Black Hole Thermodynamics in Carathéodory’s Approach

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In this letter we show that the approach of Carathéodory to thermodynamics by means of Pfaffian forms can be applied to black hole thermodynamics and strongly links black hole thermodynamics to the standard thermodynamic formalism. The Pfaffian form $\delta Q_{\text{rev}} \equiv dM - \Omega \, dJ - \Phi \, dQ$, which is assumed to be the infinitesimal heat exchanged reversibly, is shown to be integrable; moreover, we show that it is a quasi-homogeneous form. As a consequence, an integrating factor is readily calculated. It is then shown that both the entropy and the temperature of a Kerr-Newman black hole can be recovered. No a priori knowledge of the laws of black hole mechanics is required. The Hawking effect is necessary in order to give an actual thermodynamic meaning to our calculation and in order to identify a undetermined multiplicative constant in the expression of the absolute temperature and the absolute entropy of the black hole. Also the problem of extremal black holes is shortly discussed.

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Introduction: We here consider black holes of the Kerr-Newman family (we put $h = c = G = k_b = 1$, where $k_b$ is the Boltzmann constant; moreover, we work with unrationalized electrical units). We limit ourselves to recall that they correspond to stationary black hole solutions of the vacuum Einstein equations characterized by three geometric parameters, the mass $M$, the angular momentum $J$ and the charge $Q$.

We consider black hole thermodynamics in the framework of Carathéodory’s approach to thermodynamics, which postulates the integrability of the Pfaffian form $\delta Q_{\text{rev}}$ representing the infinitesimal heat exchanged reversibly [1–5]. By postulating a somehow “natural” form for $\delta Q_{\text{rev}}$, we can introduce a notion of temperature and of entropy for black holes without referring a priori to the laws of black hole mechanics. Particularly, we can generate at first a potential, which is then related to the entropy of the black hole. The method we adopt allows to find a metrical entropy and an absolute temperature which are identified with the standard ones but for a multiplicative undetermined constant.

Pfaffian form and symmetry: We define

$$\delta Q_{\text{rev}} \equiv dM - \Phi \, dQ - \Omega \, dJ,$$

(1)

where we have introduced the angular velocity

$$\Omega = \frac{J}{M} \left( \frac{1}{2 \, M^2 - Q^2 + 2 \, M \sqrt{M^2 - Q^2 - J^2/M^2}} \right),$$

(2)

and the electric potential

$$\Phi = \frac{Q \, (M + \sqrt{M^2 - Q^2 - J^2/M^2})}{2 \, M^2 - Q^2 + 2 \, M \sqrt{M^2 - Q^2 - J^2/M^2}},$$

(3)

of the black hole. Both can be assigned on a purely geometrical footing, without any a priori knowledge of black hole thermodynamics. The infinitesimal variation $dM - \Phi \, dQ - \Omega \, dJ$ is taken along stationary black hole solutions of the Kerr-Newman family, because of (2), (3); moreover, these solutions are considered as black hole equilibrium states, to be compared with equilibrium states of standard thermodynamics. Then, definition (1) is formally coherent from the thermodynamic point of view (in fact, the above Pfaffian form representing infinitesimal heat is then defined along equilibrium states). The definition (1) is somehow natural, because the (rest) mass can be identified with (a term of) the internal energy (the rest mass of a fluid can be considered as a term of the internal energy in standard thermodynamics); moreover, the work terms look as standard work terms. The thermodynamic domain is assumed to be the open non-extremal manifold $M^4 - M^2 \, Q^2 - J^2 > 0$; the extremal sub-manifold $M^4 - M^2 \, Q^2 - J^2 = 0$ is a boundary of the former, and is temporarily not taken into account. Some more discussion on this topic is found in the following. The Pfaffian form $\delta Q_{\text{rev}}$ is non-singular, i.e., there is no point of the thermodynamic manifold where all the coefficients of the differential form vanish (this property is ensured by the first term $dM$, whose coefficient is one everywhere). This non-singularity property holds also on the extremal submanifold. It is easy to show that $\delta Q_{\text{rev}}$ is smooth on the non-extremal manifold and is completely integrable, that is, it satisfies the condition

$$\delta Q_{\text{rev}} \wedge d(\delta Q_{\text{rev}}) = 0 \iff -\partial_j \, \Phi + \partial_Q \, \Omega + \Phi \, \partial_M \, \Omega - \Omega \, \partial_M \, \Phi = 0.$$

(4)

Being $\delta Q_{\text{rev}}$ a one-form in three variables, this integrability condition is surely non-trivial (it would be trivial in the case of two variables). It is also possible to find an integrating factor by using a symmetry of the Pfaffian form (1). In fact, under the quasi-homogeneous transformation [6] (also called “similarity transformation” and “stretching transformation” [7])
There are infinitesimal generator of the transformation stretching transformation is a symmetry for e.g. \([8,9]\), in the sense that

\[
(\alpha) \rightarrow (\alpha, 2\alpha)
\]

are defined to be the weights of \(M, Q, J\) respectively and they have to be determined. Let us define the so-called Euler vector field \([6]\), which is constant. The corresponding potential is

\[
\frac{\partial}{\partial Q} \left( \frac{M}{Q} \right) = \frac{1}{2\alpha} \log \left( \frac{M^2 b^2(M, Q, J) + J^2/M^2}{M^2 b^2(M_0, Q_0, J_0) + J_0^2/M_0^2} \right),
\]

where \(b(M, Q, J) \equiv (1 + \sqrt{1 - Q^2/M^2 - J^2/M^2})\). The argument of the logarithm is proportional to the black hole area \(A = 4 \pi (M^2 b^2(M, Q, J) + J^2/M^2)\). We have generated a foliation of the parameter space of Kerr-Newmann black holes. The leaves are the surfaces \(A = \text{const}\) but we cannot yet determine the so-called metrical entropy \([7]\) for black holes. The correspondence with the formalism of thermodynamics is the following. The above procedure is a generalization, discussed in Ref. [10], of the procedure one can develop for standard thermodynamics [11]. In the case of standard thermodynamics of homogeneous systems, the Pfaffian form \(\delta Q_{rev} = dU + p \, dV - \mu \, dN\) in Gibbsian variables \((U, V, N)\) is defined to be homogeneous (for the definition of homogeneous differential form see e.g. Ref. [9,12]). The generator of the symmetry is the “Liouville” operator

\[
Y = U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} + N \frac{\partial}{\partial N}
\]

and the integrating factor is \(\delta Q_{rev}(Y) = U + p \, V - \mu \, N\). For a standard thermodynamic system one finds that \(dS = dS/S\), where \(S\) is an extensive function which is shown to be the metrical entropy of the system and corresponds to the fundamental relation in the entropy representation [11]. This deduction is corroborated by using the homogeneity of \(S\) in Gibbs’ approach, which allows to find \(T \, S = U + p \, V - \mu \, N = \delta Q_{rev}(Y)\), i.e., the integrating factor coincides with \(T \, S\). We proceed by analogy with the formalism of thermodynamics just sketched (see [10] for details). The potential \(S_\alpha\) such that

\[
dS = dS_\alpha/S_\alpha\]

for the black hole case is

\[
S_\alpha = c_\alpha \, A^{1/2\alpha}
\]

where \(c_\alpha\) is an undetermined constant. We have introduced above a label \(\alpha\) which underlines that we have a one-parameter family of possible metrical entropies/fundamental relations [this ambiguity does not occur in the case of standard thermodynamics]. Notice that \(D \, S_\alpha = S_\alpha\) which is analogous to the relation \(Y \, S = S\) of standard thermodynamics. The corresponding temperature is \(T_\alpha = (\partial S_\alpha/\partial M)^{-1}\). Our result (12) agrees with the result contained in Ref. [13] but we work in a more general framework where no reference to the laws of black hole mechanics is made [notice also that in our expression for \(S\) no additive constant appears]. Here we see also the difference with the case of standard thermodynamics: there is an ambiguity in the definition of the entropy of the black hole, because there is, a priori, no reason to choose the value \(\alpha = 1/2\). The point is that there is a freedom in the definition of the quasi-homogeneity symmetry which is not present for the case of the homogeneity symmetry. It is useful to realize that

\[
S_\alpha = c_\alpha \, c_{1/2}^{1/2\alpha} A^{1/2\alpha} = c_\alpha \, c_{1/2}^{1/2\alpha} S_{1/2}^{1/2\alpha}
\]

\[
T_\alpha = 2 c_\alpha \, c_{1/2}^{1/2\alpha} S_{1/2}^{-1/2\alpha} T_{1/2}.
\]
For any $\alpha$ one gets $T_\alpha dS_\alpha = dM - \Phi \ dQ - \Omega \ dJ$. The black hole area is known to be a superadditive function of $M, Q, J$. Superadditivity of the entropy, which plays a fundamental role when one considers the merging of two black holes, does not fix $\alpha$. It simply requires $0 < \alpha \leq 1/2$. A thermodynamic argument allowing to fix $\alpha = 1/2$ is the following. Let us consider two black holes which are very far away the one from the other, so that their mutual interaction can be neglected (even on a very long (but not infinite) time-scale). If one considers such a couple of black holes as analogous to a couple of very weakly interacting ordinary thermodynamic systems, one can conclude that $S_{12} \simeq S_1 + S_2$, that is, the entropy of the distant couple is additive. In order to fix $\alpha$, one has to realize that the total area $A_{12}$ of the couple is $A_{12} = A_1 + A_2$. Then an additive metrical entropy can be obtained only by fixing $\alpha = 1/2$. Moreover, as a consequence of the quasi-homogeneity of black hole entropy, one gets the following generalized Gibbs-Duhem equation [10]:

$$M^2 d\left(\frac{1}{2MT}\right) - Q^2 d\left(\frac{\Phi}{2QT}\right) - J d\left(\frac{\Omega}{T}\right) = 0. \quad (15)$$

In order to solve the above ambiguity for $\alpha$, one could be tempted to write $\delta Q_{rev}$ in terms of $M^2, Q^2, J$, as an homogeneous form of degree 1/2

$$\delta Q_{rev} = \frac{1}{2M} dM^2 - \frac{\Phi}{2Q} dQ^2 - \Omega \ dJ; \quad (16)$$

this is equivalent to the choice $\alpha = 1/2$. Then, by pursuing the formal analogy with the standard homogeneous systems formalism, one could find immediately the entropy and the temperature of the black hole. [Note also that $S$, as a function of $M^2, Q^2, J$, is a concave function (it is not concave, as well known, as a function of $M, Q, J$)]. Notwithstanding, one could also obtain an homogeneous form by introducing $Z = \sqrt{\mathcal{J}}$ and by writing $\delta Q_{rev} = dM - \Phi dQ - 2\Omega dZ$. The latter case is equivalent to choosing $\alpha = 1$. Moreover, the Reissner-Nordström case displays an extensive $\delta Q_{rev}$ where no straightforward path towards $\alpha = 1/2$ is allowed, as it can be easily verified.

The Hawking effect is necessary in order to give us an actual thermodynamic meaning to our calculation and corroborates our “additivity ansatz” which fixes the above ambiguity, in fact, in order to identify the temperature of the black hole with the Hawking one it is mandatory to choose $\alpha = 1/2$. There is a multiplicative constant (namely, $c_{1/2}$) which is undetermined in our thermodynamic approach. By comparison with the Hawking effect, one finds that $c_{1/2} = 1/4$. Phenomenologically, one should determine $M, Q, J$ and then plot $T(M, Q, J)$ from measurements of the temperature. $\alpha = 1/2$ and $c_{1/2} = 1/4$ should come out again.

Comparison with bh mechanics: Contrarily to the naive expectation, the laws of black hole mechanics give no unique hints about the value of $\alpha$, they don’t fix uniquely the metrical entropy and the absolute temperature of the black hole. For any $\alpha$ one gets $T_\alpha dS_\alpha = dM - \Phi dQ - \Omega dJ$, to be compared with the differential form of the first law. Moreover, one finds that $f = T_\alpha S_\alpha$, which implies $\alpha (M - \Phi Q - 2 \Omega J) = T_\alpha S_\alpha = 2\alpha T_{1/2} S_{1/2}$. By comparison with the first law in the finite form one realizes that $T_{1/2} = k/(8\pi c_{1/2})$. The choice of a generic $\alpha$ is equivalent to the the substitutions $A \mapsto \tilde{A}_\alpha = A^{1/2\alpha}$ and $k \mapsto \tilde{k}_\alpha = 2\alpha k/(A^{1/2\alpha-1})$ which implement both the differential form and the finite form of the first law (the latter appears as $\tilde{k}_\alpha \tilde{A}_\alpha = 8\pi \alpha (M - \Phi Q - 2 \Omega J)$ which is equivalent to the well-known one). Notice that $\tilde{k}_\alpha$ is constant on the horizon, thus the zeroth law of black hole mechanics is not sufficient in order to select $\alpha = 1/2$.

The extremal boundary: The extremal submanifold is very problematic. It is easy to show that $\delta Q_{rev} = 0$ on the extremal submanifold, i.e. the extremal submanifold is still an integral submanifold of the Pfaffian form [14]. Nevertheless, there is an important property which fails in the case of states belonging to the extremal submanifold. In fact, given a point of the extremal submanifold, there exist two kinds of adiabatic paths having the given state as initial point. One is a path lying on the extremal submanifold, the other is an “isoareal” path, i.e. a path starting from the extremal submanifold and reaching non-extremal states each of which has the same area as the initial extremal state [14]. In absence of the latter class of solutions, the extremal states would represent a leaf of a foliation, thus they would be adiabatically disconnected from the non-extremal states. Instead, if one consider the equation $\delta Q_{rev} = 0$ with initial point on the extremal boundary e.g. in the Reissner-Nordström case, the uniqueness of solutions of the Cauchy problem for ordinary differential equations fails (the Lipschitz condition fails) because our Pfaffian form $\delta Q_{rev}$ is smooth on the non extremal manifold but it is only continuous on the extremal one [14]. Analogous considerations can be made in case of failure of the third law in standard thermodynamics [15]. The intersection between adiabatic paths along extremal states and adiabatic paths joining extremal states and non-extremal ones is a problem for thermodynamics, because the second law can be jeopardized, even if only along special paths. It is indeed possible, in line of principle, to obtain a Carnot cycle in which the work produced is equal to the heat absorbed simply by considering a Carnot cycle with the lower isotherm at $T = 0$ [14]. A detailed discussion of this topic and of the third law in black hole thermodynamics is the subject of Ref. [14]. Here we limit ourselves to notice that one obtains from the above discussion a thermodynamic suggestion to explore carefully the ther-
modynamic status of the extremal boundary, which does not allow to obtain a foliation of the whole thermodynamic domain into non-intersecting adiabatic surfaces; moreover, one could also consider as compelling the introduction of a discontinuity in $S$ between non-extremal states and extremal ones, and a consequent withdrawn of the Bekenstein-Hawking law for extremal states.

The approach to black hole thermodynamics by means of Pfaffian forms (Carathéodory’s formalism) represents a further corroboration of the fact that black hole thermodynamics is a form of thermodynamics, even if to large extent exceptional. The difference between this approach and the one of Smarr [16] is that the entropy is not postulated a priori to be proportional to the area $A$, but it is constructed from the integrability of $\delta Q_{\text{rev}}$, without considering the laws of black hole mechanics. The only inputs are the “intensive quantities” $\Phi, \Omega$, which are furnished by General Relativity. The metrical entropy is found in a one-parameter family of possible entropies by means of a thermodynamic argument, in which the realizing that $\int \omega / f$ is proportional to the logarithm of the black hole area $A$ plays a role. Notice that our approach can be extended in a straightforward way to KN-AdS black holes [17]. Also in this case, there is a quasi-homogeneity structure in the Pfaffian form, as it can be easily realized.

We wish to underline here that: (a) the behavior of GR is, to some extent, intermediate with respect to a macroscopic theory, like classical thermodynamics, and a microscopic theory, like statistical mechanics. A macroscopic (“thermodynamic”) point of view is adopted in treating variables like $M, Q, J$; on the other hand, GR equations furnish $\Phi, \Omega$ and an integrability condition which, for standard systems, should be an outcome of statistical mechanics [statistical mechanics should calculate the analytic form of the functions $\Phi, \Omega$ (they are phenomenological interpolations for thermodynamics); moreover, it should justify an integrability condition which is only a postulate in standard thermodynamics, enhanced e.g. in Ref. [18], is corroborated by our framework.

In concluding, we underline that a form of quasi-homogeneous thermodynamics for self-gravitating systems is not a peculiar property of black holes, but can be found also in other systems, like non-relativistic fermionic matter and self-gravitating radiation. See Ref. [10] and references therein.

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