Superluminal Localized Solutions to Maxwell Equations propagating along a waveguide: The finite-energy case

Michel Zamboni Rached,  
*D.M.O., Faculty of Electrical Engineering, UNICAMP, Campinas, SP, Brasil.*

Flavio Fontana  
*R&d Sector, Pirelli Labs, Milan, Italy*

and

Erasmo Recami  
*Facoltà di Ingegneria, Università statale di Bergamo, Dalmine (BG), Italy; INFN—Sezione di Milano, Milan, Italy; and CCS, State University of Campinas, Campinas, S.P., Brasil.*

**Abstract** — In a previous paper we have shown localized (non-evanescent) solutions to Maxwell equations to exist, which propagate without distortion with Superluminal speed along normal-sized waveguides, and consist in trains of “X-shaped” beams. Those solutions possessed therefore infinite energy. In this note we show how to obtain, by contrast, finite-energy solutions, with the same localization and Superluminality properties.

**PACS nos.** 41.20.Jb; 03.50.De; 03.30.+p; 84.40.Az; 42.82.Et .  
**Keywords:** Wave-guides; Localized solutions to Maxwell equations; Superluminal waves; Bessel beams; Limited-dispersion beams; Finite-energy waves; Electromagnetic wavelets; X-shaped waves; Evanescent waves; Electromagnetism; Microwaves; Optics; Special relativity; Localized acoustic waves; Seismic waves; Mechanical waves; Elastic waves; Guided gravitational waves.

---

(*) Work partially supported by CAPES (Brazil), and by MIUR/MURST and INFN (Italy).  E-mail address for contacts: recami@mi.infn.it
1. Introduction: Localized solutions to the wave equation

Already in 1915 Bateman[1] showed that Maxwell equations admit (besides of the ordinary solutions, endowed in vacuum with speed c) of wavelet-type solutions, endowed in vacuum with group-velocities $0 \leq v \leq c$. But Bateman’s work went practically unnoticed. Only few authors, as Barut et al.[2], followed such a research line; incidentally, Barut et al. constructed even a wavelet-type solution travelling with Superluminal group-velocity[3] $v > c$.

In more recent times, however, many authors discussed the fact that all (homogeneous) wave equations admit solutions with $0 < v < \infty$: see, e.g., refs.[4]. Most of those authors confined themselves to investigate (sub- or Super-luminal) localized non-dispersive solutions in vacuum: namely, those solutions that were called “undistorted progressive waves” by Courant & Hilbert. Among localized solutions, the most interesting appeared to be the so-called “X-shaped” waves, which —predicted to exist even by Special Relativity in its extended version[5]— had been mathematically constructed by Lu & Greenleaf[6] for acoustic waves, and by Ziolkowski et al.[6], and later Recami[6], for electromagnetism. Let us recall that such “X-shaped” localized solutions are Superluminal (i.e., travel with $v > c$ in the vacuum) in the electromagnetic case; and are “super-sonic” (i.e., travel with a speed larger than the sound-speed in the medium) in the acoustic case. The first authors to produce X-shaped waves experimentally were Lu & Greenleaf[7] for acoustics, Saari et al.[7] for optics, and Mugnai et al. for microwaves[7].

In a recent paper of ours, appeared in this journal[8], we showed that solutions to the Maxwell equations exist, that displace themselves with Superluminal speed even along a normal waveguide: where one ordinarily expects to meet propagating, subluminal modes only. Actually, a segment of “undersized” waveguide constitutes an evanescence region[9], and evanescent waves are known to travel Superluminally[5,9-11]; however, it was rather unexpected that (localized) waves could propagate Superluminally down a normal-sized waveguide. In fact, the dispersion relation in undersized guides is $\omega^2/c^2 - \beta^2 = -K^2$, so that the standard formula $v \simeq d\omega/d\beta$ yields a $v > c$ group-velocity[12]; by contrast, in normal guides the dispersion relation becomes $\omega^2/c^2 - \beta^2 = +K^2$, so that it seems to yield values $v < c$ only. Instead, in our paper[8] we have shown that localized solutions
to Maxwell equations do exist, propagating with \( v > c \) even in normal waveguides; but their group-velocity \( v \) cannot be given\(^\#1\) by the approximate formula \( v \simeq d\omega/d\beta \). [Let us recall that the group-velocity is well defined only when the pulse has a clear bump in space; but it can be calculated by the approximate, elementary relation \( v \simeq d\omega/d\beta \) only when \( \omega \) as a function of \( \beta \) is also clearly bumped].

2. – The infinite-energy solutions

Namely, in ref.[8] we constructed localized solutions to the Maxwell equations (which propagate undistorted, with Superluminal speed along a cylindrical waveguide located along the \( z \)-direction) for the TM (transverse magnetic) case and for a dispersion-free medium. The case with dispersion has been treated elsewhere[13], as well as the case of a co-axial cable[14]. Here, let us call attention to two points, which received just a mention in ref.[8], with regard to eq.(9) and Fig.2 therein: (i) those solutions consist in trains of pulses (similar to the one depicted in Fig.2 of ref.[1]); (ii) each of such pulses is X-shaped: See our Fig.1 below. Let us notice, incidentally, that we are referring ourselves to the electromagnetic case, but the same would hold for all situations in which a fundamental role is played by the wave equation (as in acoustics, geophysics, gravitational wave physics, etc.).

For instance, in the case of axial symmetry, let us consider a metallic waveguide with radius[8]

\[ r \equiv R \, . \]

Let us also put \( \rho \equiv (x,y) \), and \( \rho = |\rho| \) and the boundary condition \( \Psi(\rho = r,z;t) = 0 \). In the previous paper[8] we constructed the following solution,

\[
\Psi(\rho,z;t) = \sum_{n=1}^{N} \left( \frac{2}{a^2 \sin^2 \theta J_1^2(\lambda_n)} \right) J_0(K_n \rho) \cos \left( \frac{\omega_n}{V} (z - V t) \right), \tag{1}
\]

where \( \Psi \) represents the longitudinal component of the electric field, \( E_z \), while \( N \) is an integer, the quantities \( \lambda_n \) are the roots of the Bessel function, \( K_n = \lambda_n/R \), \( \omega_n = K_n c/\sin \theta \) and \( V = c/\cos \theta \). These solutions are therefore Fourier-Bessel–type sums over different
propagating modes with angular frequencies $\omega_n$. One can moreover notice that, in eq.(1), quantity $\theta$ is an arbitrary angle: by varying it, one obtains different train speeds and different distance between the pulses. Actually, our solutions propagate rigidly down the guide with (Superluminal) speed $V = c/\cos \theta$. In Fig.1 we depict one of the trains of X-shaped waves, obtained by numerical evaluation of eq.(1) for a waveguide radius $R = 5 \text{ cm}$, with $\theta = \pi/3$ (and, therefore, group-velocity $V = 2c$).

It is interesting to mention also that the integer $N$ determines the space-time width of the pulses: the higher $N$ is, the smaller the pulse “spatio-temporal” width will be. Let us emphasize that each eq.(1) represents a multimodal (but localized) propagation, as if the geometric dispersion compensated for the multimodal dispersion.

We mentioned that $\Psi$ represents the electric field component $E_z$. Let us add that, by following the procedure adopted by us in ref.[8], the other electromagnetic field components in the considered TM case result to be

$$E_\perp = i \frac{V}{V^2 - 1} \sum_{n=1}^{\infty} \frac{c}{\omega_n} \nabla_\perp \Psi ,$$

where

$$\frac{V}{V^2 - 1} \equiv \frac{\cos \theta}{\sin^2 \theta} ;$$

and

$$H_\perp = \varepsilon_0 V \hat{z} \wedge E_\perp .$$

Equation (1) allows for a physical interpretation, which suggests a very simple way to get it. Each pulse train is a sum of the first $N$ modes of our expansion (and for each $N$ we get a different train, at our choice), whose frequencies have been suitably chosen as corresponding to the intersections of the modal curves (i.e., the various branches of the dispersion-relation) with the single straight line $\omega = V \beta$ whose slope depends on $\theta$ only: see Fig.2. In such a case, all the modes correspond to the same (Superluminal) phase-velocity $V_{\text{ph}}$, it being independent of the mode index $n$; but, when the phase-velocity is
independent of the frequency, it becomes the group-velocity, which is the velocity “tout court” of the considered pulse. Let us repeat once more that we thus got (non-evanescent) solutions to the Maxwell equations, which are waves propagating undistorted along normal waveguides with Superluminal speed, even if in normal-sized waveguides the dispersion relation for each mode, i.e. for each term of our Fourier-Bessel expansion, is the ordinary “subluminal” one, \( \omega^2/c^2 - \beta^2 = +K^2 \). Let us repeat that, in fact, that the global velocity \( v \) (or group-velocity \( v_g \equiv v \)) of the pulses corresponding to eq.(1) is not to be evaluated by the ordinary formula \( v_g \approx d\omega/d\beta \), valid for quasi-monochromatic signals). This is at variance with the common situation in optical and microwave communications, when the signal is usually superimposed to a carrier wave whose frequency is generally much higher than the signal bandwidth. In that case the standard formula for \( v_g \) yields the correct velocity to deal with (e.g., when propagation delays are studied). Our case, on the contrary, is much more reminiscent of a baseband modulated signal, as those studied in ultrasonics: the very concept of a carrier becomes meaningless here, as the elementary “harmonic” components have widely different frequencies.

The fact that our Superluminal solutions travel rigidly, down the waveguide, is at variance also with what happens for truncated (Superluminal) solutions[15,7], which travel almost rigidly only along their finite “field depth” and then abruptly decay.

It may be finally underlined that the coefficients in eq.(1) can be varied so to keep the pulse spectrum inside the desired frequency range. This point will be discussed again soon.

3. – The finite-energy solutions

In this note, we have called attention to the fact that solutions (1) are infinite trains of pulses, with infinity energy. This is not a real problem (plane-waves too have infinite energy), provided that we are able to truncate them in space and time without destroying their good properties. We shall go on following the previous assumptions: what we are going to do holds, however, for both the TM and the TE case. Let us anticipate that, in order to get finite total-energy solutions, we shall have to replace each characteristic frequency \( \omega_n \) [cf. eq.(1), or Fig.2] by a small frequency band \( \Delta\omega \) centered at \( \omega_n \), always choosing the same \( \Delta\omega \) independently of \( n \). In fact, since all the modes entering the
Fourier-type expansion (1) possess the same phase-velocity $V_{ph} \equiv V = c / \cos \theta$, each small bandwidth packet associated with $\omega_n$ will possess the same group-velocity $v_g = c^2 / V_{ph}$, so that we shall have as a result a wave whose envelope travels with the subluminal group-velocity $v_g$. However, inside that subluminal envelope, one or more pulses will be travelling with the dual (Superluminal) speed $V = c^2 / v_g$. Such well-localized peaks will have nothing to do with the ordinary (sinusoidal) carrier-wave, and will be regarded as constituting the relevant wave. When integrating each term of expansion (1) over its corresponding frequency-band, one may choose, e.g., Gaussian spectra.

Before going on, let us mention that previous work related to FTESs can be found—as far as we know—only in refs.[16,14,17].

More formally, let us consider our ordinary solutions for a metallic waveguide, written in the form

$$\psi_n(\rho, z; t) = A_n R_n(\rho) \cos[\beta(\omega) z - \omega t] ,$$

where coefficients $A_n$ and functions $R_n$ are given by the coefficients and the (transverse) functions entering eq.(1); namely:

$$A_n = \frac{2}{a^2 \sin^2 \theta \sqrt{J_1^2(\lambda_n)}} \quad R_n(\rho) = J_0(K_n \rho) ; \quad K_n = \frac{\lambda_n}{R} .$$

Then, let us adopt the spectral functions

$$W_n \equiv \exp[-q^2(\omega - \omega_n)^2] , \quad (4)$$

where the weight-parameter $q$ is always the same, so that $\Delta \omega$ too is independent of $n$ [in fact, it is $\Delta \omega = 1/q$]; and where

$$\omega_n \equiv \frac{K_n c}{\sin \theta} .$$

quantity $\sin \theta$ having a fixed but otherwise arbitrary value. Notice that the last relation implies the wavenumbers $\beta_n$ of the longitudinal waves to be given, in terms of the
corresponding $\omega_n$, by $\beta_n = \omega_n \cos \theta/c$. We can construct FTESs, $\mathcal{F}(\rho, z; t)$, of the type

$$\mathcal{F}(\rho, z; t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} d\omega \psi_n W_n,$$

(5)

with arbitrary $N$. Notice that we are not using a single Gaussian weight, but a different Gaussian function $W_n$ for each $\omega_n$-value. Such weights $W_n$ are well localized around the corresponding $\omega_n$, so that one can expand (for each value of $n$, in the above sum) the function $\beta(\omega)$ in the neighbourhood of the corresponding $\omega_n$-value as follows:

$$\beta(\omega) \simeq \beta(\omega_n) + \frac{\partial \beta}{\partial \omega} \bigg|_{\omega_n} (\omega - \omega_n) + \ldots$$

where $\beta(\omega_n) = \omega_n \cos \theta/c$, and the further terms are neglected [since, let us repeat, $\Delta \omega$ has been assumed to be small]. Therefore, we are now facing no longer a set of phase-velocities, but the set of group-velocities

$$\frac{1}{v_{gn}} = \frac{\partial \beta}{\partial \omega} \bigg|_{\omega_n},$$

which result to be independent of $n$, all of them possessing therefore the same value

$$v_{gn} \equiv v_g = c \cos \theta.$$

(6)

By performing the integration in eq.(5), we eventually obtain

$$\mathcal{F}(\rho, z; t) = \sqrt{\frac{\pi}{q}} \exp \left[ -\frac{(z - v_g t)^2}{4q^2 v_g^2} \right] \Psi(\rho, z - Vt),$$

(7)

where $\Psi(\rho, z - Vt)$ is the pulse train given by eq.(1); and we had recourse to the identity

$$\int_{-\infty}^{\infty} df \exp[-q^2 f^2] \cos[f(z - v_g t)/v_g] = \frac{\sqrt{\pi}}{q} \exp \left[ -\frac{(z - v_g t)^2}{4q^2 v_g^2} \right].$$

*When integrating over $\omega$ from $-\infty$ to $+\infty$ also the non-physical (non-causal) components could contribute, that travel backwards in space[17,14]. But their actual contribution is totally negligible, since the weight-functions $W_n$ are strongly localized in the vicinity of the $\omega_n$-values (which are all positive: see, e.g., Fig.2). In any case, one could integrate from 0 to $\infty$ at the price of increasing a little the mathematical complexity: we are preferring the present formalism for simplicity’s sake.
It is rather interesting that our FTESs are related to the X-shaped waves, since eq. (5) has been written in the form (7), where the function $\Psi(\rho, z-Vt)$ is any one of our previous solutions in eq. (1) above, at our choice.

4. – Conclusions

In conclusion, looking for finite total-energy solutions, we have found a Gaussian envelope that travels with a subluminal velocity $v = c \cos \theta$. However, inside it, we have got a train of pulses travelling Superluminally (with $V = c^2/v = c/\cos \theta$). And we can control the number of pulses inside the envelope just by varying the value of $q$.

We have actually shown that, if we choose the $\omega_n$-values as in Fig. 2, all the small-bandwidth packets centered in the $\omega_n$’s will have the same phase-velocity $V > c$ and therefore the same group-velocity $v_g < c$ [since for metallic waveguides the quantities $K_n^2 = \omega_n^2/c^2 - \beta^2$ are constant for each mode, and $v_g \equiv \partial \omega / \partial \beta$, so that it is $V v_g = c^2$]. This means that the envelope of solution (5)-(7) moves with slower-than-light speed; the envelope length† $\Delta \ell$ depending on the chosen $\Delta \omega$, and being therefore proportional to $qv_g$. However, inside such an envelope, one has a train of (X-shaped) pulses —having nothing to do with the ordinary carrier wave,‡— travelling with the Superluminal speed $V$.

An interesting point is that we can choose the envelope length so that it contains only one (X-shaped wave) peak. Even if the (global) speed of the envelope is subluminal in the finite-energy case, while Superluminal speeds are met only locally (internally), nevertheless in the present case the Superluminal speed $V = c^2/v_g$ of such a “single” pulse might be regarded as the actual velocity of the wave. In order to have just one peak inside the envelope, the envelope length is to be chosen smaller than the distance between two successive peaks of the (infinite total energy) train (1). The amplitude of such a single X-shaped pulse (which remains confined inside the envelope boundary) first increases, and afterwards decreases, while travelling; till when it practically disappears. While the considered pulse tends to vanish on the right (i.e., under the right tail of the envelope), a second pulse starts to be created on the left; and so on [from eq. (7) it is clear,

---

†One may call “envelope length” the distance between the two points in which the envelope height is, for instance, 10% of its maximum height.

‡Actually, they can be regarded as a sum of carrier waves.
in fact, that our finite-energy solutions are nothing but an (infinite-energy) solution of the type in eq.(1), multiplied by a Gaussian function]. We illustrate such a behaviour in Figs.3, namely, in the set of eight figures from Fig.3a to Fig.3h. We have depicted a similar behaviour in the last set of figures of ref.[13], when studying the case of a co-axial guide.

Let us finally remark that similar considerations could be extended to all the situations where a waveguide supports several modes. Tests at microwave frequencies, for instance, should be rather easy to perform; by contrast, experiments in the optical domain are made difficult, at present, by the limited extension of the spectral windows corresponding to not too large attenuations: we shall discuss this point elsewhere. It is rather interesting that our FTESs are related to the X-shaped waves, since in eq.(7) the function $\Psi(\rho, z - Vt)$ is any one of our previous solutions in eq.(1) above, at our choice.

Acknowledgements

The authors are very grateful to Hugo E.Hernández-Figueroa and K.Z.Nóbrega (Fac. of Electric Engineering, Unicamp), and to Amr Shaarawi (Cairo University). Useful discussions are moreover acknowledged with T.F.Arecchi and C.Dartora, as well as with V.Abate, A.Attiya, F.Bassani, C.Becchi, M.Brambilla, C.Cocca, R.Collina, R.Colombi, C.Conti, G.Costa, P.Cotta-Ramusino, G.Degli Antoni, R.Garavaglia, L.C.Kretly, G.Kurizki, D.Mugnai, G.Privitera, V.Petrillo, A.Ranfagni, A.Salanti, G.Salesi, J.W.Swart, M.T.Vasconcelos and M.Villa.
Figure Captions

**Fig.1** – This figure depicts one of our (infinite total-energy) localized solutions, given in eq.(1). It consists, as expected, in a *train* of X-shaped waves; and propagates rigidly along the considered, normal-sized circular waveguide, with radius $r = 5\, \text{cm}$, with the Superluminal speed $v = c/\cos \theta$. In this figure the chosen value of $\theta$ is $\theta = \pi/3$.

**Fig.2** – Dispersion curves for the symmetrical TM$_{0l}$ modes in a perfect cyclindrical waveguide, and location of the frequencies whose corresponding modes have equal phase-velocity. See the text.

**Figs.3** – This set of figures depicts one of our *finite* total-energy localized solutions, given in eqs.(5) and (7). Indeed, they show the time evolution of a finite total energy solution: Choosing $q = 2.041 \times 10^{-10}$ and $\theta = \pi/3$ (and normalized units), there is only *one* X-shaped pulse inside the subluminal envelope: see the text. The pulse and envelope velocities are given by $V = 1/cos \theta$ and $v_g = 1/V$. The (global) speed $v$ of the envelope is therefore subluminal in the finite-energy case, while Superluminal speeds $V$ are met only *locally* (internally). Nevertheless in the present case the Superluminal speed $V = c^2/v_g$ of such a “single” pulse could be regarded as the actual velocity of the wave. Figures (a), (b), (c), (d), (e), (f), (g) and (h) show a complete cycle of the pulse; they correspond to the time instants written at the top of each of them.
References


