Lochlainn O’Raifeartaigh, Fluids, and Noncommuting Fields

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Abstract

Lochlainn O’Raifeartaigh and his work are recalled; the connection between fluid mechanics – his last research topic – and noncommuting gauge fields is explained.

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Lochlainn O’Raifeartaigh was an eminent theoretical/mathematical physicist, the most recent member of a cohort of outstanding Irish physicists that includes Hamilton, Stokes, Father Callan (the inventor of the induction coil), Fitzgerald, Walton, Synge, and others. I came to know him in the 1970s during his visits to my colleague the mathematician Irving Segal. Of course, I already knew his famous work from the 1960s, wherein he put a stop to a search for a group-theoretical combination of internal and kinematical relativistic symmetries. At that time, particle physicists were impressed by Wigner’s successful combination of spin SU(2) and isospin SU(2) into an SU(4) of nuclear physics, and they tried to find something similar for the quark model. The obvious choice was an SU(6) built from spin and Gell-Mann’s flavor SU(3), but this was a nonrelativistic construction and the search was on for a relativistic version. Various authors pointed out various difficulties with the idea, but the definitive paper was Lochlainn’s Physical Review Letter in which he proved that unacceptable mass degeneracies follow when the putative invariance group contains internal symmetries and Poincaré symmetries, mixed in a nontrivial way. This story is documented in Dyson’s collection Symmetry Groups in Nuclear and Particle Physics.

Although more definitive no-go theorems were later constructed, serious research on “relativistic SU(6)” stopped with the publication of “O’Raifeartaigh’s theorem”. But the desire of physicists to combine Poincaré and internal symmetries did not stop, and this is an interesting example of how the force of physics, which derives from Nature, overcomes mathematical obstacles, which are put up by human ingenuity! Of course, I am referring to supersymmetry, which evades all no-go theorems and does combine internal with space-time symmetries through the simple expedient of mixing bosons and fermions – something that does not happen for transformations belonging to ordinary symmetry groups. Again, Lochlainn contributed decisively to this newly evolved idea. Supersymmetry still entails unacceptable mass degeneracies, between bosons and fermions. In his most-cited paper on the “O’Raifeartaigh mechanism”, he constructed simple examples in which supersymmetry is spontaneously broken, and the unwanted mass-degeneracies are removed.

In later years Lochlainn wrote extensively on historical and pedagogical topics, centered on gauge theories. I recommend especially his two books: Group Structure of Gauge Theories and The Dawning of Gauge Theory, and two articles: “Gauge Theory: The Gentle Revolution” and “Gauge Theory: Historical Origins and Some Modern Developments”, the last with Straumann. These works reflect his life-long desire to teach, clarify, and explain physics intricacies, and this is also seen from the vast number of schools, symposia, and meetings to which he contributed – I counted twenty-five in the last decade. Both the research and the teaching were fittingly recognized by the Wigner medal, which he received just before he died.

Lochlainn and I never collaborated on actual research, but our interests paralleled each other to a significant degree. Both he and I explored anomalies; we both wrote
on effective potentials; he worked extensively on monopoles, and I determined the quantum-mechanical implications of these and other classical solutions. His last published paper with Sreedhar\textsuperscript{9} concerns fluid mechanics, and I too have recently been studying this topic because of its unexpected connections to extended objects in field theory and to noncommutative gauge theories.\textsuperscript{10} Lochlainn did not have the opportunity to work on noncommuting gauge theories, but I suspect that he would have liked to, because that subject fits so well with everything that he did before. So I shall conclude my talk by informing you about the relevance of fluids to noncommuting gauge fields.

The suggestion that configuration-space coordinates may not commute

\[ [x^i, x^j] = i\theta^{ij} \]

where \( \theta^{ij} \) is a constant, anti-symmetric two-index object, has arisen recently from string theory, but in fact it has a longer history. Like many interesting quantum-mechanical ideas, it was first suggested by Heisenberg, in the late 1930s, who reasoned that coordinate noncommutativity would entail a coordinate uncertainty and would ameliorate short-distance singularities, which beset quantum fields. He told his idea to Peierls, who eventually made use of it when analyzing electronic systems in an external magnetic field, so strong that projection to the lowest Landau level is justified. After this projection, the coordinates fail to commute (since the state space has been truncated).\textsuperscript{11} But this phenomenological realization of Heisenberg’s idea did not address issues in fundamental science, so Peierls told Pauli about it, who in turn told Oppenheimer, who asked his student Snyder to work it out and this led to the first published paper on the subject.\textsuperscript{12} Today’s string-theory origins of noncommutativity are very similar to Peierls’s application – both rely on the presence of a strong background field.

When confronting the noncommutativity postulate (1), it is natural to ask which (infinitesimal) coordinate transformations

\[ \delta x^i = -f^i(x) \]

leave (1) unchanged. The answer is that the (infinitesimal) transformation vector function \( f^i(x) \) must be determined by a scalar through the expression\textsuperscript{13}

\[ f^i(x) = \theta^{ij}\partial_j f(x) . \]

Since \( \partial_j f^i(x) = 0 \), these are recognized as volume-preserving transformations. (They do not exhaust all volume preserving transformations, except in two dimensions. In dimensions greater two, (3) defines a subgroup of volume-preserving transforms that also leave \( \theta^{ij} \) invariant.)

The volume-preserving transformations form the link between noncommuting coordinates and fluid mechanics. Since the theory of fluid mechanics is not widely
known outside the circle of fluid mechanicians, let me put down some relevant facts. 
There are two, physically equivalent descriptions of fluid motion: One is the Lagrange 
formulation, wherein the fluid elements are labeled, first by a discreet index \( n \): \( X_n(t) \) 
is the position as a function of time of the \( n \)th fluid element. Then one passes to a 
continuous labeling variable \( n \to x : X(t, x) \), and \( x \) may be taken to be the position 
of the fluid element at initial time \( X(0, x) = x \). This is a comoving description. 
Because labels can be arbitrarily rearranged, without affecting physical content, the 
continuum description is invariant against volume-preserving transformations of \( x \), 
and in particular, it is invariant against the specific volume-preserving 
transformations (3), provided the fluid coordinate \( X \) transforms as a scalar:

\[
\delta f X = f^i(x) \frac{\partial}{\partial x^i} X = \theta^{ij} \partial_i X \partial_j f .
\]

The common invariance of Lagrange fluids and of noncommuting coordinates is a 
strong hint of a connection between the two.

Formula (4) will take a very suggestive form when we rewrite it in terms of a 
bracket defined for functions of \( x \) by

\[
\{ O_1(x), O_2(x) \} = \theta^{ij} \partial_i O_1(x) \partial_j O_2(x) .
\]

Note that with this bracket we have

\[
\{ x^i, x^j \} = \theta^{ij} .
\]

So we can think of bracket relations as classical precursors of commutators for a 
noncommutative field theory – the latter obtained from the former by replacing 
brackets by \(-i\) times commutators, à la Dirac. More specifically, the noncommuting 
field theory that emerges from the Lagrange fluid is a noncommuting \( U(1) \) gauge 
theory.

This happens when the following steps are taken. We define the evolving portion 
of \( X \) by

\[
X^i(t, x) = x^i + \theta^{ij} \hat{A}_j(t, x) .
\]

(It is assumed that \( \theta^{ij} \) has an inverse.) Then (4) is equivalent to the suggestive 
expression

\[
\delta f \hat{A}_i = \partial_i f + \{ \hat{A}_i, f \} .
\]

When the bracket is replaced by \((-i\) times the commutator, this is precisely the 
gauge transformation for a noncommuting \( U(1) \) gauge potential \( \hat{A}_i \). Moreover, the 
gauge field \( \hat{F}_{ij} \) emerges from the bracket of two Lagrange coordinates

\[
\{ X^i, X^j \} = \theta^{ij} + \theta^{im} \theta^{jn} \hat{F}_{mn}
\]

\[
\hat{F}_{mn} = \partial_m \hat{A}_n - \partial_n \hat{A}_m + \{ \hat{A}_m, \hat{A}_n \} .
\]
Again (10) is recognized from the analogous formula in noncommuting gauge theory.

What can one learn from the parallelism of the formalism for a Lagrange fluid and a noncommuting gauge field? One result that has been obtained addresses the question of what is a gauge field’s covariant response to a coordinate transformation. This question can be put already for commuting, non-Abelian gauge fields, where conventionally the response is given in terms of a Lie derivative $L_f$:

$$\delta_f x^\mu = - f^\mu(x)$$
$$\delta_f A_\mu = L_f A_\mu \equiv f^{\alpha} \partial_\alpha A_\mu + \partial_\mu f^{\alpha} A_\alpha .$$

But this implies

$$\delta_f F_{\mu\nu} = L_f F_{\mu\nu} \equiv f^{\alpha} \partial_\alpha F_{\mu\nu} + \partial_\mu f^{\alpha} F_{\alpha\nu} + \partial_\nu f^{\alpha} F_{\mu\alpha}$$

which is not covariant since the derivative in the first term on the right is not the covariant one. The cure in this, commuting, situation has been given some time ago:\textsuperscript{14} Observe that (12) may be equivalently presented as

$$\delta_f A_\mu = L_f A_\mu = f^{\alpha} \left( \partial_\alpha A_\mu - \partial_\mu A_\alpha - i[A_\alpha, A_\mu] \right)$$
$$+ f^{\alpha} \partial_\mu A_\alpha - i[A_\mu, f^{\alpha} A_\alpha] + \partial_\mu f^{\alpha} A_\alpha$$
$$= f^{\alpha} F_{\alpha\mu} + D_\mu (f^{\alpha} A_\alpha) .$$

Thus, if the coordinate transformation generated by $f^{\alpha}$ is supplemented by a gauge transformation generated by $- f^{\alpha} A_\alpha$, the result is a gauge covariant coordinate transformation

$$\delta_f A_\mu = f^{\alpha} F_{\alpha\mu}$$

and the modified response of $F_{\mu\nu}$ involves the gauge-covariant Lie derivative $L'_f$:

$$\delta_f' F_{\mu\nu} = L'_f F_{\mu\nu} \equiv f^{\alpha} D_\alpha F_{\mu\nu} + \partial_\mu f^{\alpha} F_{\alpha\nu} + \partial_\nu f^{\alpha} F_{\mu\alpha} .$$

In the noncommuting situation, loss of covariance in the ordinary Lie derivative is even greater, because in general the coordinate transformation functions $f^{\alpha}$ do not commute with the fields $A_\mu, F_{\mu\nu}$; moreover, multiplication of $x$-dependent quantities is not a covariant operation. All these issues can be addressed and resolved by considering them in the fluid mechanical context, at least, for volume-preserving diffeomorphisms. The analysis is technical and I refer you to the published papers,\textsuperscript{13,15}

Instead, I shall discuss the Seiberg-Witten map,\textsuperscript{16} which can be made very transparent by the fluid analogy. The Seiberg-Witten map replaces the noncommuting vector potential $\hat{A}_\mu$ by a nonlocal function of a commuting potential $a_\mu$ and of $\theta$; i.e., the former is viewed as a function of the latter. The relationship between the two follows from the requirement of stability against gauge
transformations: a noncommuting gauge transformation of the noncommuting gauge potential should be equivalent to a commuting gauge transformation on the commuting vector potential on which the noncommuting potential depends. Moreover, when the action and the equations of motion of the noncommuting theory are transformed into commuting variables, the dynamical content is preserved: the physics described by noncommuting variables is equivalently described by the commuting variables, albeit in a complicated, nonlocal fashion.

The Seiberg-Witten map is intrinsically interesting in the unexpected equivalence that it establishes. Moreover, it is practically useful for the following reason. It is difficult to extract gauge invariant content from a noncommuting gauge theory because quantities constructed locally from \( \hat{F}_{\mu\nu} \) are not gauge invariant; to achieve gauge invariance, one must integrate over space-time. Yet for physical analysis one wants local quantities: profiles of propagating waves, etc. Such local quantities can be extracted in a gauge invariant manner from the physically equivalent, Seiberg-Witten mapped commutative gauge theory.\(^{17}\)

Let me now use the fluid analogy to obtain an explicit formula for the Seiberg-Witten map; actually, we shall present the inverse map, expressing commuting fields in terms of noncommuting ones. For our development we must refer to a second, alternative formulation of fluid mechanics, the so-called Euler formulation. This is not a comoving description, rather the experimenter observes the fluid density \( \rho \) and velocity \( \mathbf{v} \) at given point in space-time \((t, \mathbf{r})\). The current is \( \rho \mathbf{v} \) and satisfies with \( \rho \) a continuity equation

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) = 0 .
\]  

(17)

The theory is completed by positing an “Euler equation” for \( \partial \mathbf{v}/\partial t \), but we shall not need this here.

Of interest to us is the relation between the Lagrange description and the Euler description. This is given by the formulas

\[
\rho(t, \mathbf{r}) = \int dx \, \delta(X(t, x) - \mathbf{r})
\]  

(18a)

\[
\rho(t, \mathbf{r})\mathbf{v}(t, \mathbf{r}) \equiv \mathbf{j}(t, \mathbf{r}) = \int dx \, \frac{\partial}{\partial t} X(t, x) \delta(X(t, x) - \mathbf{r}) .
\]  

(18b)

(The integration and the \( \delta \)-function carry the dimensionality of space.) Observe that the continuity equation (17) follows from the definitions (18), which can be summarized by

\[
j^{\mu}(t, \mathbf{r}) = \int dr \, \frac{\partial}{\partial t} X^{\mu} \delta(X - \mathbf{r})
\]  

(19)

\[
X^{0} = t
\]

\[
\partial_{\mu} j^{\mu} = 0 .
\]  

(20)
The (inverse) Seiberg-Witten map, for the case of two spatial dimensions, can be extracted from (19), (20).\textsuperscript{13} (The argument can be generalized to arbitrary dimensions, but there it is more complicated.\textsuperscript{13}) Observe that the right side of (19) depends on $\hat{A}$ through $X$ [see (7)]. It is easy to check that the integral (19) is invariant under the transformations (4): equivalently viewed as a function of $\hat{A}$, it is gauge invariant [see (8)]. Owing to the conservation of $j^\mu$ [see (20)], its dual $\varepsilon_{\alpha\beta\mu}j^\mu$ satisfies a conventional, commuting Bianchi identity, and therefore can be written as the curl of an Abelian vector potential $a_\alpha$, apart from proportionality and additive constants:

\[
\partial_\alpha a_\beta - \partial_\beta a_\alpha + \text{constant} \propto \varepsilon_{\alpha\beta\mu} \int \mathrm{d}x \frac{\partial}{\partial t} X^\mu \delta(X - r) \\
\partial_i a_j - \partial_j a_i + \text{constant} \propto \varepsilon_{ij} \int \mathrm{d}x \delta(X - r) = \varepsilon_{ij}\rho.
\]

This is the (inverse) Seiberg-Witten map, relating the $a$ to $\hat{A}$.

Thus far operator noncommutativity has not been taken into account. To do so, we must provide an ordering for the $\delta$-function depending on the operator $X^i = x^i + \theta^{ij} \hat{A}_j$. This we do with the Weyl prescription by Fourier transforming. The final operator version of equation (21), restricted to the two-dimensional spatial components, reads

\[
\int \mathrm{d}r e^{ik \cdot r} (\partial_i a_j - \partial_j a_i) = -\varepsilon^{ij} \left[ \int \mathrm{d}x e^{ik \cdot X} - (2\pi)^2 \delta(k) \right].
\]

Here the additive and proportionality constants are determined by requiring agreement for weak noncommuting fields, and the integral on the right is interpreted as a trace over the operators.

Formula (22) has previously appeared in a direct analysis of the Seiberg-Witten relation.\textsuperscript{18} Here we recognize it as the (quantized) expression relating Lagrange and Euler formulations for fluid mechanics.

I think Lochlann would have liked this.

References


