It is now well established that gravitational collapse in a self-similar symmetric potential, such as the one used in this work, is quiescent. The collapse of a self-similar symmetric potential, such as the one used in this work, is quiescent. The collapse of a self-similar symmetric potential is quiescent. The collapse of a self-similar symmetric potential,
The reduced action becomes
\[ S = \frac{1}{2G} \int d^2 x \sqrt{-g} \left[ \phi R(g) + V^{(n)}(\phi, \Lambda) \right] \]
\[ = \frac{1}{2G} \int d^2 x \sqrt{-g} \ H^{(n)}(\phi) |\delta \Lambda|^2 \] (6)
where \( G \equiv 2\pi n/(n-1) \), the scalar field has been rescaled (\( \chi \to \chi / \sqrt{G(n)} \)) in order to make it dimensionless, and the overall factor of the \( n \)-sphere volume has been dropped from the action. In addition we have defined
\[ H^{(n)}(\phi) \equiv \frac{8(n-1)}{n} \phi, \]
\[ V^{(n)}(\phi, \Lambda) \equiv \frac{1}{n} \left[ \frac{8(n-1)}{n} \right] \phi^2 + \frac{n^2}{8} \left( \frac{8(n-1)}{n} \right) \frac{\phi^2}{\Lambda} - \frac{1}{2} \phi^2 \Lambda \] (8)

Spacetime dimension information now appears only in the parameter \( n \).

Now with the metric parametrized as
\[ ds^2 = -2g(u, v)\, dt(\phi, v) \, dudv \] (9)
the field equations are
\[ \dot{\phi} = -\frac{1}{2} V^{(n)}(\phi) g \phi' \]
\[ \ddot{\phi} - \dot{g}/2 \]
\[ \ddot{h} = \frac{1}{2} (h - \chi) \left( g \phi V^{(n)} - \bar{g} \right), \] (12)
where prime and dot denote the \( v \) and \( u \) derivatives respectively.

The evolution equations may be put in a form more useful for numerical solution by defining the variable
\[ h = \frac{\phi \chi'}{\phi'}, \]
which replaces the scalar field \( \chi \) by \( h \). The evolution equations become
\[ \dot{\phi} = -\bar{g}/2 \]
\[ \ddot{\phi} = \frac{1}{2} (h - \chi) \left( g \phi V^{(n)} - \bar{g} \right), \]
\[ g = \exp \left[ 4\pi \int_u^v dt' \frac{\dot{\phi}'}{\phi'} (h - \chi)^2 \right], \]
\[ \bar{g} = \int_u^v (g \phi' V^{(n)}) \, dt', \]
\[ \chi = \frac{1}{2} \sqrt{\phi} \int_u^v \frac{\dot{\phi}'}{\sqrt{\phi}}. \] (18)

This is the final form of the equations used for numerical evolution. At this stage the spacetime parameters \( n \) and \( \Lambda \) appear only in the dilaton potential \( V^{(n)} \).

The numerical scheme uses a \( v \) (‘space’) discretization to obtain a set of coupled ODEs:
\[ h(u, v) \to h_i(u), \quad \phi(u, v) \to \phi_i(u). \] (19)
where \( i = 0, \ldots, N \) specifies the \( v \) grid. Initial data for these two functions is prescribed on a constant \( v \) slice, from which the functions \( h_i(u), \phi_i(u) \) are constructed. Evolution in the ‘time’ variable \( u \) is performed using the 4th order Runge-Kutta method. The general scheme is similar to that used in [9], together with some refinements used in [10]. This procedure was also used for the 3-dimensional collapse calculations in [9].

The initial scalar field configuration \( \chi (\phi, u = 0) \) is most conveniently specified as a function of \( \phi \) rather than \( r \). (Recall that \( \phi \propto r^n \).) This together with the initial arrangement of the radial points \( \phi(v, u = 0) \) fixes all other functions. We used the initial specification \( \phi(0, v) = v \).

Most of our computations are for the initial scalar field configurations of the Gaussian form
\[ \chi_G(u = 0, \phi) = a \phi \exp \left[ -\left( \frac{\phi - \phi_0}{\sigma} \right)^2 \right], \]
with attention restricted to variations of the amplitude \( a \). However we also used the “cosh” initial data
\[ \chi_{G}(u = 0, \phi) = a \cosh[b(\phi - \phi_0)]^{-2}, \] (21)
to study convergence of our code and to test universality with non-zero \( \Lambda \).

The initial values of the other functions are determined in terms of the above by computing the integrals for \( y_n \) and \( \tilde{y}_n \) using Simpson’s rule.

The boundary conditions at fixed \( u \) are
\[ \phi_k = 0, \quad \bar{\phi}_k = 0, \quad h_k = 1. \] (22)
where \( k \) is the index corresponding to the position of the origin \( \phi = 0 \). (In the algorithm used, all grid points \( 0 \leq i < k-1 \) correspond to ingoing rays that have reached the origin and are dropped from the grid). These conditions are equivalent to \( r(u, v) = 0, \ g_{r,v} = g(u, v) = 1, \) and guarantee regularity of the metric at \( r = 0 \). Notice that \( \phi \) for our initial data, \( \phi_k \) and hence \( h_k \) are initially zero, and therefore remain zero at the origin because of Eqn. (15).

At each \( u \) step, a check is made to see if an apparent horizon has formed by observing the function
\[ a h \equiv g^{\alpha \beta} \partial_\alpha \partial_\beta \phi = -\frac{\dot{\phi}}{\bar{g}} \] (23)
whose vanishing signals the formation of an apparent horizon. For each run of the code with fixed amplitude \( a \), this function is scanned from larger to smaller radial
values after each Runge-Kutta iteration, and evolution is terminated if the value of this function reaches $10^{-4}$. The corresponding radial coordinate value is recorded as $R_{th}$. In the subcritical case, it is expected that all the radial grid points reach zero without detection of an apparent horizon. This is the signal of pulse reflection. The results $(a, R_{th})$ are collated by seeking a relationship of the form

$$R_{th} \propto (a - a_s)^\gamma.$$ \hfill (24)

The code was tested for grid sizes ranging from 2000 to 20000 points, and with the $u$ and $v$ step sizes ranging from $10^{-2}$ to $10^{-6}$, for the two types of initial data used, as well as the vacuum case of vanishing scalar field. These tests established that the code converges.

For most of the runs we used $\phi_0 = 1$ and $\sigma = 0.3$ for the Gaussian initial data, and $b = 5.0$, $\phi_0 = 0.5$ for the cosh data. We also varied $\phi_0$ in the Gaussian data from 0.1 to 20 to see if there was dependence of $\gamma$ on starting position of the pulse. This is important to study because the cosmological constant sets a scale in the problem.

Our results for the two types of data appear in Tables 1 and 2, which list the computed $\gamma$ values for the corresponding (negative) values of the cosmological constant $\Lambda$. The trend is clear: $\gamma$ does not depend on $\Lambda$. The first value provides a crucial test of our formalism and code since it reproduces the known $\Lambda = 0$ value of the exponent [1].

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.001</td>
<td>0.370-0.375</td>
</tr>
<tr>
<td>-5</td>
<td>0.37-0.38</td>
</tr>
<tr>
<td>-10</td>
<td>0.37-0.39</td>
</tr>
<tr>
<td>-20</td>
<td>0.36-0.38</td>
</tr>
</tbody>
</table>

Table 1. Computed ranges of $\gamma$ for Gaussian data.

<table>
<thead>
<tr>
<th>$\Lambda$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>0.37-0.38</td>
</tr>
<tr>
<td>-20</td>
<td>0.36-0.38</td>
</tr>
<tr>
<td>-50</td>
<td>0.37-0.40</td>
</tr>
</tbody>
</table>

Table 2. Computed ranges of $\gamma$ for cosh data.

The ranges of $\gamma$ in Tables 1 and 2 are arrived at by assessing our uncertainty in the determination of the critical amplitudes $a^\pm$: the $a^\pm$ values are determined to lie within certain domains, and the end points of these domains are used to determine the range of $\gamma$ values. For example, for $\Lambda = -20$ and cosh data, we find $a^\pm$ lies in the range $6.00344 \times 10^{-3} - 6.00360 \times 10^{-3}$, which gives $\gamma$ in the range indicated in Table 2. Our determination of the $a^\pm$ window gets coarser with increasing $\Lambda$, which leads to the larger error bars on the $\gamma$ values, as indicated in the tables.

Figure 1 is the In-In graph of the apparent horizon radius vs initial data amplitude, for $\Lambda = -5$. This is representative of the type of results from which the critical exponent ranges in Tables 1 and 2 are deduced.

Figure 2 shows plots of the apparent horizon function (23) and the scalar field $h$. This is typical of the type of data used to extract horizon radius information at the supercritical onset of black hole formation. These graphs show impending black hole formation at $\phi \sim 6.3 \times 10^{-4}$ ($r \sim 5 \times 10^{-3}$) for a supercritical amplitude with $\Lambda = -5$. We also observed that the dip in the apparent horizon function oscillates toward and away from the origin as it approaches the $\phi$-axis, and appears to be the cause
of the small oscillations about the least squares fit line in Figure 1. Although these latter oscillations have been noted in earlier work, it appears that they are the manifestation of the dip oscillations in the apparent horizon function (23) near criticality. This in turn is connected with the discrete self-similarity of the scalar field, which not surprisingly, is also manifested in the apparent horizon function. Figure 1 contains two complete oscillations, and may be used to obtain a rough estimate of the echoing period: it gives $\Delta \sim 3.5$. This value is close to the $\Lambda = 0$ value $\Delta = 3.44$ computed in earlier work[1, 2, 3]. Thus, both the numbers $\gamma$ and $\Delta$ associated with critical behaviour appear to be independent of $\Lambda$.

It is known, and verified again here, that $\gamma$ is independent of initial data profiles and parameter values. In addition we find that for fixed $\Lambda$, critical exponents are independent of characteristic pulse width and starting location in relation to the $\Lambda$ scale.

The $\Lambda$ independence is a new and perhaps surprising feature, given that $\Lambda$ sets a scale in the problem.

There are however intuitive arguments that suggest this result is reasonable: it is known for perfect fluid collapse with equation of state $P = \rho k$ that critical exponents vary with the parameter $k$ [4], (which is dimensionless in $G = c = 1$ units). The cosmological constant is effectively a perfect fluid with $\rho = \Lambda = -P$, for fixed $k = -1$. Viewed this way, $\Lambda$ is an initial data parameter describing the (constant) density profile, in addition to being a parameter in the equations of motion. Furthermore, had it turned out that $\gamma$ depended on $\Lambda$, we would be presented with the puzzle of explaining why some scales (e.g. pulse width and initial position) are irrelevant but the $\Lambda$ scale is not. Viewing $\Lambda$ as an initial data parameter provides a natural understanding of the results.

It is interesting to contrast this state of affairs with collapse studies of the minimally coupled scalar field of mass $\mu$, which also has a scale in the problem. Here too types of critical behaviour are observed at the threshold of black hole formation, depending on initial pulse width $\sigma$ in comparison with $\mu^{-1}$ [12]: for $\sigma \gtrsim \sigma_0 = \mu^{-1}$ there is a mass gap, whereas for $\sigma \ll \sigma_0$ there is no mass gap and the exponent $\gamma \sim 0.378$ is computed[12].

There are two differences between $\Lambda$ and $\mu$ worth emphasising. The first is specific to our results, and the second is general: (i) We find no evidence of a mass gap for a range of values of pulse width and initial position in relation to the cosmological length scale $1/\sqrt{\Lambda}$. In the absence of a mass gap, the critical exponent is measured in the limit of small horizon values. In our simulations, the typical size of the horizon at the onset of black hole formation ranged between $10^{-2}$ and $10^{-4}$. By contrast, the smallest cosmological length scale that the code can handle is $1/\sqrt{50} \sim 0.14$. Under these circumstances, it is not surprising that the cosmological constant has no effect. (ii) Unlike $\mu$, $\Lambda$ appears in the stress-energy tensor independent of the scalar field. Therefore it is not unreasonable that $\Lambda$ and $\mu$ give qualitatively different results.

For positive $\Lambda$ we are so far not able to obtain accurate results for $\gamma$ because of two competing effects in our procedure: since our grid is evolving, we observe an outward deSitter expansion of the grid, which confines the interesting features of the collapsing collapse to an ever shrinking region near the origin. Nevertheless, on the intuitive ground mentioned above, we expect that the sign of $\Lambda$ will not change our results.

It is known that critical exponents may be computed via a linear perturbation analysis of the critical solution [4, 11]. This has so far been done only for $\Lambda = 0$. For non-zero $\Lambda$, it is reasonable to expect that both the critical solution and its perturbation equation depend on $\Lambda$, which after all is a parameter in the equations of motion. It would be useful to see how $\Lambda$ drops out of this type of calculation, yielding a $\Lambda$ independent exponent.

In summary, our numerical simulations of massless scalar field collapse in spherical symmetry in four dimensions, show that the critical exponent associated with the collapse is independent of (negative) $\Lambda$ values. This result extends the scope of universality to include the cosmological constant, and suggests that including $\Lambda$ with other matter types, such as the perfect fluid, will also not change the critical exponent.

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