Asymptotically embedded defects

Nathan F. Lepora*
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For many cases, the conditions to fully embed a classical solution of one field theory within a larger theory cannot be met. Instead, we find it useful to embed only the solution’s asymptotic fields as this relaxes the embedding constraints. Such asymptotically embedded defects have a simple classification that can be used to construct classical solutions in general field theories.


1. Introduction

Embedded defects [1] are a useful way of describing classical solutions to spontaneously broken Yang-Mills theories. The idea is to take a defect in one symmetry breaking and then embed this breaking in that of the full theory:

\[ G \to H \]

\[ \bigcup G_{emb} \to H_{emb}. \]

(1)

Then, if certain conditions are met, the embedded defect solves the full field equations.

A convenient feature of such embedded defects is that their construction allows the degeneracy of the solutions to be easily found. Then the embedded classification is complementary to the topological classification [2], where only the existence of the classical solutions is usually clear. Degeneracies of classical solutions are relevant to non-Abelian duality [3], the collective coordinate quantization [4], zero modes and the global properties of gauge transformations [5].

The construction of embedded defects has been closely examined for vortices [6] and monopoles [7]. For both cases, one of the embedding conditions gives a geometric constraint on (1). This constraint is simply stated in terms of the Lie algebras of the two theories.

In addition to this geometric constraint, there is another condition on the fields of the embedding. This condition was shown, in reference [1], to hold when

\[ D(G_{emb})V_{emb} = V_{emb} \]

(2)

with \( D \) the representation and \( V_{emb} \) the embedded vector space of scalar field values. Essentially, (2) constrains the pairs \( (V_{emb}, G_{emb}) \) that define the embedded theory. However, when examining embedded monopoles [7], we found condition (2) to be so constraining that it only holds in the simplest cases. Because of this, we instead considered asymptotically embedded monopoles, which are only embedded at infinity and can behave differently in the core. Such asymptotically embedded monopoles are not constrained by (2).

The aim of this paper is to generalize this idea of asymptotic embedding to other defects and then study how these defects are constrained. Just like embedded monopoles, we find that many defects can only be asymptotically embedded and, then, the problematic condition (2) does not apply.

The plan of this paper is as follows. In section 2, we review the theory of embedded defects, then in section 3 we discuss how the embedding (1) is constrained — generalizing the discussion from references [6,7]. After giving an example of a non-embedded vortex in section 4, we consider asymptotically embedded defects and their properties in section 5. To conclude, we summarize our main results in section 6.

2. Embedded defects

Consider a spontaneously broken Yang-Mills theory with compact gauge group \( G \) and a scalar field \( \Phi \in V \) in the \( D \) representation of \( G \)

\[ \mathcal{L}[\Phi, A^\mu] = -\frac{1}{4}(F_{\mu\nu}^A, F^{\mu\nu}) + \frac{1}{2}(D_\mu\Phi, D^\mu\Phi) - V[\Phi], \]

(3)

\[ F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \]

(4)

\[ D_\mu\Phi = \partial_\mu\Phi + d(A^\mu)\Phi. \]

(5)

In this Lagrangian density, the derived representation \( d(X) \) is defined by \( e^{d(X)} = D(e^X) \) with \( X \) in the Lie algebra \( \mathcal{G} \). Then \( \mathcal{L}[\Phi, A^\mu] \) is invariant under the gauge transformation

\[ \Phi \mapsto D(g)\Phi, \quad A^\mu \mapsto \text{Ad}(g)A^\mu - (\partial^\mu g)g^{-1} \]

(6)

with \( g(x) \in G \) and \( x = (r, t) \).

If the potential \( V[\Phi] \) is minimized at some value \( \Phi_0 \), the residual gauge symmetry is

\[ H = \{ h \in G : D(h)\Phi_0 = \Phi_0 \}, \]

(7)

where \( H \) is the Lie algebra of \( H \). This defines an \( \text{Ad}(H) \)-invariant decomposition of \( \mathcal{G} \) into massless and massive gauge boson generators

\[ \mathcal{G} = \mathcal{H} \oplus \mathcal{M}, \quad \text{Ad}(H)\mathcal{H} \subseteq \mathcal{H}, \quad \text{Ad}(H)\mathcal{M} \subseteq \mathcal{M}. \]

(8)

Here \( \text{Ad} \) refers to the adjoint representation of \( G \) on \( \mathcal{G} \), where \( \text{Ad}(g)X = gXg^{-1} \) and the derived representation is \( \text{ad}(X)Y = [X, Y] \) for \( X, Y \in \mathcal{G} \).
In this paper, we use a coordinate independent notation, which we believe better reflects the geometry behind most of our results \[6,8\]. Then the gauge kinetic term in (3) is defined by an inner-product on \(G\)

\[
(X, Y) = \frac{1}{f_a} \{X, Y\}_a, \quad X, Y \in G_a
\]

(9)

with \(f_a\) a coupling constant and \(\{X, Y\}_a\) a real, \(\text{Ad}(G_a)\)-invariant inner-product on each simple or \(u(1)\) subgroup \(G_a \subseteq G\) \[9\]. Likewise, the scalar kinetic term is defined by a Euclidean inner-product \(\langle \Phi_1, \Phi_2 \rangle\) on \(V\) \[10\].

Then, in this notation, the field equations are

\[
D_\mu D^\mu \Phi = -\frac{\partial V}{\partial \Phi}, \quad D_\mu F^{\mu\nu} = J^\nu,
\]

(10)

where the current and covariant derivative are

\[
\langle J^\nu, Y \rangle = \langle d(Y) \Phi, D^\nu \Phi \rangle - \langle D^\nu \Phi, d(Y) \Phi \rangle, \quad D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}].
\]

(11, 12)

Embedded defects \[1\] are \((2 - k)\)-dimensional defects that remain solutions when embedded into a larger gauge theory (so \(k = 0, 1, 2\) for domain walls, vortices and monopoles). They are defined by an embedding of their gauge symmetry breaking in that of the full theory

\[
G \rightarrow H
\]

\[
\cup \cup
\]

\[
G_{\text{emb}} \rightarrow H_{\text{emb}}.
\]

(13)

Such embedded defects have fields fully embedded over their space-time domain

\[
\Phi_{\text{emb}}(r, t) \in V_{\text{emb}}, \quad A^{\mu}_{\text{emb}}(r, t) \in G_{\text{emb}}, \quad r \in \mathbb{R}^{1+k},
\]

(14)

where \(V_{\text{emb}} \subseteq V\) is a vector subspace. Note that it will often be useful to embed topological defects, which have \(\pi_k(G_{\text{emb}}/H_{\text{emb}}) \neq 0\).

In reference \[1\] another condition was given for a defect to be embedded:

\[
D(G_{\text{emb}})V_{\text{emb}} = V_{\text{emb}},
\]

(15)

because the constraint in (b), below, is then satisfied. Later in this paper we find this condition to be so constraining that it only holds for the simplest cases. This motivates our discussion of asymptotically embedded defects, for which condition (15) is relaxed.

An embedded defect is a solution of the full theory if the field equations reduce to consistent field equations on its embedded theory \[1\]. This gives four constraints from the two field equations in (10):

(a) The current, found from \(\Phi_{\text{emb}}\) and \(A^{\mu}_{\text{emb}}\), satisfies

\[
\langle J^\nu, G_{\text{emb}}^\perp \rangle = 0,
\]

(16)

where \(G = G_{\text{emb}} \oplus G_{\text{emb}}^\perp\). This constrains how \(G_{\text{emb}} \subseteq G\).

(b) The kinetic scalar term satisfies

\[
\langle D_\mu D^\mu \Phi_{\text{emb}}, V_{\text{emb}}^\perp \rangle = 0,
\]

(17)

where \(V = V_{\text{emb}} \oplus V_{\text{emb}}^\perp\). As mentioned above, this always holds when (15) is satisfied \[1\].

(c) The scalar potential, found from \(\Phi_{\text{emb}}\), satisfies

\[
\frac{\partial V}{\partial \Phi} V_{\text{emb}}^\perp = 0.
\]

(18)

This constrains the potential — for example, it holds in the BPS limit.

(d) The gauge kinetic terms satisfies

\[
\langle D_\mu F^{\mu\nu}, G_{\text{emb}}^\perp \rangle = 0.
\]

(19)

This always holds by algebraic closure of \(G_{\text{emb}}\) \[1\].

3. Constraining the spectrum of embedded defects

The condition \(\langle J^\nu, G_{\text{emb}}^\perp \rangle = 0\) in (a) above has been shown to constrain the spectrum of embedded vortices \[6\] and embedded monopoles \[7\]. Here we give a general proof of this constraint for embedded defects.

The following proof relies on reducing \(\mathcal{M}\) into irreducible subspaces under the adjoint action of \(H\). These correspond to irreducible representations of \(H\) on \(\mathcal{M}\):

\[
\mathcal{M} = M_1 \oplus \cdots \oplus M_n, \quad \text{Ad}(H)M_a \subseteq M_a.
\]

(20)

Physically, each \(M_a\) defines a gauge multiplet of massive gauge bosons (for example, the W- and Z-bosons in electroweak theory).

To examine condition (16), we firstly rewrite (13) as

\[
\mathcal{G} = \mathcal{H} \oplus \mathcal{M}
\]

\[
\cup \cup \cup
\]

\[
\mathcal{G}_{\text{emb}} = \mathcal{H}_{\text{emb}} \oplus \mathcal{N}
\]

(21)

and split \(\mathcal{N}\) into its \(\text{Ad}(H_{\text{emb}})\)-irreducible subspaces

\[
\mathcal{N} = N_1 \oplus \cdots \oplus N_m, \quad \text{Ad}(H_{\text{emb}})N_a \subseteq N_a
\]

(22)

in a similar way to \(\mathcal{M}\) in (20). It will also be useful to consider an embedded defect with scalar field generated by the action of \(G\) upon \(\Phi_0\) \[11\]

\[
\Phi_{\text{emb}}(x) = h(x)D[g(x)]\Phi_0, \quad g(x) \in G
\]

(23)

with \(h(x)\) a function of \(x = (r, t)\). Consequently, a gauge transformation (6) with group element \(g^{-1}(x)\) takes the embedded defect to a unitary gauge \[13\]

\[
\Phi_{\text{emb}} = h(x)\Phi_0, \quad A^{\mu}_{\text{emb}} = C^{\mu}_{\text{emb}}(x) + W^{\mu}_{\text{emb}}(x),
\]

(24)

where the gauge field naturally separates into massless \(C^{\mu}_{\text{emb}} \in \mathcal{H}_{\text{emb}}\) and massive \(W^{\mu}_{\text{emb}} \in \mathcal{N}\) parts.
Then a substitution of $J^\nu$ from (11) into (16) constrains the ansatz (24) by

$$(d(G_{emb}^+)\Phi_0, D^\nu \Phi) = \langle D^\nu \Phi, d(G_{emb}^+)\Phi \rangle = 0.$$  \hspace{1cm} (25)

Now, in a unitary gauge

$$D^\nu \Phi = (\partial^\nu h)\Phi_0 + h d(W^{\nu}_{emb})\Phi_0.$$  \hspace{1cm} (26)

Using this and noting $\langle d(G)\Phi_0, \Phi_0 \rangle = 0$, we see that (25) holds when

$$\langle d(G_{emb}^+)\Phi_0, d(N)\Phi_0 \rangle = \langle d(N)\Phi_0, d(G_{emb}^+)\Phi_0 \rangle = 0,$$  \hspace{1cm} (27)

which algebraically constrains the embedding (21).

This algebraic constraint (27) is expressed more simply by applying the following result [6]:

$$\langle d(X_\alpha)\Phi_0, d(Y_\alpha)\Phi_0 \rangle = \lambda_\alpha \lambda_\beta \langle X_\alpha, Y_\beta \rangle,$$

$$\lambda_\alpha = \frac{\|d(X_\alpha)\Phi_0\|}{\|X_\alpha\|}, \quad X_\alpha \in M_a, Y_\beta \in M_b,$$  \hspace{1cm} (28)

which says the representation respects orthogonality between the irreducible subspaces $M_a$ in (20). Therefore when $\lambda_\alpha \neq \lambda_\beta$, we have

$$N_c \subseteq M_a$$  \hspace{1cm} (29)

with $N_c$ a subspace from (22). If $\lambda_\alpha = \lambda_\beta$, the embedding can also be between gauge families (see reference [6]).

When $N$ is irreducible under $Ad(H_{emb})$, so (22) is trivial, the constraint (29) becomes more simply $N \subseteq M_a$. This is the case with embedded vortices and embedded monopoles, as we now discuss.

For embedded vortices, which are Nielsen-Olesen vortices [12] embedded like (13), the embedding is [11]

$$G \rightarrow H$$  \hspace{1cm} (30)

$$u(1) \rightarrow 0$$

with $U(1) = \exp(X \phi)$ and $X \in M$. In the radial gauge, the embedded ansatz (14) is

$$\Phi_{emb}(r, \phi) = f(r) D(e^{X \phi})\Phi_0,$$  \hspace{1cm} (31)

$$A_{emb}(r, \phi) = \frac{g(r)}{r} X \hat{\phi}.$$  \hspace{1cm} (32)

In the unitary gauge, this ansatz becomes [13]

$$\Phi_{emb}(r, \phi) = f(r)\Phi_0,$$  \hspace{1cm} (33)

$$A_{emb}(r, \phi) = \frac{g(r) + 1}{r} X \hat{\phi}.$$  \hspace{1cm} (34)

Then (29) constrains the vortex's embedding by [6]

$$N \subseteq M_a, \quad N = \mathbb{R}X$$  \hspace{1cm} (35)

with $M_a$ an irreducible subspace from (20).

For embedded monopoles, which are 't Hooft-Polyakov monopoles [14] embedded like (13), the embedding is

$$G \rightarrow H$$  \hspace{1cm} (36)

$$u(1) \rightarrow 0$$

with $su(2)$ basis $[t_a, t_b] = \epsilon_{abc} t_c$ and $U(1) = \exp(t_3 \theta)$. In the radial gauge, the embedded ansatz (14) is

$$\Phi(r) = \frac{H}{r} D(g(r))\Phi_0,$$  \hspace{1cm} (37)

$$A(r) = -A_D t_3 - \frac{K-1}{r} \epsilon_{iab}\hat{\theta}_a t_b,$$  \hspace{1cm} (38)

where $g(r) = e^{r \epsilon \tau_3} e^{\eta \tau_2} e^{\varphi \tau_1}$ in spherical polars. In the unitary gauge, this ansatz becomes [13,15]

$$\Phi(r) = \frac{H}{r} \Phi_0, \quad A(r) = -A_D t_3 - \frac{K}{r} \hat{n}_s t_s$$  \hspace{1cm} (39)

with $A_D = \hat{\phi}(1 - \cos \theta)/r \sin \theta$ the Dirac gauge potential ($\nabla \cdot A_D = 0, \nabla \wedge A_D = \hat{r}/r^2$) and

$$\hat{n}_1 = \sin \varphi \hat{\theta} + \cos \varphi \hat{\phi}, \quad \hat{n}_2 = -\cos \varphi \hat{\theta} + \sin \varphi \hat{\phi}$$  \hspace{1cm} (40)

are two orthonormal unit-vectors orthogonal to $\hat{r}$. Then (29) constrains the monopole's embedding by [7]

$$N \subseteq M_a, \quad N = \mathbb{R}t_1 + \mathbb{R}t_2,$$  \hspace{1cm} (41)

where $M_a$ is an irreducible subspace from (20).

4. An illustrative example: vortices in $SU(3) \rightarrow U(2)$

Let us consider a spontaneously broken $SU(3)$ Yang-Mills theory with scalar field in the adjoint representation. This theory should give a typical example of a non-Abelian gauge theory in which to construct embedded defects.

It will be convenient to define the following diagonal generators

$$T_1 = i \text{diag}(1,1,-2), \quad T_2 = i \text{diag}(1,-1,0).$$  \hspace{1cm} (42)

Then a vacuum $\Phi_0 = vT_1$ gives a residual gauge symmetry $H = U(2)$ that satisfies $Ad(H)\Phi_0 = \Phi_0$ with Lie algebra

$$H = \begin{pmatrix}
\text{su}(2) & \cdots \\
\cdots & \cdots \\
0 & \cdots
\end{pmatrix} \oplus \text{u}(1)T_1, \quad \text{u}(1)T_1 = \mathbb{R}T_1.$$  \hspace{1cm} (43)

This algebra defines a split $\text{su}(3) = H \oplus M$, where

$$M = \{X(v) : v \in \mathbb{C}^2\}, \quad X(v) = \begin{pmatrix}
0_2 & \cdots & v \\
\cdots & \cdots & \cdots \\
-v^\dagger & \cdots & 0
\end{pmatrix}$$  \hspace{1cm} (44)

with $M$ irreducible under $Ad(H)$. 

3
Using the methods in section 2, we consider an embedded vortex ansatz in a radial gauge

$$\Phi_{\text{emb}} = f(r) \text{Ad}(e^{\partial X_1}) \Phi_0, \quad A_{\text{emb}} = \frac{g(r)}{r} X_1 \hat{\theta}$$

(42)

with generator $X_1 = X \left( \frac{1}{\sqrt{3}} e_1 \right)$, where $e_1 = (1, 0)$. Then $\mathcal{V}_{\text{emb}}$, the vector space of $\Phi_{\text{emb}}$ values, is constructed by expanding

$$\text{Ad}(e^{\partial X_1}) \Phi_0 = \Phi_0 + \partial \text{Ad}(X_1) \Phi_0 + \frac{1}{2} \partial^2 \text{ad}(X_1) \text{ad}(X_1) \Phi_0 + \cdots$$

(43)

In this expansion

$$\text{ad}(X_1) \Phi_0 = -3vY_1, \quad \text{ad}(X_1) \text{ad}(X_1) \Phi_0 = -9vX_1,$$

(44)

where $Y_1 = X \left( \frac{1}{\sqrt{3}} e_1 \right)$. Therefore $\mathcal{V}_{\text{emb}}$ is spanned by at least three independent generators $T_1, X_1, Y_1$.

However, (42) is supposed to be an embedded vortex, for which $\mathcal{V}_{\text{emb}}$ should be isomorphic to either $\mathbb{C}$ or $\mathbb{R}^2$ in a radial gauge. Therefore we conclude that (42) is not an embedded vortex.

Another way to see that (42) is not an embedded vortex is by calculating

$$\text{Ad}(e^{\partial X_1}) T_1 = \left( \frac{1}{4} \cos \theta + \frac{1}{2} \right) T_1$$

$$+ \left( \frac{1}{4} \cos \theta - \frac{3}{2} \right) T_2 - \frac{1}{2} \sin \theta Y_1.$$ 

(45)

Evidently, this forms a complicated curve that is not contained within a two-dimensional vector space.

5. Asymptotically embedded defects

Essentially, the vortex in section 4 is not embedded because the condition $D(G_{\text{emb}}) \mathcal{V}_{\text{emb}} = \mathcal{V}_{\text{emb}}$ does not hold for any $\mathcal{V}_{\text{emb}} \cong \mathbb{R}^2$. However, the simple classification in section 3 still applies, so let us see if we can generalize the definition of an embedded defect in a useful way.

For this definition, we consider an asymptotically embedded defect to have asymptotic fields

$$\Phi_{\text{asy}} \sim \Phi_{\text{emb}}(x) \in \mathcal{V}_{\text{emb}}, \quad A_{\text{asy}}^\mu \sim A_{\text{emb}}^\mu(x) \in G_{\text{emb}}.$$ 

(46)

These are asymptotically similar to an embedded defect, but can differ from that form elsewhere. We also suppose that the scalar field is like (23), so in a unitary gauge

$$\Phi_{\text{asy}} \sim h(x) \Phi_0, \quad A_{\text{asy}}^\mu \sim C_{\text{emb}}^\mu(x) + W_{\text{emb}}^\mu(x)$$

(47)

with $h(x) \to 1$ as $r \to \infty$.

To simplify matters, we take $h = h(r)$ to be a radial function and suppose the gauge field satisfies

$$\partial_\mu W_{\text{emb}}^\mu = 0, \quad \partial^i W_{\text{emb}}^i = 0.$$ 

(48)

All of the above conditions are motivated by the properties of embedded vortices and embedded monopoles.

Now, we examine condition (b) in section 2, which requires $\langle D_\mu D^\nu \Phi_{\text{emb}}, V_{\text{emb}}^\perp \rangle = 0$. In this condition, the second-order covariant derivative is

$$D_\mu D^\nu \Phi_{\text{emb}} = (\nabla^2 h) \Phi_0 + h g_{\mu\nu} d(W_{\text{emb}}^\mu) d(W_{\text{emb}}^\nu) \Phi_0$$

$$+ h d(\partial_\mu W_{\text{emb}}^\mu) / \Phi_0 - \partial^2 h d(W_{\text{emb}}^\mu) / \Phi_0.$$ 

(49)

Since $h = h(r)$ is radial, then $\partial^2 h = h'/r^2$ and so by (47)

$$D_\mu D^\nu \Phi_{\text{emb}} = (\nabla^2 h) \Phi_0 + h g_{\mu\nu} d(W_{\text{emb}}^\mu) d(W_{\text{emb}}^\nu) \Phi_0.$$ 

(50)

Then condition (b) in section 2 holds when

$$g_{\mu\nu} d(W_{\text{emb}}^\mu) d(W_{\text{emb}}^\nu) \Phi_0 \propto \Phi_0.$$ 

(51)

This is very constraining on both the defect’s fields and embedding $N \subset M$.

Therefore when both conditions (29) and (51) hold, there are embedded defect solutions like (14). These solve the full field equations.

Alternatively, if (51) does not hold, the defect may then be asymptotically embedded. This situation happens when the second term of (50) is negligible compared to the first; namely, when asymptotically

$$O(\nabla^2 h_{\text{asy}}) > O(W_{\text{asy}}^2).$$ 

(52)

Whether this is the case depends on the nature of the defect and the parameters of the theory. Such asymptotically embedded defects are constrained only by the analysis in section 3, which requires $N_a \subseteq M_b$.

To illustrate these results, we now consider embedded vortices and embedded monopoles.

For embedded vortices, condition (51) becomes

$$d(X) d(X) \Phi_0 \propto \Phi_0,$$ 

(53)

where $X$ is the vortex generator in (42). This equation is very constraining on $X$ and does not always hold [for example, see equation (44) in section 4]. When (53) does not hold, the vortex could instead be asymptotically embedded; this depends on the scalar and gauge masses in the embedded theory, $m_S$ and $m_V$, that define the asymptotic Nielsen-Olesen profiles [12,16]

$$f(r) - v = \left\{ \begin{array}{ll} O[r^{-1/2} \exp(-m_S r)], & m_S \leq 2m_V, \\ O[r^{-1/2} \exp(-2m_V r)], & m_S \geq 2m_V. \end{array} \right.$$ 

(54)

$$g(r) + 1 = O[r^{1/2} \exp(-m_V r)].$$ 

(55)

Therefore the vortex can be asymptotically embedded when $m_S < 2m_V$.

For embedded monopoles, condition (51) becomes

$$d(t_1) d(t_1) \Phi_0 + d(t_2) d(t_2) \Phi_0 \propto \Phi_0.$$ 

(56)

Again this is very constraining and does not generally hold. When (56) is not the case, the monopole may instead be asymptotically embedded; this depends upon the asymptotic ’t Hooft-Polyakov profiles.
Therefore the monopole can be asymptotically embedded when $m_S < 2m_V$, as found in reference [7].

To finish this section, we discuss a couple of points about asymptotically embedded vortices and monopoles.

The first point is that condition (56) essentially generalizes (53) to two generators. This generalization relates to the property that any monopole embedding contains vortex embeddings:

\[
\begin{align*}
\mathfrak{u}(1) & \subseteq \mathfrak{su}(2) \subseteq \mathcal{G} \\
\subseteq & \mathcal{H} \\
0 & \subseteq \mathfrak{u}(1) \subseteq \mathcal{H}
\end{align*}
\]

For further discussion, we refer to reference [7].

The other point is that conditions (53) and (15) are the same in many situations. For example, the relation $d(X)d(X)\Phi_0 = c^2\Phi_0$ is the same as

\[
D(e^{X\theta})\Phi_0 = \cos \theta \Phi_0 + \sin \theta d(X/c)\Phi_0.
\]

Then (60) means that $\mathcal{V}_{\text{emb}} = \mathbb{R} D(e^{X\theta})\Phi_0$ is a two-dimensional vector space with $D(U(1))\mathcal{V}_{\text{emb}} = \mathcal{V}_{\text{emb}}$. Thus (53) and (15) are equivalent for embedded vortices.

In this paper, however, we find the constraint (53) more convenient because it more clearly relates to the definition of asymptotically embedded defects.

### 6. Conclusions

To conclude, we summarize our results and make a comment.

(a) Our arguments relate to the following decomposition

\[
\mathcal{G} = \mathcal{H} \oplus \mathcal{M}, \quad \mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n
\]

with each $\mathcal{M}_a$ irreducible under $\text{Ad}(H)$.

(b) Then defect embeddings are constructed by

\[
\mathcal{G}_{\text{emb}} = \mathcal{H}_{\text{emb}} \oplus \mathcal{N}
\]

and satisfy a constraint

\[
\mathcal{N}_a \subset \mathcal{M}_b,
\]

where $\mathcal{N} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_m$ under $\text{Ad}(H_{\text{emb}})$.

(c) Whether defects are fully embedded or asymptotically embedded depends on whether the embedded ansatz

\[
\Phi = h(r)\Phi_0, \quad A_{\text{emb}}^\mu = C_{\text{emb}}^\mu + W_{\text{emb}}^\mu
\]

satisfies the field equations everywhere or only asymptotically. The defect is fully embedded when

\[
g_{\mu\nu} d(W_{\text{emb}}^\mu) d(W_{\text{emb}}^\nu) \Phi_0 \propto \Phi_0
\]

(with certain conditions on $W_{\text{emb}}^\mu$); otherwise, the defect is asymptotically embedded when

\[
O(\nabla^2 h_{\text{asy}}) < O(W^2_{\text{asy}}),
\]

which depends on the scalar and gauge masses within the embedded theory, $m_S$ and $m_V$. Such asymptotically embedded defects are constrained only by (b).

(d) For embedded Nielsen-Olesen vortices

\[
\Phi_{\text{emb}} = f(r) \text{Ad}(e^{\theta X_1})\Phi_0, \quad A_{\text{emb}} = g(r)_r X_1 \partial_r,
\]

the vortex embedding is therefore constrained by

\[
\mathcal{N} \in \mathcal{M}_a, \quad \mathcal{N} = \mathbb{R} X.
\]

Furthermore, the vortex is fully embedded when

\[
d(X)d(X)\Phi_0 \propto \Phi_0
\]

and, otherwise, asymptotically embedded if $m_S < 2m_V$.

(e) For embedded ’t Hooft-Polyakov monopoles

\[
\Phi(r) = \frac{H}{r} D(g(r))\Phi_0, \quad A^I(r) = \frac{K - 1}{r} \epsilon_{iab} \hat{x}_b t_a,
\]

the monopole embedding is therefore constrained by

\[
\mathcal{N} \in \mathcal{M}_a, \quad \mathcal{N} = \mathbb{R} t_1 + \mathbb{R} t_2.
\]

Furthermore, the monopole is fully embedded when

\[
d(t_1)d(t_1)\Phi_0 + d(t_2)d(t_2)\Phi_0 \propto \Phi_0
\]

and, otherwise, asymptotically embedded if $m_S < 2m_V$.

(f) For a final comment, we note that the formalism in (a) to (c) is general. Therefore more complicated embeddings can be constructed — for example, electroweak theory in flipped-SU(5)

\[
\text{su}(5) \oplus \text{u}(1) \rightarrow \text{su}(3) \oplus \text{su}(2) \oplus \text{u}(1)
\]

such an embedding could be useful for considering the counterparts of Z-strings, W-strings, sphalerons and Nambu monopoles in a grand unified theory.


[9] For example, $-2 \text{tr} [\text{ad}(X)\text{ad}(Y)]$ or $-\Re(\text{tr} [X^\dagger Y]/\text{tr} [1_a])$ with $1_a \in G_a$ the identity.

[10] We have used same notation for both inner-products because the one referred to should be clear from the context.

[11] Note there are more general ansatzé with solutions not generated from a single value $\Phi_0$ — for example, see reference [16] for a vortex with such an ansatz in $SO(3) \to 1$.


[13] While the field equations and embedding conditions are equivalent in the radial and unitary gauges, care should be taken with the stability analysis. This is because the physical fluctuations are non-singular in the radial gauge, but can be gauge transformed to singular fluctuations in the unitary gauge.


* email address: n_lepora@hotmail.com