

1. INTRODUCTION
recent investigations concentrate mostly on IFS’s acting in the space of pure states \( \mathcal{M} \), we advocate a more general setup, in which IFS’s act in the space of mixed quantum states.

The main issue of this paper is to propose a general definition of quantum iterated function system (QIFSs). Formally, it suffices to consider the standard definition of IFS and to take for \( \Omega \) an \( N \)-dimensional Hilbert space \( \mathcal{H}_N \). Instead of functions \( f_i \), \( i = 1, \ldots, k \), representing classical maps, one should use linear functions \( F_i : \mathcal{H}_N \to \mathcal{H}_N \), which represent the corresponding quantum maps. Alternatively, one may consider the space \( \mathcal{M}_N \) of density matrices of size \( N \) and construct an iterated function system out of \( k \) positive maps \( G_i : \mathcal{M}_N \to \mathcal{M}_N \). The QIFSs defined in this way can be used to describe processes of quantum measurements, decoherence, and dissipation. Moreover, QIFSs offer an attractive field of research on the semiclassical limit of quantum random systems. In particular, it is interesting to explore quantum analogues of classical IFSs, which lead to fractal invariant measures, and to investigate, how do quantum effects smear fractal structures out.

This paper is organized as follows. In the next section we recall the definition and basic properties of the classical IFSs, and discuss several examples. In Sec. III we propose the definition of QIFSs, investigate their properties, and relate them to the notion of quantum channels and complete positive maps used in the theory of quantum dynamical semigroups. The quantum-classical correspondence is a subject of Sec. IV, in which we compare dynamics of exemplary IFSs and the related QIFSs. Concluding remarks are presented in Sec. V.

II. CLASSICAL ITERATED FUNCTION SYSTEMS

Consider a compact metric space \( \Omega \) and \( k \) functions \( f_i : \Omega \to \Omega \), where \( i = 1, \ldots, k \). Let us specify \( k \) probability functions \( p_i : \Omega \to [0, 1] \) such that for each point \( x \in \Omega \) the condition \( \sum_{i=1}^{k} p_i(x) = 1 \) is fulfilled. Then the functions \( f_i \) may be regarded as classical maps, which act randomly with probabilities \( p_i \). The set \( \mathcal{F}_C := \{ \Omega, f_i, p_i : i = 1, \ldots, k \} \) is called an iterated function system (IFS).

Let \( \mathcal{M}(\Omega) \) denotes the space of all probability measures on \( \Omega \). The IFS \( \mathcal{F}_C \) generates the following Markov operator \( P \) acting on \( \mathcal{M}(\Omega) \)

\[
(P\mu)(B) = \sum_{i=1}^{k} \int_{f_i^{-1}(B)} p_i(\lambda) d\mu(\lambda),
\]

where \( B \) is a measurable subset of \( \Omega \) and a measure \( \mu \) belongs to \( \mathcal{M}(\Omega) \). This operator represents the corresponding Markov stochastic process defined on the code space consisting of \( k \) letters which label each map \( f_i \). On the other hand, \( P \) describes the evolution of probability measures under the action of \( \mathcal{F}_C \).

Consider an IFS defined on an interval in \( \mathbb{R} \) and consisting of invertible \( C^1 \) maps \( \{ f_i : i = 1, \ldots, k \} \). This IFS generates the associated Markov operator \( P \) on the space of densities \( \mathcal{D}^+ \), which describes one step evolution of a classical density \( \gamma \)

\[
P\gamma(x) = \sum_{i} p_i(f_i^{-1}(x)) \gamma(f_i^{-1}(x)) \left( \frac{df_i^{-1}(x)}{dx} \right),
\]

where for \( x \in \Omega \) the sum goes over \( i = 1, \ldots, k \), such that \( x \in f_i(\Omega) \).

Let \( d(x, y) \) denotes the distance between two points \( x \) and \( y \) in the metric space \( \Omega \). An IFS \( \mathcal{F}_C \) is called hyperbolic, if it fulfills the following conditions for all \( i = 1, \ldots, k \):

(i) \( f_i \) are Lipschitz functions with the Lipschitz constants \( L_i < 1 \), i.e., they satisfy the contraction condition \( d(f_i(x), f_i(y)) \leq L_i d(x, y) \) for all \( x, y \in \Omega \);

(ii) the probabilities \( p_i \) are Hölder continuous, i.e., they fulfill the condition \( |p_i(x) - p_i(y)| \leq K \cdot (x, y) \alpha \) for some \( \alpha \in (0, 1) \), \( K_i \in \mathbb{R}^+ \) for all \( x, y \in \Omega \);

(iii) all probabilities are positive, i.e., \( p_i(x) > 0 \) for any \( x \in \Omega \).

The Markov operator \( P \) associated with a hyperbolic IFS has a unique invariant probability measure \( \mu \), satisfying the equation \( P\mu = \mu \). This measure is attractive, i.e., \( P^n \mu \) converges weakly to \( \mu \), for every \( \mu \in \mathcal{M}(\Omega) \) as \( n \to \infty \). In other words, \( \int_{\Omega} d(P^n \mu) \) tends to \( \int_{\Omega} d\mu \) for every continuous function \( u : \Omega \to \mathbb{R} \). Let us mention that the hyperbolicity conditions (i)-(iii) are not necessary to assure the existence of a unique invariant probability measure - some other, less restrictive, sufficient assumptions were analyzed in [2].

Observe, that in the above case, in order to obtain the exact value of an integral \( \int_{\Omega} d(P^n \mu) \), it is sufficient to find the limit of the sequence \( \int_{\Omega} d(P^n \mu) \) for an arbitrary initial measure \( \mu \). This method of computing integrals over the invariant measure \( \mu \), is purely deterministic. Sometimes it is possible to perform the integration over the invariant measure analytically, even though \( \mu \) displays fractal properties \( \mathcal{F} \). Alternatively a random iterated algorithm may be employed by generating a random sequence \( x_j \in \Omega \) by the IFS, \( j = 0, 1, \ldots \), which originates from an arbitrary
initial point \( x_0 \). Due to the ergodic theorem for IFSs, the mean value \( \frac{1}{n} \sum_{j=0}^{n-1} u(x_j) \) converges with probability one in the limit \( n \to \infty \) to the desired integral \( \int_{A} u \, d\mu \).

If probabilities \( p_i \) are constant we say that an IFS is of the first kind. Such IFSs are often studied in the mathematical literature (see \( \text{ref} \) and references therein). Moreover they have also some applications in physics. For example, they were used to construct multifractal energy spectra of certain quantum systems, and to investigate second order phase transitions. On the other hand, with some classical and quantum dynamical systems IFSs with place-dependent probabilities can be associated. By analogy with the position-dependent gauge transformations such IFSs may be called \textit{iterated function systems of the second kind}.

If \( \Omega \) is a compact subset of \( \mathbb{R}^n \), while \( d_R(x, y) \) represents the Euclidean distance, or \( \Omega \) is a compact manifold (e.g. sphere \( S^2 \) or torus \( T^2 \)) equipped with the natural (Riemannian) distance \( d_R \), then an IFS will be called \textit{classical}. For concreteness we provide below some examples of classical IFSs. The first example demonstrates that even simple linear maps \( f_i \) may lead to a nontrivial structure of the invariant measure.

\textbf{Example 1.} \( \Omega \subseteq [0, 1], k = 2, p_1 = p_2 = 1/2 \) and two affine transformations are given by \( f_1(x) = x/3 \) and \( f_2(x) = x/3 + 2/3 \) for \( x \in \Omega \). Since both functions are continuous contractions with Lipschitz constants \( L_1 = L_2 = 1/3 < 1 \), this IFS is hyperbolic. Thus there exists a unique attracting invariant measure \( \mu_* \). It is easy to show that \( \mu_* \) is concentrated uniformly on the Cantor set of the fractal dimension \( d = 2 \ln 2/ \ln 3 \).

The next example presents an IFS of the second kind.

\textbf{Example 2.} As before, \( \Omega = [0, 1], k = 2, f_1(x) = x/3 \) and \( f_2(x) = x/3 + 2/3 \) for \( x \in \Omega \). The probabilities are now place dependent, \( p_1(x) = x \) and \( p_2(x) = 1 - x \). Although this IFS is not hyperbolic (condition (iii) is not fulfilled), a unique invariant measure \( \mu_* \) still exists. It is also concentrated on the Cantor set, but now in a non-uniform way.

The measure \( \mu_* \) displays in this case multifractal properties, since its generalized dimension depends on the Rényi parameter.

\textbf{Example 3.} \( \Omega = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2, k = 4, p_1 = p_2 = p_3 = p_4 = 1/4 \). Four affine transformations are given by

\[
\begin{align*}
    f_1 \left( \begin{array}{c} x \\ y \end{array} \right) &= \left( \begin{array}{cc} 1/3 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right), \\
    f_2 \left( \begin{array}{c} x \\ y \end{array} \right) &= \left( \begin{array}{cc} 1/3 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} 2/3 \\ 0 \end{array} \right), \\
    f_3 \left( \begin{array}{c} x \\ y \end{array} \right) &= \left( \begin{array}{cc} 1 & 0 \\ 0 & 1/3 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right), \\
    f_4 \left( \begin{array}{c} x \\ y \end{array} \right) &= \left( \begin{array}{cc} 1 & 0 \\ 0 & 1/3 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} 0 \\ 2/3 \end{array} \right). 
\end{align*}
\]

(3)

Also this IFS is not hyperbolic, since the transformations \( f_i \) are not globally contracting, the former two contract along \( x \)-axis, while the latter two along \( y \)-axis only. An invariant measure \( \mu_* \) for this IFS is presented in Fig. 1.

The support of \( \mu_* \) covers the Cartesian product of two Cantor sets. Thus, its fractal dimension is \( d = 2 \ln 2/ \ln 3 \).

\textbf{Example 4.} Let \( \Omega = S^2 \). Take \( k = 2, p_1 = p_2 = 1/2 \), and choose \( f_1 \) to be the rotation along \( z \)-axis by angle \( \chi_1 \) (later referred to as \( R_z(\chi_1) \)). In the standard spherical coordinates, \( f_1(\theta, \phi) = (\theta, \phi + \chi_1) \). The second function \( f_2 \) is a rotation by angle \( \chi_2 \) along an axis inclined by angle \( \beta \) with respect to \( z \)-axis. Since both classical maps are isometries this IFS is by no means hyperbolic. The properties of the Markov operator depend on the angle \( \beta \), and the commensurability of the angles \( \chi_i \). However, the Lebesgue measure on the sphere is always an invariant measure for this IFS.

\textbf{Example 5.} \( \Omega = [0, 1], k = 2, p_1 = p_2 = 1/2, f_1(x) = 2x \) for \( x < 1/2 \) and \( f_1(x) = 2(1 - x) \) for \( x \geq 1/2 \) (tent map); \( f_2(x) = 2x \) for \( x < 1/2 \) and \( f_2(x) = 2x - 1 \) for \( x \geq 1/2 \) (Bernoulli map). Both classical maps are expanding (and chaotic), thus the IFS is not hyperbolic. The Lebesgue measure in \([0, 1]\) is a unique absolutely continuous invariant measure \( \mu_* \) for this IFS.

### III. Quantum Iterated Function Systems

#### A. Pure states QIFSs

To describe a quantum dynamical system we consider a complex Hilbert space \( \mathcal{H} \). When the corresponding classical phase space \( \Omega \) is compact, the Hilbert space \( \mathcal{H}_N \) is finite dimensional and its dimension \( N \) is inversely proportional to the Planck constant \( \hbar \) measured in the units of the volume of \( \Omega \). Analyzing quantum systems, \( N \) is usually treated as a free parameter, and the semiclassical limit is studied by letting \( N \to \infty \).

A quantum state can be described by an element \( |\psi\rangle \) of \( \mathcal{H}_N \) normalized according to \( \langle \psi | \psi \rangle = 1 \). Since for any phase \( \alpha \) the element \( |\psi\rangle = e^{i\alpha} |\psi\rangle \) describes the same physical state as \( |\psi\rangle \), we identify them, and so the space of all
pure states $\mathcal{P}_N$ has $2N - 2$ real dimensions. From the topological point of view, it can be represented as the complex projective space $\mathbb{C}P^{N-1}$ equipped with the Fubini–Study (FS) distance
\[
D_{FS}(|\phi\rangle, |\psi\rangle) = \arccos |\langle \phi |\psi \rangle|.
\]
It varies from zero for $|\phi\rangle = |\psi\rangle$ to $\pi/2$ for any two orthogonal states. In the simplest case of a two-dimensional Hilbert space $\mathcal{H}_2$ the space of pure states $\mathcal{P}_2$ reduces to the Bloch sphere, $\mathbb{C}P^1 \simeq S^2$, and the FS distance between any two quantum states equals to the natural (Riemannian) distance between the corresponding points on the sphere of radius $1/2$.

**Definition 1.** To define a (pure states) quantum iterated function system (QIFS) it is sufficient to use the general definition of IFS revolved in Sect. II taking for $\Omega$ the space $\mathcal{P}_N$. We specify two sets of $k$ linear invertible operators:

- $V_i : \mathcal{H}_N \rightarrow \mathcal{H}_N (i = 1, \ldots, k)$, which generates maps $F_i : \mathcal{P}_N \rightarrow \mathcal{P}_N (i = 1, \ldots, k)$ by
\[
F_i(|\phi\rangle) := \frac{V_i(|\phi\rangle)}{||V_i(|\phi\rangle)||}.
\]

for any $|\phi\rangle \in \mathcal{P}_N$, and

- $W_i : \mathcal{H}_N \rightarrow \mathcal{H}_N (i = 1, \ldots, k)$, forming an operational resolution of identity, $\sum_{i=1}^{k} W_i^2 W_i = \mathbb{I}$, which generates probabilities $p_i : \mathcal{P}_N \rightarrow [0, 1]$ ($i = 1, \ldots, k$) by
\[
p_i(|\phi\rangle) := ||W_i(|\phi\rangle)||^2.
\]

Clearly, for any $|\phi\rangle \in \mathcal{P}_N$ the normalization condition $\sum_{i=1}^{k} p_i(|\phi\rangle) = 1$ is fulfilled. In this situation a QIFS may be defined as a set
\[
\mathcal{F}_N = \{ \mathcal{P}_N ; \ F_i : \mathcal{P}_N \rightarrow \mathcal{P}_N ; \ p_i : \mathcal{P}_N \rightarrow [0, 1] : i = 1, \ldots, k \}.
\]

Such a QIFS may be realized by choosing an initial state $|\phi_0\rangle \in \mathcal{P}_N$ and generating randomly a sequence of pure states $|\phi_i\rangle \in \mathcal{P}_N$. The state $|\phi_0\rangle$ is transformed into $|\phi_1\rangle = F_i(|\phi_0\rangle)$ with probability $p_i(|\phi_0\rangle)$, later $|\phi_1\rangle$ is mapped into $|\phi_2\rangle = F_j(|\phi_1\rangle)$ with probability $p_j(|\phi_1\rangle)$, and so on. If we choose $W_i = \sqrt{p_i}\mathbb{I}$, then the probabilities are constant: $p_i(|\phi\rangle) = p_i$ for $i = 1, \ldots,k$. An arbitrary QIFS $\mathcal{F}_N$ determines by formula $\mathbb{M}$ the operator $P$ acting on probability measures on $\mathcal{P}_N$.

The general theorem concerning invariant measures for hyperbolic IFS allows us to formulate the following result as its special case:

**Proposition 1.** If a QIFS is hyperbolic (i.e. the quantum maps $F_i$ are contractions with respect to the Fubini–Study distance in $\mathcal{P}_N$ and the probabilities $p_i$ are Hölder continuous and positive), then there exists a unique invariant measure $\mu$ for the associated operator $P$ on $\mathcal{M}(\mathcal{P}_N)$.

**Example 6.** $\Omega = \mathcal{P}_N \simeq \mathbb{C}P^{N-1}$, $k = 2$, $p_1 = p_2 = 1/2$, $F_1(|\psi\rangle) = U_1(|\psi\rangle)$ and $F_2(|\psi\rangle) = U_2(|\psi\rangle)$, where the operators $U_i$ ($i = 1, 2$) are unitary. This QIFS is not hyperbolic, since both quantum maps are isometries. Thus the natural Riemannian (Fubini–Study) measure in $\mathcal{P}_N$ is invariant, but as we shall see in the next section, its uniqueness depends on the choice of $U_1$ and $U_2$.

**B. Mixed states QIFSs**

Mixed states are described by $N$–dimensional density operators $\rho$, i.e., positive Hermitian operators acting in $\mathcal{H}_N$ with trace normalized to unity, $\rho = \rho^\dagger$, $\rho \geq 0$ and $tr\rho = 1$. They may be represented (in a non-unique way) as a convex combination of projectors. We shall denote the space of density operators by $\mathcal{M}_N$.

**Definition 2.** Now we can formulate the general definition of a mixed states quantum iterated function system (QIFS) as a set
\[
\mathcal{F}_N := \{ \mathcal{M}_N ; \ G_i : \mathcal{M}_N \rightarrow \mathcal{M}_N ; \ p_i : \mathcal{M}_N \rightarrow [0, 1] : i = 1, \ldots, k \}.
\]

where the maps $G_i$, $i = 1, \ldots, k$ transform density operators into density operators, and for every density operator $\rho \in \mathcal{M}_N$ the probabilities are normalized, i.e., $\sum_{i=1}^{k} p_i(\rho) = 1$. 
The above definition of QIFS is more general than the previous one, since in particular \( G_i \) and \( p_i \) may be defined by

\[
G_i(\rho) = \frac{V_i \rho V_i^\dagger}{\text{tr} \left( V_i \rho V_i^\dagger \right)}
\]  

(9)

and

\[
p_i(\rho) = \text{tr} \left( W_i \rho W_i^\dagger \right)
\]

(10)

for \( i = 1, \ldots, k \) and \( \rho \in \mathcal{M}_N \), where the linear maps \( V_i \) and \( W_i \) are as in Definition 1. Thus, each QIFS on \( \mathcal{P}_N \) can be extended to a QIFS on \( \mathcal{M}_N \). Note that in this case \( p_i(\rho) = \text{tr}(W_i^\dagger W_i) \). Hence we can alternatively define the probabilities by \( p_i(\rho) = \text{tr} (\mathcal{L}_i \rho) \) (\( i = 1, \ldots, k \), \( \rho \in \mathcal{M}_N \)), where the linear operators \( \mathcal{L}_i \) are Hermitian, positive, and fulfill the identity \( \sum_{i=1}^k \mathcal{L}_i = 1 \).

Now the dynamics takes place in the convex body all density matrices \( \mathcal{M}_N \). In contrast to the \((2N-2)\)-dimensional space of pure states \( \mathcal{P}_N \), the space of mixed states \( \mathcal{M}_N \) has \( N^2 - 1 \) real dimensions. For \( N = 2 \) it is just the 3-dimensional Bloch ball, i.e., the volume bounded by the Bloch sphere.

The special class of QIFSs is a class of homogenous QIFSs introduced in more general setting by one of the authors [17]. A QIFS is called homogenous if both \( p_i \) and \( G_i \cdot p_i \) are affine maps for \( i = 1, \ldots, k \). The mixed states QIFS being a generalization of a pure state QIFS and defined by formulas \( \text{(9.1)} \) and \( \text{(9.2)} \) is homogenous if \( W_i = V_i \) for \( i = 1, \ldots, k \). Beautiful examples of such systems acting on the Bloch sphere where recently analyzed by Jadczyk and Öberg [17].

A homogenous QIFS \( p_i \) and \( G_i \) may be interpreted in terms of a discrete measurement process as the probability that the measurement outcome is \( i \), and the state of the system after the measurement if the result was actually \( i \), respectively.

A homogenous QIFS generates not only the Markov operator \( P \) acting in the space of probability measures on \( \mathcal{M}_N \), but also the linear, trace-preserving, and positive operator \( \Lambda : \mathcal{M}_N \rightarrow \mathcal{M}_N \) defined by

\[
\Lambda(\rho) = \sum_{i=1}^k p_i(\rho) G_i(\rho)
\]

(11)

for \( \rho \in \mathcal{M}_N \).

A mixed state \( \tilde{\rho} \) is \( \Lambda \)-invariant if and only if it is the barycenter of some \( P \)-invariant measure \( \tilde{\mu} \), i.e.,

\[
\tilde{\rho} = \int_{\mathcal{M}_N} \rho d\tilde{\mu}(\rho),
\]

(12)

see [27].

**Example 7.** \( \Omega = \mathcal{M}_N, k = 2, p_1 = p_2 = 1/2, G_1(\rho) = U_1 \rho U_1^\dagger \) and \( G_2(\rho) = U_2 \rho U_2^\dagger \). This is just Example 6 in other casting; the normalized identity matrix, \( \rho_* = 1/N \) is \( \Lambda \)-invariant irrespectively of the form of unitary operators \( U_i, i = 1, 2 \). Note that \( \tilde{\rho} = \rho_* \) may be represented as \( \text{[40]} \), where the measure \( \tilde{\mu} \), uniformly spread over \( \mathcal{P}_N \) (the Fubini-Study measure), is \( P \)-invariant.

To define hyperbolic QIFSs one needs to specify a distance in the space of mixed quantum states. There exist several different metrics in \( \mathcal{M}_N \), which may be applicable (see e.g. [28] and references therein). The standard distances: the Hilbert-Schmidt distance

\[
D_{\text{HS}}(\rho_1, \rho_2) = \sqrt{\text{tr}[\rho_1 - \rho_2]^2},
\]

(13)

the trace distance

\[
D_{\text{tr}}(\rho_1, \rho_2) = \sqrt{\text{tr}[\rho_1 - \rho_2]^2} = ||\rho_1 - \rho_2||_{\text{tr}},
\]

(14)

and the Bures distance [31]

\[
D_{\text{Bures}}(\rho_1, \rho_2) = \sqrt{2(1 - \text{tr}(\rho_1^{1/2} \rho_2 \rho_1^{1/2})^{1/2})}
\]

(15)

the latter based on the idea of purification of mixed quantum states [31] [32], are mutually bounded [33]. They generate the same natural topology in \( \mathcal{M}_N \). Having endowed the space of mixed state with a metric, we may formulate
immediate conclusion from the theorem on hyperbolic IFSs. We define a hyperbolic QIFS as in the previous section, and clearly an analogue of Proposition 1 holds.

Proposition 2. If a QIFS $\Omega$ is homogenous and hyperbolic (that is the quantum maps $G_i$ are contractions with respect to one of the standard distances in $\mathcal{M}_N$), $p_i$ are Hölder continuous and positive), the associated Markov operator $P$ possesses a unique invariant measure $\mu$. This invariant measure determines a unique $\Lambda$-invariant mixed state $\rho \in \mathcal{M}_N$ given by (16).

Note that for a homogenous hyperbolic QIFS, the sequence $\Lambda^n(p_0)$ tends in the limit $n \to \infty$ to a unique invariant state $\rho$ irrespectively of the choice of an initial state $p_0$ (16).

Example 8. $\Omega = \mathcal{M}_N$, $k = 2, p_1 = p_2 = 1/2$. $G_1(p) = (p + 2p_1)/3$ and $G_2(p) = (p + 2p_2)/3$, where we choose the both projectors $p_1 = |1\rangle \langle 1|$ and $p_2 = |2\rangle \langle 2|$ to be orthogonal. Since both homotheties $G_i$ are contractions (with the Lipschitz constants 1/3) this QIFS is hyperbolic and a unique invariant measure $\mu$ exists. By analogy with the IFS discussed in Example 1 we see that a support of $\mu$ covers the Cantor set at the line joining both projectors $p_1$ and $p_2$. However, this is nothing else as a rather sophisticated representation of the maximally mixed two-level state $\rho_* = (p_1 + p_2)/2$, which follows from the symmetry of the Cantor set and may be formally verified by performing the integration prescribed by (16).

C. Completely positive maps and unitary QIFSs

From the mathematical point of view it may be sufficient to require that the map $\Lambda$ is positive, that is, it transforms a positive operator into another positive operator. From the physical point of view it is desirable to require a stronger condition of complete positivity related to a possible coupling of the quantum system under consideration with an environment. A map $\Lambda$ is completely positive (CP-map), if the extended map $\Lambda \otimes I$ is positive for any extension of the initial Hilbert space, $\mathcal{H}_N \to \mathcal{H}_N \otimes \mathcal{H}_E$, which describes coupling to the environment (15).

It is well known that each trace preserving CP-map $\Lambda$ (sometimes called quantum channel), can be represented (non uniquely) in the following Stinespring-Kraus form

$$\rho' = \Lambda_\kappa(p) = \sum_{j=1}^k V_j \rho V_j^\dagger, \quad \text{with} \quad \sum_{j=1}^k V_j V_j^\dagger = I,$$

where linear operators $V_j$ ($j = 1, \ldots, k$) are called Kraus operators (16). For any quantum channel acting in an $N$-dimensional Hilbert space the number of operators $k$ needs not exceed $N^2$ (15). It is clear that each quantum channel can be treated as a pure or mixed states homogenous QIFS.

If, additionally, $\sum_{j=1}^k V_j V_j^\dagger = I$ holds, then $\Lambda_\kappa(I/N) = I/N$, and the map $\Lambda$ is called unital. It is the case if all Kraus operators are normal, $V_j V_j^\dagger = V_j^\dagger V_j$ ($j = 1, \ldots, k$), however, this condition is not necessary. A unital trace preserving CP-map is called bistochastic. An example of a bistochastic channel is given by random external fields (15) defined by

$$\rho' = \Lambda_U(p) = \sum_{i=1}^k p_i U_i \rho U_i^\dagger,$$

where $U_i, i = 1, 2, \ldots, k$ are unitary operators and the vector of non-negative probabilities is normalized, i.e., $\sum_{i=1}^k p_i = 1$. The Stinespring-Kraus form may be regarded as a homogenous QIFS of the first kind (with constant probabilities) with $k$ unitary maps $G_i(p) = U_i \rho U_i^\dagger$ ($i = 1, \ldots, k$). In particular, Example 7 belongs to this class. In the sequel such QIFSs will be called unitary. For a unitary QIFS not only $\rho_*$ is an invariant state of $\Lambda_U$, but also the measure $\delta_{\rho_*}$ is invariant for the operator $P_U$ induced by this QIFS.

Although a unitary QIFS consists of isometries, the operator $\Lambda_U$ needs not preserve the standard distances between any two mixed states. For the Hilbert-Schmidt we have

$$D_{\text{HS}}(\Lambda_U(p_1), \Lambda_U(p_2)) \leq D_{\text{HS}}(p_1, p_2).$$

In fact this statement is true for any bistochastic channels as shown by Uhlmann (15), but it is false for arbitrary CP maps, since the Hilbert-Schmidt metric is not monotone (16). On the other hand, $\Lambda_U$ is a contraction in a sense of the Bures distance (Riemannian) and the trace distance (not Riemannian), which are monotone and do not grow under the action of any CP map (15). Choosing for $p_2$ the maximally mixed state $\rho_* = I/N$, which is invariant with respect to $\Lambda_U$ for any unitary QIFS, we see in particular that the distance of any state $p_1$ to $\rho_*$ does not increase in
time. Similarly, the von Neumann entropy, \( H(\rho) = \text{tr}(\rho \ln \rho) \), does not decrease during the time evolution \( \rho(t) \). On the other hand, the inequality in \cite{1} is weak, and in some cases the distance may remain constant. The question, under which conditions this inequality is strong is related to the problem, for which unitary QIFSs the maximally mixed state \( \rho_s \) is a unique invariant state of \( \Lambda_U \). This is not the case, if all operators \( U_i \) commute, since then all density matrices diagonal in the eigenbase of \( U_i \) are invariant. Such a situation may occur also in subspaces of smaller dimension. To describe such a case we shall call unitary matrices of the same size common block-diagonal, if they are block-diagonal in the same basis and with the same blocks. The uniqueness of the invariant state of a unitary QIFS is then characterized by the following proposition, the proof of which is provided in the appendix A.

**Proposition 3.** Assume that all probabilities \( \rho_i \) \( (i = 1, \ldots, k) \) are strictly positive. Then the maximally mixed state \( \rho_s \) is not a unique invariant state for the operator \( \Lambda_U \) if and only if unitary operators \( U_i \) \( (i = 1, \ldots, k) \) are common block-diagonal.

It follows from the proof of this proposition that in this case there exists \( \rho \neq \rho_s \) such that \( \delta_{\rho} \) is an invariant measure for the operator \( \rho(t) \) induced by the QIFS.

To show an application of Proposition 3 consider a two level quantum system, called *qubit*, which may be used to carry a piece of quantum information. Let us assume it is subjected to a random noise, described by the following map:

\[
\rho \rightarrow \rho' = \Lambda_U(\rho) = (1 - p)\rho + \frac{p}{3} \left( \sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2 + \sigma_3 \rho \sigma_3 \right).
\]

(19)

This bistochastic map, defined by the unitary Pauli matrices \( \sigma_i \), is called depolarizing quantum channel \cite{2}, and the parameter \( p \) plays the role of the probability of error. This map transforms any vector inside the Bloch ball toward the center, so the length of the polarization vector decreases. In formalism of QIFSs this quantum channel is equivalent to:

**Example 9.** \( \Omega = \mathbb{P}_2, k = 4, U_1 = \mathbb{1}, U_2 = \sigma_x, U_3 = \sigma_y, U_4 = \sigma_z, p_1 = 1 - p \) and \( p_2 = p_3 = p_4 = p > 0 \). Since the Pauli matrices are not common block-diagonal, the maximally mixed state \( \rho_s \) is a unique invariant state of the operator \( \Lambda_U \) associated with this unitary QIFS.

To introduce an example of QIFS arising from atomic physics consider a two level atom in a constant magnetic field \( B_z \) subjected to a sequence of resonant pulses of electromagnetic wave. The length of each wave pulse is equal to its period \( T \) and it interacts with the atom by the periodic Hamiltonian \( V(t) = V(t + T) \). Let us assume that each pulse occurs randomly with probability \( p \). Thus the evolution operator transforms any initial pure state

\[
U_1 = \exp(-iH_0T/\hbar)
\]

(20)
in the absence of the pulse, or by the operator

\[
U_2 = C \exp \left[-\frac{i}{\hbar} \left( H_0T + \int_0^T V(t)dt \right) \right]
\]

(21)
in the presence of the pulse. The unperturbed Hamiltonian \( H_0 \) is proportional to \( B_z J_z \) \( (J_z \) is \( z \) component of the angular momentum operator) and \( C \) denotes the chronological operator. Thus this random system may be described by

**Example 10.** \( \Omega = \mathbb{P}_2, k = 2, p_1 = 1 - p \) and \( p_2 = p \), the Floquet operators \( U_1 \) \cite{3} and \( U_2 \) \cite{4} as specified above. The maximally mixed state \( \rho_s = \mathbb{1}/2 \), corresponding to the center of the Bloch ball, is the invariant state of the system. In the case of a generic perturbation \( V \), the matrices \( U_1 \) and \( U_2 \) are not common block-diagonal, and so \( \rho_s \) is the unique invariant state.

The QIFSs arise in a natural way if considering a quantum system acting on \( \mathcal{H}_N \) coupled with an ancilla: a state in an auxiliary \( m \)-dimensional Hilbert space \( \mathcal{H}_m \), which describes the environment. Initially, the composite state describing the system and the environment is in the product form, \( \sigma = \rho_A \otimes \rho_B \), where \( \rho_B = \mathbb{1}/m \) is the maximally mixed state, but the global unitary evolution couples two subsystems together. A unitary matrix \( U \) of size \( N \times m \) acting on the tensor space \( \mathcal{H}_N \otimes \mathcal{H}_m \) may be represented in its Schmidt decomposition form as \( U = \sum_{i=1}^K \sqrt{q_i} V^A_i \otimes V^B_i \), where the number of terms is determined by the size of the smaller space, \( K = \min\{N^2, m^2\} \); the operators \( V^A_i \) and \( V^B_i \) act on \( \mathcal{H}_N \) and \( \mathcal{H}_m \) respectively, and the Schmidt coefficients are normalized as \( \sum_{i=1}^K q_i = 1 \). Restricting our attention to the system \( A \) one needs to trace out the variables of the environment \( B \) which gives the following QIFS:

\[
\rho_A = \Lambda(\rho_A) = \text{tr}_B(U \sigma U^\dagger) = \sum_{i=1}^K q_i V^A_i \rho_A V^A_i \dagger.
\]

(22)

Since \( \Lambda(\rho_A) = \text{tr}_B(U (\rho_A \otimes \rho_B^B) U^\dagger) = \rho_A^B \), the CP-map \( \Lambda \) is bistochastic.
IV. QUANTUM–CLASSICAL CORRESPONDENCE

To investigate various aspects of the semiclassical limit of the quantum theory it is interesting to compare a given discrete classical dynamical system generated by $f : \Omega \rightarrow \Omega$ with a family of the corresponding quantum maps, usually defined as $F_N : \mathcal{H}_N \rightarrow \mathcal{H}_N$ with an integer $N$. Several alternative methods of quantization of classical maps in compact phase space have been applied to construct quantum maps corresponding to baker map on the torus and Arnold cat map, and other automorphisms on the torus, periodically kicked top and baker map on the sphere.

To specify in which manner the classical and the quantum maps are related, it is convenient to introduce a set of coherent states $|y\rangle \in \mathcal{H}_N$, defined for each classical point $y$ of the phase space $\Omega$. (For more properties of coherent states and a general definition consult the book of Perelomov.) They satisfy the resolution of identity formula:

$$\int_{\Omega} |y\rangle \langle y| dy = \mathbb{I},$$

and allow us to represent any state $\rho$ by its Husimi representation, $H(y) = \langle y| \rho |y\rangle$. Quantization of a classical map $f$, which leads to a family of quantum maps $F_N$ is called regular, if for almost all classical points $x$ the classical and the quantum images are connected in the sense that the normalized Husimi distribution of the state $F_N|y\rangle$ integrated over a finite vicinity of the point $f(y)$ tends to unity in the limit $N \rightarrow \infty$. Another method of linking a classical map with a family of quantum maps is based on the Egorov property, which relates the classical and the quantum expectation values.

In a similar way we may construct QIFSs related to certain classical IFSs. More precisely, a sequence of pure states $QIFS \mathcal{F}_N = \{P_N; F_{i,N}, R_{i,N} : i = 1, \ldots, k\}$ induced by two sets of linear maps $V_{i,N}, W_{i,N} : \mathcal{H}_N \rightarrow \mathcal{H}_N (i = 1, \ldots, k)$ (see [1] and [2]) is a quantization of a classical IFS $\mathcal{F}_C = \{\Omega; F_i, R_i : i = 1, \ldots, k\}$, when:

- the functions $F_{i,N}$ are quantum maps obtained by quantization of the classical maps $f_i$;
- the probabilities $p_{i,N}$ computed at coherent states $|y\rangle$ fulfill

$$p_{i,N}(|y\rangle) = \|W_{i,N}(|y\rangle)\|^2 \underset{N \rightarrow \infty}{\rightarrow} p_i(y) \quad \text{for} \quad y \in \Omega \text{ and } i = 1, \ldots, k. \quad (23)$$

To illustrate the procedure let consider random rotations on the sphere, performed along $x$ or $z$–axis. This special case of Example 4 may be easily quantized with the help of the components $J_i (i = x, y, z)$ of the angular momentum operator $J$, satisfying the standard commutation relations, $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$. The size of the Hilbert space is determined by the quantum number $j$ as $N = 2j + 1$.

**Example 11.** $k = 2$, random rotations

a) classical, $\mathcal{F}_{C1} = \{\Omega = S^3, f_1 = R_x(\theta_1), f_2 = R_y(\theta_2), p_1 = p_2 = 1/2\}$. The Lebesgue measure on the sphere in an invariant measure of this IFS.

b) quantum, $\mathcal{F}_N = \{\Omega = P_N, F_1 = \exp(i\theta_1 J_z), F_2 = \exp(i\theta_2 J_z), p_1 = p_2 = 1/2\}$. Both unitary operators are not common–block diagonal, due to Proposition 2, the maximally mixed state $\rho_\ast$ is a unique invariant state.

A quantization of an IFS of the second kind is given by the following modification of the previous example:

**Example 12.** $k = 2$, random rotations on the sphere with varying probabilities depending on the latitude $\theta$ computed with respect to the $z$–axis.

The spaces and the functions are as in Example 11, but

a) classical IFS $\mathcal{F}_{C1}$: $p_1 = (1 + \cos \theta)/2$ and $p_2 = (1 - \cos \theta)/2$;

b) quantum IFS $\mathcal{F}_N$: $p_1 = 1/2 + (J_z)/2j$ and $p_2 = 1/2 - (J_z)/2j$ with $N = 2j + 1$. Interestingly, this modification influences the number of invariant states of the IFS. Since $p_2$ vanish at the north pole, $\theta = 0$, of the classical sphere $S^2$, this point is invariant with respect to $\mathcal{F}_{C1}$. Similarly, the corresponding quantum state $|j, j\rangle$ localized at the pole is invariant with respect to $\mathcal{F}_N$.

The above examples of unitary QIFS dealt with simple regular maps — rotations on the sphere. However an IFS may also be constructed out of nonlinear maps, which may lead to deterministic chaotic dynamics. For instance one may consider the map describing periodically kicked top. It consists of a linear rotation with respect to $x$–axis by angle $\alpha$ and a nonlinear rotation with respect to $z$–axis by an angle depending on the $z$ component. In a compact notation the classical top reads, $T_Q(\alpha, \beta) = R_z(\beta) R_x(\alpha)$, while its quantum counterpart, acting in the $N = 2j + 1$–dimensional Hilbert space can be defined by $T_Q(\alpha, \beta) = \exp(-i\beta J_z^2/2j) \exp(-i\alpha J_x)$ [15]. This quantum map becomes one of the important toy model often studied in research on quantum chaos. A certain modification of this model, in which the kicking strength parameter $\beta$ was chosen randomly out of two values, was proposed and investigated by Scharf and Sundaram [16]. This random system may be put into the QIFS formalism.

**Example 13.** Randomly kicked top.

a) classical, $\mathcal{F}_{C1} = \{\Omega = S^1, f_1 = T_C(\alpha, \beta), f_2 = T_C(\alpha, \beta + \Delta), p_1 = p_2 = 1/2\}$. 


b) quantum, \( \mathcal{F}_N = \{ \Omega = \mathcal{P}_N, F_1 = T_Q(\alpha, \beta), F_2 = T_Q(\alpha, \beta + \Delta), p_1 = p_2 = 1/2 \}. \) For \( \alpha > 0 \) and a positive \( \Delta \) both unitary operators are non- block-diagonal, so the maximally mixed state \( \rho_\mu \) is a unique invariant state of this unitary QIFS. Our numerical results obtained for \( \alpha = \pi/4, \beta = 2 \) and \( \Delta = 0.05 \) suggest that the trajectory of any pure coherent state converges to the equilibrium exponentially fast.

To discuss a quantum analogue of an IFS with a fractal invariant measure consider the classical IFS presented in Example 3. The classical phase space \( \Omega \) is equivalent to the torus. For pedagogical purpose, let us rename both variables \( x, y \) into \( q, p \), which may be looked at as canonically coupled position and momentum. We shall work in \( N = 3L \)-dimensional Hilbert space. Let \( |j\rangle_i \) with \( i = 1, ..., N \) be eigenstates of the position operator, and similarly \( |l\rangle_p \) with \( l = 1, ..., N \) be eigenstates of the momentum operator. Both bases are related by \( |l\rangle_p = \sum_{j=1}^{N} W_{ij} |j\rangle_i \), where the matrix \( W \) is the \( N \) point discrete Fourier transformation, \( W_{ij} = (1/N) e^{i2\pi jl/N} \). The classical map \( f_1 \) in \( \mathbb{R}^4 \) representing a three-fold contraction in the \( x \) direction, corresponds to the transformation \( G_1 \) of the density operator given by

\[
G_1(\rho) = \sum_{i,j=1}^{L} |j\rangle_i \left( \sum_{m,n=0}^{2} \langle 3i + m | \rho | 3j + n \rangle \right) \langle j|_i .
\]  

(24)

In a similar way, the quantum map \( G_2 \) corresponding to \( f_2 \) is defined by

\[
G_2(\rho) = \sum_{i,j=1}^{3L} |j\rangle_i \left( \sum_{m,n=0}^{2} \langle 3i + m | \rho | 3j + n \rangle \right) \langle j|_i .
\]  

(25)

The maps \( G_3 \) and \( G_4 \) are obtained in analogous way like \( G_1 \) and \( G_2 \), using the eigenstates of momentum operator \( |k\rangle_p \).

\[
G_3(\rho) = \sum_{k,l=1}^{L} |k\rangle_p \left( \sum_{m,n=0}^{2} \langle 3k + m | \rho | 3l + n \rangle \right) \langle l|_p .
\]  

(26)

\[
G_4(\rho) = \sum_{k,l=1}^{3L} \sum_{m,n=0}^{2} |k\rangle_p \left( \sum_{m,n=0}^{2} \langle 3k + m | \rho | 3l + n \rangle \right) \langle l|_p .
\]  

(27)

The random system defined below may be considered as a QIFS related to the IFS introduced in Example 3.

**Example 14.** Quantum tartan specified by the following QIFS: \( \mathcal{F}_N = \{ \Omega = \mathcal{P}_N, k = 4, G_1, G_2, G_3, G_4; p_1 = p_2 = p_3 = p_4 = 1/4 \} \).

An invariant states for the maps \( \Lambda \) induced by this QIFS are illustrated in Fig. 1 for \( N = 3^4, N = 3^5 \) and \( N = 3^6 \). Invariant quantum state \( \rho \) is shown in the generalized Husimi representation

\[
H_\rho(p, q) = \frac{1}{2\pi} \langle q, p | \rho | q, p \rangle ,
\]  

(28)

based on the set of coherent states on the torus \( |q, p \rangle = Y^{-Nq}X^{-Np} |k \rangle \). The reference state \( |k \rangle \) is chosen as an arbitrary state localized in \( (1/2, 1/2) \)

\[
|k \rangle = \left( \frac{2}{N} \right)^{-1/4} e^{-\pi(n-N/2)^2/N} ,
\]  

(29)

while \( X \) denotes the operators of shift in position \( X|j\rangle = |j + 1\rangle \), with an identification \( |j + N\rangle = |j\rangle \). Similarly \( Y \) shifts the momentum eigenstates, \( Y|l\rangle = |l + 1\rangle \) and \( |l + N\rangle = |l\rangle \). The quantum state \( |q, p \rangle \) is well localized in the vicinity of the classical point \( (q, p) \) on the torus \( \mathbb{R}^2 \). This representation of any quantum states corresponding to the classical system on the torus was used in the analysis of an irreversible quantum baker map \( \mathbb{R}^2 \).

The larger value of \( N \), the finer structure of the invariant state \( \rho_{\mu} \) is visible in the phase space. In the semiclassical limit \( N \to \infty \), (which means \( h \to 0 \)) the invariant measure of the QIFS tends to be localized at the fractal support of the invariant measure of the classical IFS, shown for comparison in Fig. 1c. Strictly speaking, for any finite \( N \), the Husimi distribution of the quantum state \( \rho_{\mu} \) does not posses fractal character, since self-similarity has to terminate at the length scale comparable with \( \sqrt{N} \). In other words, quantum effects are responsible for smearing out the fractal structure of the classical invariant measure. However, the classical fractal structures may be approximated with an arbitrary accuracy by quantum objects in the semiclassical limit \( \mathbb{R}^2 \).
Figure 1: "Tartan-like" invariant density of the QIFS defined in Example 14 for (a) \( N = 3^4 \), (b) \( N = 3^5 \), and (c) \( N = 3^6 \) - dimensional Hilbert space, shown in the generalized Husimi representation. Invariant measure of the corresponding classical IFS on the torus occupies a fractal set (d).

V. CLOSING REMARKS

Classical iterated function systems display several interesting mathematical properties and may be applied in various problems from different branches of physics. In this work we have generalized the formalism of IFSs introducing the concept of QIFSs. Quantum iterated function systems may be defined in the space of pure states on a finite dimensional Hilbert space \( \mathcal{H}_N \), or more generally, in the space of density operators acting on \( \mathcal{H}_N \). As their classical analogues, QIFSs allow a certain degree of stochasticity, in the sense that at each step of time evolution the choice of one of the prescribed quantum maps is random.

This formalism is useful to describe several problems of quantum mechanics, including non-unitary dynamics, processes of decoherence and quantum measurements. In fact, the large class of quantum channels, called random external fields may serve directly as examples of a QIFS. Furthermore, for several classical IFSs one may construct the corresponding QIFSs and analyze the similarities and differences between them. As shown in the last example, one may focus on the fractal properties of invariant measures of some classical IFSs and study their quantum counterpart. Thus the concept of QIFS allows one to investigate the semiclassical limit of random quantum systems.

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Appendix A: PROOF OF PROPOSITION 3

We start from the following lemma:
Lemma 1. Let $U = (U_{nm})_{n,m=1,...,N}$ be an $N$-dimensional unitary matrix. Assume that there exist two non-empty sets of indices $A$ and $B$ such that: $A \cup B = 1 := \{1, \ldots, N\}$ and $A \cap B = \emptyset$. Then, $U_{nm} = 0$ for $n \in A$ and $m \in B$, implies $U_{nm} = 0$ for $n \in B$ and $m \in A$.

Proof of the lemma: We compute the number of elements of the set $A$:

$$|A| = \sum_{n \in A} \sum_{m \in A} |U_{nm}|^2$$

$$= \sum_{n \in A} \sum_{m \in A} |U_{nm}|^2 + \sum_{n \in A} \sum_{m \in B} |U_{nm}|^2$$

$$= \sum_{n \in A} \sum_{m \in A} |U_{nm}|^2$$

$$= \sum_{n \in A} \sum_{m \in A} |U_{nm}|^2 - \sum_{n \in B} \sum_{m \in A} |U_{nm}|^2$$

$$= |A| - \sum_{n \in B} \sum_{m \in A} |U_{nm}|^2,$$

and so $\sum_{n \in B} \sum_{m \in A} |U_{nm}|^2 = 0$, as required.

Now we turn to the proof of Proposition 3.

$\Rightarrow$ Let $U_i (i = 1, \ldots, k)$ be block-diagonal in the common base, and let dimension of the blocks be $\alpha_1, \ldots, \alpha_L$, where $\sum_{j=1}^L \alpha_j = N$. Define a diagonal density matrix as a direct sum

$$\rho := \bigoplus_{j=1}^L \alpha_j \sigma_j.$$

(A1)

where $\sum_{j=1}^L \alpha_j = 1$. Then $U_i \rho U_i^\dagger = \rho$ for every $i = 1, \ldots, k$. Hence $\rho$ is $\Lambda_U$-invariant and $\delta_\rho$ is a $P_U$-invariant measure on $P_N$ for an arbitrary choice of $(\sigma_j)_{j=1,\ldots,L}$.

$\Leftarrow$ Let $\rho$ be an invariant state for $\Lambda_U$ such that $\rho \neq \rho_*$. Then $\rho$ can be written in the form

$$\rho = \sum_{n=1}^N \sigma_n |\Psi_n\rangle\langle \Psi_n|,$$

(A2)

where $|\Psi_m\rangle \in P_N$, $\langle \Psi_m|\Psi_m\rangle = \delta_{nm}$ $(n, m = 1, \ldots, N)$, and $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_N$; $\sigma_1 \leq 1/N$. For $\gamma \in [0,1]$ the operator $\rho' = \gamma \rho + (1-\gamma) \rho_*$ is $\sum_{n=1}^N \sigma_n' |\Psi_n\rangle\langle \Psi_n|$, where $\sigma_n' = \gamma \sigma_n + (1-\gamma) N^{-1} (n = 1, \ldots, N)$ is also an invariant state for $\Lambda_U$. Put $\gamma := 1/(1 - \sigma_1 N)$. This choice implies $\sigma'_n = 0$ and $\sum_{n=1}^N \sigma'_n = 1$. Assume that $\sigma'_n = 0$ for $n = 1, \ldots, n'$ and $\sigma'_n > 0$ for $n = n'+1, \ldots, N$, where $n' \geq 1$. The equation $\Lambda_U(\rho') = \rho'$ can be rewritten in the form

$$\sigma'_n = \sum_{i=1}^k \sum_{m=1}^N |(U_i)_{nm}|^2 \sigma'_m,$$

(A3)

where $(U_i)_{nm} (n, m = 1, \ldots, N)$ are the elements of matrices $U_i (i = 1, \ldots, k)$ in the basis $|\Psi_n\rangle_{n=1,\ldots,N}$.

For $n = 1, \ldots, n'$ we get

$$0 = \sum_{i=1}^k \sum_{m=n'+1}^N |(U_i)_{nm}|^2 \sigma'_m.$$

(A4)

Hence $(U_i)_{nm} = 0$ for $n = 1, \ldots, n'$ and $m = n'+1, \ldots, N$. Using Lemma 1, we deduce that $(U_i)_{nm} = 0$ for $n = n'+1, \ldots, N$ and $m = 1, \ldots, n'$. Thus $U_i (i = 1, \ldots, k)$ are common block-diagonal.
