theta) * A'_\mu(x, \theta), \\
\frac{e}{2} \bar{\psi}_6(x, \theta) * \gamma^\mu (A'_\mu(x, \theta) * \psi_6(x, \theta) - \psi_6(x, \theta) * A_\mu(x, \theta)).
\end{align*}
\]

\(^{11}\)In general one can only say that $\Delta_\mu \to 0$ at $\theta \to 0$. 

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They all conserve $C$ if we define

\[
C\psi(x, \theta)C^{-1} \equiv \psi^c(x, \theta) = C\bar{\psi}^T(x, \theta),
\]

\[
C\bar{\psi}(x, \theta)C^{-1} \equiv \bar{\psi}^c(x, \theta) = -\psi^T(x, \theta)C^{-1},
\]

\[
CA_\mu(x, \theta)C^{-1} \equiv A^c_\mu(x, \theta) = -A_\mu(x, \theta),
\]

\[
CA'_\mu(x, \theta)C^{-1} \equiv A'^c_\mu(x, \theta) = -A'_\mu(x, \theta),
\]

(35)

where $C$ is the charge conjugation matrix satisfying $C^{-1}\gamma^\mu C = -\gamma^\mu T$, provided $C$ reverses the order of the operators (the order of Moyal product). For instance, the gauge coupling for the spinor $\chi_1$ upon integration is invariant under $C$

\[
-\frac{e}{2} \int d^4x d^6\theta W(\theta) [\bar{\chi}_1^c(x, \theta) \gamma^\mu T A^c_\mu(x, \theta) \chi_1^c(x, \theta) - \bar{\chi}_1^c(x, \theta) A^c_\mu(x, \theta) \chi_1^c(x, \theta) - \bar{\chi}_1^c(x, \theta) A^c_\mu(x, \theta) \chi_1^c(x, \theta)]
\]

\[
= \frac{e}{2} \int d^4x d^6\theta W(\theta) [\bar{\chi}_1(x, \theta) \gamma^\mu A_\mu(x, \theta) \chi_1(x, \theta) - \bar{\chi}_1(x, \theta) A_\mu(x, \theta) \chi_1(x, \theta)].
\]

(36)

On the other hand, the gauge coupling for the spinor $\psi_5$

\[
\frac{e}{2} \int d^4x d^6\theta W(\theta) [\bar{\psi}_5(x, \theta) \gamma^\mu A_\mu(x, \theta) \psi_5(x, \theta) + \bar{\psi}^c_5(x, \theta) \gamma^\mu A'_\mu(x, \theta) \psi_5^c(x, \theta)],
\]

(37)

is invariant under the transformation $(\psi_5 \rightarrow \psi)$

\[
C'\psi(x, \theta)C'^{-1} = \psi^c(x, \theta) = C\bar{\psi}^T(x, \theta),
\]

\[
C'\bar{\psi}(x, \theta)C'^{-1} = \bar{\psi}^c(x, \theta) = -\psi^T(x, \theta)C^{-1},
\]

\[
C'A_\mu(x, \theta)C'^{-1} = A'_\mu(x, \theta), \quad C'A'_\mu(x, \theta)C'^{-1} = A_\mu(x, \theta),
\]

(38)

provided the order of the operators (the order of Moyal product) are not reversed. Similarly for the spinor $\psi_6$. We call the transformation

\[
\psi(x, \theta) \leftrightarrow \psi^c(x, \theta), \quad \bar{\psi}(x, \theta) \leftrightarrow \bar{\psi}^c(x, \theta), \quad A_\mu(x, \theta) \leftrightarrow A'_\mu(x, \theta),
\]

(39)

for the spinors $\psi = \psi_5, \psi_6$, $C'$ transformation\cite{11} provided no reversal of operators (the order of the Moyal product) is made. Then we may say that $A'_\mu(x, \theta)$ is $C'$ conjugate of $A_\mu(x, \theta)$. There are two charge conjugations $C$ and $C'$, the difference being the reversal and unreversal of the operators,
respectively. In the commutative limit, the order of the operators become irrelevant so that \( C' \rightarrow C \) at \( \theta = 0 \). Hence follows (32) by definition.

To conclude this section we define \( C' \)-invariant NC Maxwell action by

\[
\hat{S}_M = -\frac{1}{8} \int d^4x d^6\theta W(\theta)(F_{\mu\nu}(x, \theta) * F^{\mu\nu}(x, \theta) + F'_{\mu\nu}(x, \theta) * F''^{\mu\nu}(x, \theta)),
\]

\[
F_{\mu\nu}(x, \theta) = \partial_\mu A_\nu(x, \theta) - \partial_\nu A_\mu(x, \theta) - \frac{ie}{2} (A_\mu(x, \theta) * A_\nu(x, \theta) - A_\nu(x, \theta) * A_\mu(x, \theta)),
\]

\[
F'_{\mu\nu}(x, \theta) = F_{\mu\nu}|_{A_\mu(x, \theta) \rightarrow A'_\mu(x, \theta)}.
\]

(40)

Each term \( F^2 \) and \( F'^2 \) are separately \( C \)-invariant,

\[
F'_{\mu\nu}(x, \theta) = -F_{\mu\nu}(x, -\theta),
\]

\[
A_\mu(x, \theta) * A_\nu(x, \theta)|_{\theta \rightarrow -\theta} = A_\nu(x, -\theta) * A_\mu(x, -\theta).
\]

(42)

That is, \( C' \)-invariant NC Maxwell Lagrangian is even in \( \theta \).

4 An invariant damping factor

The original motivation of introducing quantized space-time\cite{16} was to remove UV divergence troubles in quantum field theory by replacing point-like interactions of elementary particles with specific Lorentz-invariant nonlocal interactions. On the other hand, Filk\cite{17} showed that, based on the algebra

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu},
\]

the nonplanar diagrams receive an oscillating damping factor due to the space-time non-commutativity, whereas UV divergence of the planar diagrams remain unchanged. We would like to emphasize that the oscillating damping factor violates Lorentz invariance. In fact, the authors in Re.13) observed in non-commutative scalar model that 1PI two-point function in the one-loop approximation explicitly violates Lorentz symmetry, and exhibited a singular behavior at \( \theta \rightarrow 0 \).
limit after loop integration. Since $\theta$ appears always through the combination $\theta^{\mu\nu} q_\nu$ in the oscillating damping factor, where $q_\nu$ is the external momentum, this singular behavior necessarily implies IR-singularity\cite{13} of the amplitude at $\theta^{\mu\nu} q_\nu \to 0$. Hayakawa\cite{10} independently pointed out an explicit violation of Lorentz symmetry in the one-loop photon propagator in NCQED and investigated IR behavior together with the $\theta \to 0$ singularity in relation with UV divergence. It would be interesting to see what happens on the oscillating damping factor if we employ the Lorentz-invariant formulation.

In order to study this problem we consider a simpler example, namely, NC scalar $\lambda \phi^4$-theory in Euclidean metric. In the old Lorentz-non-invariant version it is defined by

$$S = \int d^4 x \left[ \frac{1}{2} (\partial_\mu \phi(x) \partial^\mu \phi(x) + m^2 \phi^2(x)) + \frac{\lambda}{4!} \phi(x) * \phi(x) * \phi(x) * \phi(x) \right]. \quad (43)$$

The proper self-energy part in the one-loop approximation is given by\cite{13}

$$\Sigma^{(1)}_{pl} = \frac{\lambda}{3(2\pi)^4} \int d^4 k \frac{1}{k^2 + m^2},$$

$$\Sigma^{(1)}_{npl} = \frac{\lambda}{6(2\pi)^4} \int d^4 k \frac{e^{ik_\mu \theta^{\mu\nu} p_\nu}}{k^2 + m^2}, \quad (44)$$

for planar and nonplanar diagrams, respectively. It is well-known\cite{17} that the nonplanar diagram is UV-finite due to the oscillating factor, $e^{ik_\mu \theta^{\mu\nu} p_\nu}$, where $p_\nu$ is the external momentum. This oscillating factor comes from the nonlocal interaction concealed in the star product of (43). Using the Schwinger representation,\footnote{In Minkowski space-time, the indices of $\theta^{\mu\nu}$ are lowered by the Lorentz metric, $\theta_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} \theta^{\rho\sigma}$, $(g_{\mu\nu}) = \text{diag}(+1, -1, -1, -1)$.} $\frac{1}{k^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha (k^2 + m^2)}$ and regularizing by multiplication of the factor $e^{-1/\alpha \Lambda^2}$, we get

$$\Sigma^{(1)}_{pl} = \frac{\lambda}{48\pi^2} \int_0^\infty d\alpha \frac{e^{-\frac{\Lambda^2}{\alpha \Lambda^2} - \frac{m^2}{\alpha \Lambda^2}}}{\alpha^2} = \frac{\lambda}{48\pi^2} \left[ \frac{\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2}}{m^2} + \mathcal{O}(1) \right],$$

$$\Sigma^{(1)}_{npl} = \frac{\lambda}{96\pi^2} \int_0^\infty d\alpha \frac{e^{-\frac{\Lambda^2}{\alpha \Lambda_{eff}^2} - \frac{m^2}{\alpha \Lambda_{eff}^2}}}{\alpha^2} = \frac{\lambda}{96\pi^2} \left[ \frac{\Lambda_{eff}^2 - m^2 \ln \frac{\Lambda_{eff}^2}{m^2}}{m^2} + \mathcal{O}(1) \right], \quad (45)$$

where we have defined

$$\frac{1}{\Lambda_{eff}^2} = \frac{1}{\Lambda^2} + \frac{\vec{p}^2}{4},$$

$$\vec{p}^2 = \theta^{\mu\nu} p_\mu \theta_{\mu\nu}. \quad (46)$$

Here, $\theta^{\mu\nu} = \theta_{\mu\nu}$\footnote{In Minkowski space-time, the indices of $\theta^{\mu\nu}$ are lowered by the Lorentz metric, $\theta_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} \theta^{\rho\sigma}$, $(g_{\mu\nu}) = \text{diag}(+1, -1, -1, -1)$.}. It is clear that the nonplanar diagram is UV-finite as far as $\vec{p}^2 \neq 0$. However, it is IR-singular at $\vec{p}^2 \to 0$ if we first let $\Lambda^2 \to \infty$. In order to avoid the IR-singularity in the limit
\( \Lambda^2 \to \infty \), we are tempted to take the limit \( \theta \to 0 \) simultaneously. Thus we have instead of the last expression of (45),

\[
\Sigma_{\text{npl}}^{(1)} = \frac{\lambda}{96\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\frac{1}{\alpha \Lambda^2}} \left[ 1 - \frac{\tilde{p}^2}{4\alpha} + \frac{\tilde{p}^4}{2!(4\alpha)^2} + \cdots \right]
\]

\[
= \frac{\lambda}{96\pi^2} \left[ 2(\Lambda^2 m^2)^{1/2} K_1(2\sqrt{m^2/\Lambda^2}) - \frac{\tilde{p}^2}{4} 2(\Lambda^2 m^2) K_2(2\sqrt{m^2/\Lambda^2}) \right.
\]

\[
+ \frac{\tilde{p}^4}{32} 2(\Lambda^2 m^2)^{3/2} K_3(2\sqrt{m^2/\Lambda^2}) + \cdots \right], \tag{47}
\]

where \( K_n \) is the modified Bessel function. If we put

\[
\theta^{\mu\nu} = a^2 \tilde{\theta}^{\mu\nu}, \tag{48}
\]

where \( a \) is the fundamental length with \( \tilde{\theta}^{\mu\nu} \) being dimensionless, then the commutative limit corresponds to \( a \to 0 \). Near IR region which is indistinguishable from the commutative limit in the present model, we propose to take UV limit in the following way,

\[
\Lambda^2 \to \infty,
\]

\[
a^2 \to 0,
\]

\[
\Lambda^2 a^2 : \text{fixed}. \tag{49}
\]

Then the first term in the bracket \( [ \) of (47) diverges quadratically as usual, the second becomes constant of the order \( a^4 \Lambda^4 \), the third vanishes because it behaves like \( a^8 \Lambda^6 \) and the rest follows the same fate as the third. It should be noted, however, that the drawback of the above argument is the apparent loss of the Lorentz invariance (here, Euclidean symmetry). Our argument may work only in the case of \( \Sigma_{\text{npl}}^{(1)} \) being a function of the invariant, \( p^2 \).

This shortcoming is remedied by Carlson-Carone-Zobin formulation\(^[7]\) of the model,

\[
\hat{S} = \int d^4 x d^6 \theta W(\theta) \left[ \frac{1}{2} (\partial_\mu \phi(x) \partial^\mu \phi(x) + m^2 \phi^2(x)) + \frac{\lambda}{4!} \phi(x) * \phi(x) * \phi(x) * \phi(x) \right]. \tag{50}
\]

In this new Lorentz-invariant (here, Euclidean-invariant) version we have

\[
\Sigma_{\text{pl}}^{(1)} = \frac{\lambda}{3(2\pi)^4} \int d^4 k \frac{1}{k^2 + m^2},
\]

\[
\Sigma_{\text{npl}}^{(1)} = \frac{\lambda}{6(2\pi)^4} \int d^4 k d^6 \theta W(\theta) \frac{e^{ik_\mu \theta^{\mu\nu} p_\nu}}{k^2 + m^2}, \tag{51}
\]

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where we have used the normalization (16) to obtain the same expression of $\Sigma^{(1)}_{\text{pl}}$ as before. That is, the planar diagram is no different. The $\theta$-integration in the nonplanar diagram,

$$I = \int d^6\theta W(\theta) e^{ik_\mu \theta^\mu p_\nu}$$  \hspace{1cm} (52)

can be performed as follows. Assuming

$$W(\theta) = a^{-12} w(\bar{\theta}),$$  \hspace{1cm} (53)

the integral becomes

$$I = \int d^6\bar{\theta} w(\bar{\theta}) e^{ia^2k_\mu \theta^\mu p_\nu}. \hspace{1cm} (54)$$

To proceed further we have to choose the functional form of $w(\bar{\theta}) = w(-\bar{\theta})$ with $\int d^6\bar{\theta} w(\bar{\theta}) = 1$. There is no guiding principle to determine it. In what follows, we put for computational purpose only

$$w(\bar{\theta}) = \frac{1}{\pi^3} e^{-(\bar{\theta}^0)^2-(\bar{\theta}^2)^2-(\bar{\theta}^3)^2-(\bar{\theta}^1)^2-(\bar{\theta}^1)^2}.$$  \hspace{1cm} (55)

It is then easy to obtain

$$I = e^{-a^4[k^2p^2-(k\cdot p)^2]/4}, \hspace{1cm} (56)$$

where $k^2 = k_0^2 + k_1^2 + k_2^2 + k_3^2$ (we still use the index 0 instead of 4). Thus the integral $I$ works as an invariant damping factor for the nonplanar diagram as far as $a$ and $p$ do not vanish. Inserting this result into new $\Sigma^{(1)}_{\text{pl}}$ of (51), we get Lorentz (here, Euclidean) invariant result

$$\Sigma^{(1)}_{\text{pl}} = \frac{\lambda}{6(2\pi)^4} \int_0^\infty d\alpha e^{-\frac{1}{a^4}\alpha m^2} \int d^4k e^{-\alpha k^2-a^4[k^2p^2-(k\cdot p)^2]/4}$$

$$= \frac{\lambda}{96\pi^2} \frac{\Gamma(3/2)}{\Gamma(1)} e^{-\alpha m^2}, \hspace{1cm} (57)$$

$^{13}$It is to be noted that to make $w(\bar{\theta})$ Lorentz-invariant the exponent must be at least quartic in $\bar{\theta}$ since $\bar{\theta}^\mu \bar{\theta}_\mu$ is indefinite. However, $(\bar{\theta}^\mu \bar{\theta}_\mu)^2$ is positive definite and we may use the formula,

$$\int dx e^{-x^4} = \frac{1}{4} \Gamma\left(\frac{1}{4}\right),$$

to maintain the normalization.

$^{14}$If we insist to Minkowski space-time formulation, we would get similar Lorentz-invariant damping factor for the nonplanar diagram. By the way, the oscillating factor is absent if only the projection of $p$ onto the non-commutative subspace vanishes, but our damping factor becomes unity only if all components of $p$ vanish.
which is UV finite unless $a^4 p^2 = 0$. It is, however, IR-singular, $\Sigma_{npl}^{(1)} \to \frac{8}{\alpha^* p^2} \times \frac{\lambda}{96 \pi^2}$ at $p^2 \to 0$, if we first let $\Lambda^2 \to \infty$. This is the same phenomenon as before except that our result is Lorentz (here, Euclidean) invariant. To avoid this IR-singularity we take the UV limit as defined above. Expanding the exponential containing the small parameter $a^4$ in new $\Sigma_{npl}^{(1)}$ and carrying out the $k$-integration, we obtain

$$\Sigma_{npl}^{(1)} = \frac{\lambda}{96 \pi^2} \left[ 2 \sqrt{m^2 \Lambda^2} K_1 \left( 2 \sqrt{m^2 / \Lambda^2} \right) - \frac{3}{8} a^4 p^2 2 (m^2 \Lambda^2) K_2 \left( 2 \sqrt{m^2 / \Lambda^2} \right) \right.
\left. + \frac{15}{128} a^8 p^4 2 (m^2 \Lambda^2)^{3/2} K_3 \left( 2 \sqrt{m^2 / \Lambda^2} \right) + \cdots \right].$$

(58)

Our UV limit reproduces the well-known quadratic divergence from the first term in the above expansion in addition to a constant term from the second term. The third and the rest vanish in our UV limit. We may of course retain the contributions of order $(a^4 p^2)^n (\Lambda^2)^{n+1}$, $n = 0, 1, 2, \cdots$, (including logs) in the above $(n+1)$-st term without strictly neglecting the third and the rest. Our argument to bypass the IR-singularity is to take the limit $a^2 \to 0$ at the same time as the limit $\Lambda^2 \to \infty$ so that the above expansion is effectively truncated. In this way we have successfully evaded the IR-singularity in an invariant way. Outside the region where (51) is used, $\Sigma_{npl}^{(1)}$ is UV finite.

We expect that a similar mechanism works in higher order loops and also in NCQED.

## 5 Seiberg-Witten map in NCQED

In order to investigate the commutative limit of NCWS for leptons, we have previously proposed $\theta$-expansion,

$$A_\mu (x, \theta) = A^{(0)}_\mu (x) + A^{(1)}_\mu (x) + A^{(2)}_\mu (x) + \cdots,$$

$$A'_\mu (x, \theta) = -A_\mu (x, -\theta) = -A^{(0)}_\mu (x) + A^{(1)}_\mu (x) - A^{(2)}_\mu (x) + \cdots,$$

$$F_{\mu \nu} (x, \theta) = F^{(0)}_{\mu \nu} (x) + F^{(1)}_{\mu \nu} (x) + F^{(2)}_{\mu \nu} (x) + \cdots,$$

$$F'_{\mu \nu} (x, \theta) = -F^{(0)}_{\mu \nu} (x) + F^{(1)}_{\mu \nu} (x) - F^{(2)}_{\mu \nu} (x) + \cdots,$$

$$\psi (x, \theta) = \psi^{(0)} (x) + \psi^{(1)} (x) + \psi^{(2)} (x) + \cdots,$$

(59)

where $\psi$ stands for all 8 spinors under consideration. The reason why we are led to the $\theta$-expansion even if we based our NCWS for leptons in Ref.11) on the Lorentz non-covariant algebra $[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu \nu}$, leading to fields not explicitly containing $\theta$, comes from the demand that the variational principle
should be applied in various cases exhibited in (34). We shall not repeat the argument but only note that there is a formidable increase in unknown local fields in the above expansion. Namely, although the first terms in the expansion are identified with the local fields in the commutative limit, 

\[ A_\mu(x) = A^{(0)}_\mu(x), \psi(x) = \psi^{(0)}(x), F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = F^{(0)}_{\mu\nu}(x), \]

there occur many unknown local fields in higher order terms. Nonetheless, gauge invariance of the action is proved order by order up to \( n = 2 \).

On the other hand, the authors in Ref. 14) developed a bottom-up version of NCGT. They also implicitly assumed \( \theta \) dependence of the field in Lorentz non-invariant NCGT and introduced additional parameter \( h \) to justify their formulation of NCGT using \( h \)-expansion (equivalent to \( \theta \)-expansion) and Seiberg-Witten map[15]. They showed that NCGT though violates Lorentz invariance can be formulated for arbitrary gauge group including \( SU(n) \) by employing Seiberg-Witten map[15] to determine all higher-order terms in terms of only the lowest-order term with due consideration on the gauge parameter coming from the consistency condition. Thus they expand the gauge parameter, too,

\[ \alpha(x, \theta) = \alpha^{(0)}(x) + \alpha^{(1)}(x) + \alpha^{(2)}(x) + \cdots, \]
\[ \alpha(x, -\theta) = \alpha^{(0)}(x) - \alpha^{(1)}(x) + \alpha^{(2)}(x) - \cdots, \]

where \( \alpha(x) = \alpha^{(0)}(x) \) is the same as in (29). The \( \theta \)-expansion of fields must be compatible with the \( \theta \)-expansion of the action

\[ \hat{S}_{QED} = S_{QED}^{(0)} + S_{QED}^{(1)} + S_{QED}^{(2)} + \cdots, \quad (61) \]

in the following respect. The action \( \hat{S}_{QED} \) is a functional of \( A_\mu(x, \theta), A'_\mu(x, \theta) \) and the spinors \( \chi_1(x, \theta), \cdots, \psi_6(x, \theta) \). It is invariant under the NC gauge transformations (25), (30) and (31). We now assume following Ref. 14) that each term in the above expansion is a local functional of only 4-dimensional fields, \( A_\mu(x), \chi_1(x), \cdots, \psi_6(x) \) with respected gauge invariance, (29) and

\[ A_\mu(x) \rightarrow g A_\mu(x) = \frac{2}{e} \partial_\mu \alpha(x). \quad (62) \]

Hence the \( \theta \)-expansion (61) contains both kinds of gauge invariance. This requires[14] that there exists the Seiberg-Witten map[15]

\[ A_\mu(x, \theta) = A_\mu[A(x), \theta], \]
\[ A'_\mu(x, \theta) = A'_\mu[A(x), \theta], \]
\[ \psi(x, \theta) = \psi[A(x), \psi(x), \theta], \]
\[ U(x, \theta) = U[A(x), U(x) \equiv e^{i\alpha(x)}, \theta], \quad (63) \]
where \( \psi \) stands for any spinor \( \chi_1(x, \theta), \ldots, \psi_6(x, \theta) \), such that

\[
\begin{align*}
\hat{g} A_\mu(x, \theta) &= A_\mu [\hat{g} A(x), \theta], \\
\hat{g} A'_\mu(x, \theta) &= A'_\mu [\hat{g} A(x), \theta], \\
\hat{g} \psi(x, \theta) &= \psi [\hat{g} A(x), \hat{g} \psi(x), \theta],
\end{align*}
\]

(64) hold true. This mapping must satisfy the consistency condition

\[
\begin{align*}
(g_1 g_2) A_\mu(x, \theta) &= g_1 (g_2 A_\mu(x, \theta)), \\
(g_1 g_2) A'_\mu(x, \theta) &= g_1 (g_2 A'_\mu(x, \theta)), \\
(g_1 g_2) \psi(x, \theta) &= g_1 (g_2 \psi(x, \theta)).
\end{align*}
\]

(65)

In the following we shall determine the Seiberg-Witten map only in infinitesimal gauge transformation,

\[
U(x, \theta) = (e^{i \alpha(x, \theta)})_* = 1 + i \alpha(x, \theta),
\]

\[
\begin{align*}
\delta_\alpha A_\mu(x, \theta) &= i (\alpha(x, \theta) \ast A_\mu(x, \theta) - A_\mu(x, \theta) \ast \alpha(x, \theta)) + \frac{2}{e} \partial_\mu \alpha(x, \theta), \\
\delta_\alpha A'_\mu(x, \theta) &= -i (\alpha(x, -\theta) \ast A'_\mu(x, \theta) - A'_\mu(x, \theta) \ast \alpha(x, -\theta)) - \frac{2}{e} \partial_\mu \alpha(x, -\theta), \\
\delta_\alpha \chi_1(x, \theta) &= i \alpha(x, \theta) \ast \chi_1(x, \theta) - \chi_1(x, \theta) \ast i \alpha(x, \theta), \\
\delta_\alpha \psi_1(x, \theta) &= i \alpha(x, \theta) \ast \psi_1(x, \theta), \\
\delta_\alpha \psi_6(x, \theta) &= -i \alpha(x, -\theta) \ast \psi_6(x, \theta) - \psi_6(x, \theta) \ast i \alpha(x, \theta).
\end{align*}
\]

(66)

The spinors we shall consider below are only \( \chi_1, \psi_1 \) and \( \psi_6 \). The other cases will be obvious. The infinitesimal form of (64) read

\[
\begin{align*}
\delta_\alpha A_\mu(x, \theta) &= \delta_\alpha A_\mu(x, \theta) \equiv A_\mu [A(x) + \delta_\alpha A(x), \theta] - A_\mu [A(x), \theta], \\
\delta_\alpha A'_\mu(x, \theta) &= \delta_\alpha A'_\mu(x, \theta) \equiv A'_\mu [A(x) + \delta_\alpha A(x), \theta] - A'_\mu [A(x), \theta], \\
\delta_\alpha \psi(x, \theta) &= \delta_\alpha \psi(x, \theta) \equiv \psi [A(x) + \delta_\alpha A(x), \psi(x) + \delta_\alpha \psi(x), \theta] - \psi [A(x), \psi(x), \theta],
\end{align*}
\]

(67)

while the consistency condition (65) reads

\[
i \delta_\alpha \beta[A, \theta] - i \delta_\beta \alpha[A, \theta] - \beta[A, \theta] \ast \alpha[A, \theta] + \alpha[A, \theta] \ast \beta[A, \theta] = 0.
\]

(68)
Here we define, writing \( \alpha(x, \theta) = \alpha[A(x), \theta] = \alpha[A, \theta] \),
\[
\delta_\alpha \beta[A, \theta] \equiv \beta[A + \delta_\alpha A, \theta] - \beta[A, \theta].
\]

Our purpose of this section is to show that the \( \theta \) expansion (59) of fields \( A_\mu(x, \theta) \) and \( A'_\mu(x, \theta) \) has a consistent solution given the gauge transformations (66) thanks to the expansion (60). We now know\(^{[14]}\) that the solution to the consistency condition (68) determines \( \alpha^{(n)}(x) \) iteratively. The result up to \( n = 2 \) is as follows. (In the following, we omit argument \( x \) for simplicity.)
\[
\begin{align*}
\alpha^{(0)} &= \alpha, \\
\alpha^{(1)} &= \frac{e}{4} \theta^{\rho\sigma} \partial_\rho \alpha A_\sigma, \\
\alpha^{(2)} &= -\frac{e^2}{8} \theta^{\rho\sigma} \theta^{\lambda\tau} \partial_\rho \alpha A_\lambda \partial_\tau A_\sigma.
\end{align*}
\]

It can be shown that the same solution on the gauge fields \( A_\mu^{(1)} \) and \( A_\mu^{(2)} \) comes from both \( \delta_\alpha A_\mu(x, \theta) \) and \( \delta_\alpha A'_\mu(x, \theta) \) in (66) due to (60). Thus we have
\[
\begin{align*}
A_\mu^{(0)} &= A_\mu, \\
A_\mu^{(1)} &= -\frac{e}{4} \theta^{\rho\sigma} A_\rho (\partial_\sigma A_\mu + F_{\sigma\mu}), \\
A_\mu^{(2)} &= \frac{e^2}{8} \theta^{\rho\sigma} \theta^{\lambda\tau} (A_\rho A_\lambda \partial_\tau F_{\sigma\mu} - \partial_\sigma A_\mu \partial_\lambda A_\tau + A_\rho F_{\sigma\lambda} F_{\tau\mu}).
\end{align*}
\]

As for the field strength, we find
\[
\begin{align*}
F_{\mu\nu}^{(0)} &= F_{\mu\nu}, \\
F_{\mu\nu}^{(1)} &= \frac{e}{2} \theta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{e}{2} \theta^{\rho\sigma} A_\rho \partial_\sigma F_{\mu\nu}, \\
F_{\mu\nu}^{(2)} &= \frac{e^2}{8} \theta^{\rho\sigma} \theta^{\lambda\tau} f_{\mu\nu\rho\sigma\lambda\tau}, \\
f_{\mu\nu\rho\sigma\lambda\tau} &= F_{\mu\nu} F_{\sigma\lambda} F_{\tau\rho} F_{\tau\nu} - F_{\mu\rho} F_{\sigma\lambda} F_{\tau\nu} + F_{\sigma\lambda} \partial_\tau F_{\mu\nu} - \partial_\lambda F_{\mu\nu} \partial_\sigma A_\tau \\
&\quad + A_\rho (2F_{\mu\lambda} \partial_\sigma F_{\tau\nu} - 2F_{\nu\lambda} \partial_\sigma F_{\tau\mu} + F_{\sigma\lambda} \partial_\tau F_{\mu\nu} - \partial_\lambda F_{\mu\nu} \partial_\sigma A_\tau) \\
&\quad + A_\rho A_\lambda \partial_\tau \partial_\sigma F_{\mu\nu}.
\end{align*}
\]
Consequently, NC Maxwell action (40) has the expansion

\begin{equation}
S^{(0)}_M = -\frac{1}{4} \int d^4 x F_{\mu \nu} F^{\mu \nu},
\end{equation}

\begin{equation}
S^{(1)}_M = 0,
\end{equation}

\begin{equation}
S^{(2)}_M = -\frac{e^2}{16} \int d^6 \theta W(\theta) \theta^{\rho \sigma} \theta^{\lambda \tau} \int d^4 \tilde{f}_{\rho \sigma \lambda \tau},
\end{equation}

\begin{equation}
\tilde{f}_{\rho \sigma \lambda \tau} = F_{\mu \rho} F_{\nu \sigma} F_{\chi \lambda} F^{\nu \tau} + F^{\mu \nu} (F_{\rho \mu} F_{\sigma \lambda} F_{\tau \nu} - F_{\rho \mu} F_{\sigma \lambda} F_{\tau \nu})
\end{equation}

\begin{equation}
- F_{\rho \sigma} F^{\mu \nu} F_{\mu \lambda} F_{\nu \tau}
\end{equation}

\begin{equation}
+ \frac{1}{2} F_{\rho \sigma} A_\lambda F_{\mu \nu} \partial_\tau F^{\mu \nu} + A_\rho F^{\mu \nu} F_{\sigma \lambda} \partial_\tau F_{\mu \nu},
\end{equation}

(73)

Thus $C'$-invariance excludes three-photon vertices. If we employ NC Maxwell action

\begin{equation}
\hat{S}'_M = -\frac{1}{4} \int d^4 x d^6 \theta W(\theta) F_{\mu \nu}(x, \theta) \ast F^{\mu \nu}(x, \theta),
\end{equation}

(74)

we have $S''^{(0)}_M = S^{(0)}_M$ and $S''^{(2)}_M = S^{(2)}_M$ together with

\begin{equation}
S''^{(1)}_M = -\frac{e}{4} \int d^6 \theta W(\theta) \theta^{\rho \sigma} \int d^4 \tilde{f}_{\rho \sigma \lambda \tau}[F_{\mu \rho} F_{\nu \sigma} F_{\mu \nu} - \frac{1}{4} F_{\rho \sigma} F_{\mu \nu} F_{\mu \nu}].
\end{equation}

(75)

It contributes nothing, however, because of (17) as pointed out by Carlson-Carone-Zobin\cite{7}. In Ref. 7) (17) was derived from the Lorentz invariance, while we derived it from DFR algebra itself.

\cite{15}In the old Lorentz-non-invariant version without $\theta$ integration $S''^{(1)}_M = -\frac{e}{4} \theta^{\rho \sigma} \int d^4 \tilde{f}_{\rho \sigma \lambda \tau}[F_{\mu \rho} F_{\nu \sigma} F_{\mu \nu} - \frac{1}{4} F_{\rho \sigma} F_{\mu \nu} F_{\mu \nu}]$ makes a nontrivial contribution.
Finally, we give Seiberg-Witten map for the spinors,

\[ \chi^{(1)}_1 = -\frac{e}{2} \theta^{\rho\sigma} A_\rho \partial_\sigma \chi^{(0)}_1, \]

\[ \chi^{(2)}_1 = \frac{1}{8} \theta^{\rho\sigma} \theta^{\lambda\tau} \left[ -ie \partial_\rho A_\lambda \partial_\sigma \partial_\tau \chi^{(0)}_1 - \frac{2}{e} \partial_\rho A_\lambda A_\tau \partial_\sigma \chi^{(0)}_1 \right. \]

\[ \left. -2e^2 A_\lambda F_{\rho\tau} \partial_\sigma \chi^{(0)}_1 + e^2 A_\lambda A_\rho \partial_\sigma \partial_\tau \chi^{(0)}_1 \right], \]

\[ \psi^{(1)}_1 = -\frac{e}{4} \theta^{\rho\sigma} A_\rho \partial_\sigma \psi^{(0)}_1, \]

\[ \psi^{(2)}_1 = \frac{1}{32} \theta^{\rho\sigma} \theta^{\lambda\tau} \left[ -2ie \partial_\rho A_\lambda \partial_\sigma \partial_\tau \psi^{(0)}_1 + \frac{2}{e} A_\lambda A_\phi \partial_\sigma \partial_\tau \psi^{(0)}_1 \right. \]

\[ \left. + e^2 A_\lambda F_{\rho\tau} \partial_\sigma \psi^{(0)}_1 + 2e^2 A_\rho \partial_\sigma A_\lambda A_\tau \psi^{(0)}_1 - \frac{e^2}{2} \partial_\rho A_\lambda \partial_\sigma A_\tau \psi^{(0)}_1 \right. \]

\[ \left. + ie^3 A_\rho A_\tau \partial_\lambda \partial_\sigma \psi^{(0)}_1 - i \frac{e^3}{2} A_\rho A_\tau \partial_\lambda A_\sigma \psi^{(0)}_1 \right], \]

\[ \psi^{(1)}_6 = 0, \]

\[ \psi^{(2)}_6 = \frac{1}{32} \theta^{\rho\sigma} \theta^{\lambda\tau} \left[ 4ie \partial_\rho A_\lambda \partial_\sigma \partial_\tau \psi^{(0)}_6 - 2e^2 \partial_\rho A_\lambda \partial_\sigma \partial_\tau \psi^{(0)}_6 \right. \]

\[ \left. + 4e^2 \partial_\sigma A_\lambda A_\rho \partial_\tau \psi^{(0)}_6 \right]. \]

Dirac action for the spinor \( \chi_1 \) has the expansion with upper index still attached

\[ \hat{S}_D = S_D^{(0)} + S_D^{(1)} + S_D^{(2)} + \cdots, \]

\[ S_D^{(0)} = \int d^4 x \bar{\chi}_1^{(0)} \gamma^\mu \partial_\mu \chi_1^{(0)}, \]

\[ S_D^{(1)} = \frac{ie}{2} \int d^6 \theta W(\theta) \theta^{\rho\sigma} \int d^4 x \bar{\chi}_1^{(0)} \gamma^\mu F_{\rho\mu} \partial_\sigma \chi_1^{(0)}, \]

\[ S_D^{(2)} = \int d^4 x d^6 \theta W(\theta) \left[ \bar{\chi}_1^{(1)} \gamma^\mu \partial_\mu \chi_1^{(1)} + \bar{\chi}_1^{(0)} \gamma^\mu \partial_\mu \chi_1^{(2)} + \bar{\chi}_1^{(2)} \gamma^\mu \partial_\mu \chi_1^{(0)} \right. \]

\[ \left. + \bar{\chi}_1^{(1)} \frac{ie}{2} \theta^{\rho\sigma} (i \gamma^\mu \partial_\rho A_\mu) \partial_\sigma \chi_1^{(0)} + \bar{\chi}_1^{(1)} \frac{ie}{2} \theta^{\rho\sigma} (i \gamma^\mu \partial_\rho A_\mu) \partial_\sigma \chi_1^{(0)} \right. \]

\[ \left. + \bar{\chi}_1^{(0)} \frac{ie}{2} \theta^{\rho\sigma} (i \gamma^\mu \partial_\rho A_\mu^{(1)}) \partial_\sigma \chi_1^{(0)} \right]. \]
The second term $S_D^{(1)}$ vanishes due to (17). Consequently we do not need evaluate $S_D^{(1)}$ in the following. For $\psi_1$, we get

$$\hat{S}_D = S_D^{(0)} + S_D^{(2)} + \cdots,$$

$$S_D^{(0)} = \int d^4x \bar{\psi}_1 \gamma^\mu D_\mu \psi_1, \quad D_\mu = \partial_\mu - \frac{ie}{2} A_\mu,$$

$$S_D^{(2)} = \int d^4x d^6\theta W(\theta) \{ \bar{\psi}_1 \gamma^\mu D_\mu \psi_1 + \bar{\psi}_1 \gamma^\mu D_\mu \psi_1 + \bar{\psi}_1 \gamma^\mu D_\mu \psi_1 + \bar{\psi}_1 \gamma^\mu D_\mu \psi_1 \} .$$

For $\psi_6$ we find

$$\hat{S}_D = S_D^{(0)} + S_D^{(2)} + \cdots,$$

$$S_D^{(0)} = \int d^4x \bar{\psi}_6 \gamma^\mu D_\mu \psi_6, \quad D_\mu = \partial_\mu + ie A_\mu,$$

$$S_D^{(2)} = \int d^4x \int d^6\theta W(\theta) \{ \bar{\psi}_6 \gamma^\mu D_\mu \psi_6 + \bar{\psi}_6 \gamma^\mu D_\mu \psi_6 + \bar{\psi}_6 \gamma^\mu D_\mu \psi_6 \} .$$

To put $\hat{S}_D^{(2)}$ into a final form (76) should be used. We shall not try in this paper to do phenomenological calculations based on the above Seiberg-Witten map. It would be enough to comment that Carlson-Carone-Zobin\cite{7} calculated $2\gamma \to 2\gamma$ scattering based on (73) and found a distinctive deviation from the standard model result.
6 Conclusions

As the first step toward formulating NCSM by considering the total fermion field as a NC bi-module, we have constructed in this paper Lorentz-invariant NCQED. All possible spinors are considered, which are reduced to one neutral and four charged spinors in the commutative limit. This charge quantization is tight enough that it precisely gives the correct hypercharge assignment of leptons in addition to their electric charge.

The important aspect of our NCQED is its Lorentz invariance. It was first formulated by Carlson, Carone and Zobin\cite{7}. The commutative limit of the Lorentz-invariant NCQED smoothly coincides with Lorentz-invariant QED, since we never take the limit $[\hat{x}^\mu, \hat{x}^\nu] \to 0$.

The oscillating damping factor for nonplanar diagrams first observed in Ref. 17) is replaced with an invariant damping factor. Moreover, the singular behavior of Green functions at $\theta \to 0$ found in the literature\cite{10,13} may be evaded using a new UV limit, assumed to be valid near IR region indistinguishable from the commutative limit, in an invariant way. This conjecture was confirmed in the proper self-energy diagram of NC scalar model in the one-loop approximation.

We define covariant operator fields on DFR algebra (8) and associate them with $c$-number Weyl symbols which enjoy the same Lorentz covariance. The latter are field quantities to be subsequently quantized. Even in the commutative limit, DFR algebra remains intact. Our Minkowski space-time is a parameter space like the 4-dimensional phase space. If we say that our Minkowski space-time becomes non-commutative at very short distances by assuming the commutator $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ for constant $\theta$, we immediately sacrifice the Lorentz symmetry, which could never be remedied as far as we stick to constant $\theta$ algebra. If we employ Lorentz-covariant algebra (8) following Carlson, Carone and Zobin\cite{7}, the deformation parameter becomes an integration variable. Nonetheless, it is possible to study small $\theta$ using Seiberg-Witten map because it is dimensionfull. In addition, one-loop calculation in the section 4 indicates how to obtain invariant amplitudes on non-commutative space-time.

Finally, we would like to point out that, since QED is not a closed theory but is unified with weak interactions at present energy, NCQED should also be a part of a larger theory. This subject will be a theme in the following papers.

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A Derivative operator for constant \( \theta \)

The derivative operator for the field \( \hat{\varphi}(\hat{x}) \) on the non-commutative space-time \([\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\) is defined by

\[
\hat{p}_\mu = -i\theta_{\mu\nu} \hat{x}^\nu,
\]

where \( \theta_{\mu\nu} \) is the inverse of the matrix \( \theta^{\mu\nu} \), \( \theta_{\mu\nu} \theta^{\nu\lambda} = \delta^\lambda_\mu \) so that \([\hat{p}_\mu, \hat{x}^\nu] = \delta^\nu_\mu \). It can be shown that the commutator \([\hat{p}_\mu, \hat{\varphi}(\hat{x})]\) equals (13) with \( \hat{\varphi} \) disregarded. There are two problems, here. First of all \( \theta^{\mu\nu} \) must be assumed to be invertible. The second is that the commutative limit is not smooth as pointed out in the Appendix A in Ref.11). Suppose that the matrix \((\theta^{\mu\nu})\) is invertible and put into the canonical form with only non vanishing elements \(\theta_{1,2}\) such that \(\hat{x}^1 \) and \(\hat{x}^3\) are diagonalized with the basis \(|x^1, x^3\rangle\). Then \(\hat{p}_0 = (i/\theta_1)\hat{x}^1\) and \(\hat{p}_2 = (i/\theta_2)\hat{x}^3\) become singular in the commutative limit \(\theta_1, \theta_2 \to 0\), although the commutator \([\hat{p}_\mu, \hat{\varphi}(\hat{x})]\) for any \(\mu\) is well-defined. On the other hand, if all coordinates commute, they can be simultaneously diagonalized with the different basis \(|x^\nu\rangle\). In this case, we simply put \(\hat{p}_\mu = \partial/\partial x^\mu\). This non-smoothness no longer bothers us for DFR algebra.

B Operator product and Moyal product

In this section we shall give the proof of (20). The integrand \(\hat{\varphi}_{12}(k, \sigma)\) is calculated, using the notation \((k \times k')\hat{\theta} = (k \times k')_{\mu\nu}\hat{\theta}^{\mu\nu}\) with \((k \times k')_{\mu\nu} \equiv (1/2)(k_1 k_{2\nu} - k_1 k_{2\nu})\) and \(\hat{T}(k, \sigma) = e^{ik\cdot x + i\sigma \hat{\theta}}\), as follows.

\[
\hat{\varphi}_1(\hat{x}, \hat{\theta})\hat{\varphi}_2(\hat{x}, \hat{\theta}) = \frac{1}{(2\pi)^8} \int d^4k d^6\sigma \hat{\varphi}_1(k, \sigma)\hat{T}(k, \sigma) \int d^4k' d^6\sigma' \hat{\varphi}_2(k', \sigma')\hat{T}(k', \sigma')
\]

\[
= \frac{1}{(2\pi)^8} \int d^4k d^6\sigma d^4k' d^6\sigma' \hat{\varphi}_1(k, \sigma)\hat{\varphi}_2(k', \sigma') e^{-i(k \times k')\hat{\theta}}\hat{T}(k + k', \sigma + \sigma')
\]

\[
= \frac{1}{(2\pi)^8} \int d^4K d^6\Sigma' d^4k' d^6\sigma' \hat{\varphi}_1(K - k', \Sigma' - \sigma') \times \hat{\varphi}_2(k', \sigma')\hat{T}(K, \Sigma' - \frac{1}{2}K \times K')
\]

\[
= \frac{1}{(2\pi)^8} \int d^4K d^6\Sigma \hat{\varphi}_{12}(K, \Sigma)\hat{T}(K, \Sigma),
\]

\[
\hat{\varphi}_{12}(K, \Sigma) = \frac{1}{(2\pi)^4} \int d^4k' d^6\sigma' \hat{\varphi}_1(K - k', \Sigma + \frac{1}{2}K \times k' - \sigma')\hat{\varphi}_2(k', \sigma').
\]
Hence, we have
\[
\varphi_{12}(x, \theta) = \frac{1}{(2\pi)^4} \int d^4K d^6\Sigma e^{iKx+i\Sigma\theta} \varphi_{12}(K, \Sigma)
\]
\[
= \frac{1}{(2\pi)^{20}} \int d^4K d^6\Sigma d^4k' d^6\sigma' e^{iKx+i\Sigma\theta} \int d^4x_1 d^6\theta_1 d^4x_2 d^6\theta_2 \varphi_1(x_1, \theta_1) \varphi_2(x_2, \theta_2)
\]
\[
\times e^{-i(K-k')x_1-i(\Sigma+\frac{i}{2}K\times\sigma')\theta_1} e^{-ik'x_2-i\sigma'\theta_2}
\]
\[
= \frac{1}{(2\pi)^{20}} \int d^4K d^6\Sigma d^4k' d^6\sigma' d^4x_1 d^6\theta_1 d^4x_2 d^6\theta_2
\]
\[
\times e^{iKx+i\Sigma\theta-i(K-k')x_1-i(\Sigma-\sigma')\theta_1-ik'x_2-i\sigma'\theta_2}(e^{i\frac{1}{2}(\partial_1 \times \partial_2)\theta_1} \varphi_1(x_1, \theta_1) \varphi_2(x_2, \theta_2))
\]
\[
= e^{\frac{\partial u}{\partial x_1} \frac{\partial \varphi_1}{\partial x_1}} \varphi_1(x, \theta) \varphi_2(y, \theta)|_{x=y}
\]
\[
\equiv \varphi_1(x, \theta) * \varphi_2(x, \theta).
\]

This is nothing but (20).

References


