We discuss here in the framework of black hole thermodynamics some aspects relative to the third law in the case of black holes of the Kerr-Newman family. By means of thermodynamic arguments we show that a discontinuity in the thermodynamic manifold between extremal black holes and non-extremal ones should occur. In particular, arguments which strengthen the idea that the Bekenstein-Hawking law is violated in the case of extremal black holes are suggested. In the light of the standard proof of the equivalence between the unattainability of the zero temperature and the entropic version of the third law it is remarked that the unattainability has a special character within black hole thermodynamics. Thermostatic arguments in support of the unattainability are explored. Moreover, the other zero temperature limit, obtained in the case of very massive black holes, is also discussed and a violation of the entropic version is shown in the charged case. The third law of black hole dynamics by W. Israel is then interpreted as a further strong corroboration to the picture of a discontinuity between extremal states and non-extremal ones.

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I. INTRODUCTION

The third law of thermodynamics is explored in light of the correspondences existing between standard thermodynamics and black hole thermodynamics [1–5]. We summarize some aspects of the third law in standard thermodynamics and also some related results in black hole thermodynamics. In black hole thermodynamics, the unattainability \(U\) of \(T = 0\) holds and the entropic version \(N\) fails; it is underlined that \(U\) has a special status, is clearly not equivalent to \(U\) as it is realized in standard thermodynamics. Moreover, the limit \(T \to 0\) obtained for black hole mass \(M \to \infty\) is explored. \(N\) is shown to hold in the case of uncharged black holes; at the same time, \(U\) holds also for the charged case. Then we focus our attention on extremal states. An analysis in a thermostatic framework suggests that a discontinuity in the behavior of thermodynamics when one passes from non-extremal to extremal black holes should occur. We show that it is appealing to abandon the Bekenstein-Hawking law in the case of the extremal black holes. Thermostatic reasons for the unattainability \(U\) of extremal states are then analyzed; in particular, if the entropy of extremal states is zero, then it is possible to corroborate the unattainability principle also in a thermostatic framework. Unattainability would mean simply the impossibility of a process violating the second law of thermodynamics.

The analysis of the implications of Israel’s proof [4], to be interpreted in an irreversible thermodynamic framework, corroborates the statement that extremal states have to be discontinuous with respect to the equilibrium thermodynamic space of non-extremal ones.

The plan of the paper is the following. In sect. II a review of discussions and results about third law of thermodynamics in black hole physics [1–6] and of results about extremal black hole entropy [7–10]; a review of the status of the third law in standard thermodynamics [11–15] and Nernst’s theorem is made. In sect. III the relation between \(U\) and \(N\) is recalled. In sect. IV a short summary of the relations existing between black hole mechanics and the thermodynamic formalism is made. In sect. V by an analysis of the limit \(T \to 0\) in black hole thermodynamics. A first analysis concerns extremal states and then the limit \(T \to 0\) associated with infinitely massive black holes is
explored. In sect. VI the thermodynamic properties of extremal black holes are studied. In sect. VII Carathéodory’s approach to black hole thermodynamics is applied in order to study the properties of the extremal surface $T = 0$ and its relation with the thermodynamic foliation of the non-extremal submanifold. Thermostatic arguments suggesting a discontinuity between non-extremal states and extremal ones are then discussed in sect. IX. In sect. X thermostatic arguments supporting the unattainability are discussed. In sect. XI a translation of Israel’s result [4] in an irreversible thermodynamic frame allows to corroborate the picture of a discontinuity in thermodynamics at the extremal boundary. A discussion of the violation of (N) is made too, including its implications for a search of the degrees of freedom associated with black hole thermodynamics. Appendices A, B and C concern further aspects of the physics involved in our paper.

II. THE THIRD LAW

In standard thermodynamics there are two formulations of the third law. The entropic version of Nernst’s theorem (N) states that, for every system, if one considers the entropy as a function of the temperature $T$ and of other macroscopic parameters $x^1, \ldots, x^n$, the entropy difference $\Delta_T S \equiv S(T, x^1, \ldots, x^n) - S(T, \tilde{x}^1, \ldots, \tilde{x}^n)$ goes to zero as $T \to 0$

$$\lim_{T \to 0^+} \Delta_T S = 0$$  \hspace{1cm} (1)

for any choice of $(x^1, \ldots, x^n)$ and of $(\tilde{x}^1, \ldots, \tilde{x}^n)$. Thus (N) requires that the limit $\lim_{T \to 0^+} S(T, x^1, \ldots, x^n)$ is a constant $S_0$ which does not depend on the macroscopic parameters $x^1, \ldots, x^n$. Planck’s restatement of Nernst’s postulate fixes the entropy constant $S_0$ at $T = 0$ to be zero.\footnote{This is mandatory in the case of homogeneous thermodynamics, as is shown in [18] and it is trivial to prove.} Sometimes, the (N) version is expressed by saying that the zero temperature states of a system are isentropic. The latter statement is at least ambiguous, if the entropy is allowed to be discontinuous; the statement involving the limit of $S$ as $T \to 0$ in any case to be preferred. The unattainability version (U) can be expressed as the impossibility to reach the absolute zero of the temperature by means of a finite number of thermodynamic processes. Both the above formulations are due to Nernst. It is generally assumed that the two formulations are equivalent. Actually, this equivalence is not automatic, as it results from a discussion in Ref. [13–15]. We mean to come back to this topic in sect. VI, where its relevance in black hole thermodynamics is enhanced.

The third law of thermodynamics in black hole physics has been discussed since the formulation of the laws of black hole mechanics [1]. In fact, in Ref. [1] the analogy between the standard third law, in the form of unattainability (U) of the absolute zero temperature, and the unattainability of the extremal states by means of a finite number of physical processes [2] is remarked. A more recent result about the unattainability is found in Ref. [4], where the unattainability is rigorously obtained under suitable hypotheses. We discuss in the following this result further on. (N) is explored in the framework of black hole thermodynamics e.g. in Ref. [3]. Therein it is stressed the failure of the entropic side of the third law in black hole thermodynamics.

On the side of (N), we recall also some results obtained in the framework of gravitational partition function calculations. In Ref. [7,8] the entropy of an extremal Reissner–Nordström black hole is predicted to be zero and this result is related with the boundary structure of the spacetime. Analogous statements are found in Ref. [9] and a further corroboration of $S = 0$ for extremal Reissner–Nordström black holes appears in Ref. [10], where a semiclassical calculation of the entropy in canonical quantum gravity is made. The latter approach leads to a result that introduces a violation of the Bekenstein–Hawking law and is in agreement with the requirement of isentropic zero temperature states. This isentropy is not equivalent to (N), even if it seems to match Planck’s requirement of zero entropy for any system at the absolute zero of the temperature. A deeper discussion is found in the following sections. On the other hand, superstring theory and supergravity allow again the opposite result in which $S = A/4$ for extremal black holes. We don’t discuss herein the latter approach.

Doubts about the validity of thermodynamics for values of $T$ very near the absolute zero have been raised, when finite-size systems are taken into account. A thermodynamic description of a “standard” system below a given temperature is impossible according to Planck, because of a reduction of the effective degrees of freedom making impossible
even to define an entropy. Only statistical mechanics is then viable. In Ref. [16] this breakdown of thermodynamics near the absolute zero is shown to occur because of finite size effects, which make impossible to neglect statistical fluctuations in the calculation of thermodynamic quantities like e.g. $T, S$.

On the black hole side of this topics, arguments that are in some sense of the same nature as Planck’s ones are found in Ref. [17]. In fact, therein hints against the possibility of a thermal description of near extremal states, because of the occurrence of uncontrollable thermodynamic fluctuations, are given, and are related to the finite size nature of black holes (note also that a notion of thermodynamic limit is missing in the black hole case).

Herein, we discuss the extremal limit of black hole thermodynamics, and also the third principle in black hole thermodynamics with respect to the limit $T \to 0$ obtained for infinite black hole mass $M \to \infty$.

### III. THE EQUIVALENCE (U)⇔(N) REVISITED

In the following, we first focus our attention on the relation between (N) and (U) and on the possibility to de-link the unattainability from the entropic version of Nernst’s theorem. This somehow long discussion is preliminary with respect to the next sections, in which we discuss some properties of black hole thermodynamics near the absolute zero.

#### A. Unattainability vs. entropy behavior near $T = 0$. Landsberg’s analysis

We start by discussing (U) and (N) in standard thermodynamics. The double implication $(U)\leftrightarrow(N)$, according to the analysis developed in Ref. [13,15], relies on some hypotheses that it is interesting to recall. For a detailed discussion about the third law in standard thermodynamics we refer the reader to Ref. [18,19].

1. $(U)\Rightarrow(N)$

A detailed analysis shows that in standard thermodynamics unattainability (U) implies (N) if the following conditions a), b), c) are satisfied [13,15]:

- **a)** The stability condition $(\partial S/\partial T)_x > 0$ is satisfied for any transformation such that the external parameters (or deformation coordinates; in our discussion we include constitutive coordinates in the set of deformation coordinates), collectively indicated by means of $x$, are kept fixed; these transformations are called isometric transformations [20]. As a consequence, the heat capacity $C_x$ at constant deformation parameters $x$ is to be positive for $T > 0$.

  This hypothesis is in general ensured by the convexity/concavity properties of the thermodynamic potentials; as a consequence, in Landsberg’s works [13–15] this hypothesis is actually assumed to be always satisfied, so it is not discussed between the possible causes of a failure of the double implication $(U)\leftrightarrow(N)$. Given also the peculiar thermodynamic properties of black holes, we must choose a) as a further hypothesis to be discussed.

- **b)** There are no multiple branches in thermodynamic configuration space.

  The condition b) is introduced in order to avoid some pathological situations discussed in Ref. [13,14] (no physical behavior corresponds to them; see fig. 1); an equivalent statement is “in thermodynamic space no boundary points different from the $T = 0$ ones occur” [13].

- **c)** There is no discontinuity in thermodynamic properties of the system near the absolute zero.

  In Ref. [13] a careful discussion of the conditions to be satisfied in order to ensure (U) is contained. In particular, by following Ref. [13], if a),b),c) hold and moreover (N) fails, then $T = 0$ is attainable. If a),b) and c) hold, then (U) implies (N). If a),b) hold and (N) fails, then (U) implies that a discontinuity near the absolute zero has to occur, and such a discontinuity has to prevent the attainability of $T = 0$ (violation of c)) [13]. Anyway, in standard thermodynamics a violation of c) is ruled out [13], and (U) is associated with the impossibility to get states at $T > 0$ isentropic to states at $T = 0$. 

3
Notice that, in the case of black hole thermodynamics, the violation of \( (N) \) implies that \( (U) \) cannot be interpreted as absence of isoentropic transformations allowing to reach \( T = 0 \).

[FIG. 1. (a) Multi-branches structure of the thermodynamic space. According to Landsberg, it implies the validity of \( (U) \) and the violation of \( (N) \). (b) Violation of \( (N) \) that implies a violation of \( (U) \), due to the presence of the isoentropic AB. Landsberg conjectures that \( (U) \) holds if a discontinuity near \( T = 0 \) occurs. See also the text. In (a) and (b) the dashed regions are forbidden.

A further condition “entropies don’t diverge as \( T \to 0 \)” is also introduced in Ref. [13,15] in order to take into account the actual behavior near zero temperature of the standard thermodynamic systems. In fact, a priori, \( (N) \) could hold also if one should find a divergence in the entropy as \( T \to 0 \) not depending on \( x \) [13]. However, we don’t introduce this further hypothesis in view of our analysis in sect. VB.

About possible failures of the implication \( (U) \Rightarrow (N) \) an important reference is also Ref. [21].

2. \( (N) \Rightarrow (U) \)

A full implication \( (N) \Rightarrow (U) \) is possible in the case of thermodynamic processes which consist of an alternate sequence of quasi-static adiabatic transformations and quasi-static isothermal transformations (class \( P(x) \) according to Ref. [13,14]). Actually, a more general notion of unattainability can be assumed, that is, “zero temperature states don’t occur in the specification of attainable states of systems”. This is almost literally the \( (U4) \) principle as of Ref. [13,14]. \( (U4) \) states that no process allows to reach states at \( T = 0 \), even as transient non-equilibrium states. Then \( (N) \) can fail and \( (U) \) can still be valid. In general, the latter hypothesis allows a de-linking of \( (U) \) and \( (N) \) and implies that \( (N) \neq (U) \) and \( (U) \neq (N) \) [13,14]. But such a de-linking occurs under particular conditions. The failure of the implication \( (U) \Rightarrow (N) \) requires again a rejection of one of the hypotheses b),c) above, whereas \( (N) \Rightarrow (U) \) fails if processes not belonging to the aforementioned class \( P(x) \) allow to reach \( T = 0 \) [13,14].

Concluding this section, we recall that the standard approach to Nernst’s theorem involves heat capacities and runs e.g. as in Ref. [22,23].

IV. THE FUNDAMENTAL EQUATION IN BLACK HOLE THERMODYNAMICS

The fundamental equation in black hole thermodynamics for Kerr-Newman family, characterized by the mass \( M \), the angular momentum \( J \) and the charge \( Q \), is represented by Smarr’s formula [24]

\[
M = M(A, Q, J) = \frac{kA}{8\pi} + 2\Omega_{bh}J + \Phi_{bh}Q. \tag{2}
\]

The extensive parameters appearing in \( M \) are the charge \( Q \), the angular momentum \( J \) and the area \( A \) of the black hole

\[
A = 4\pi \left( M + \sqrt{M^2 - J^2 - Q^2} \right)^2 + J^2 M^2. \tag{3}
\]

and the intensive ones are the surface gravity \( k \), associated with the geometrical temperature \( T_{bh} = k/(2\pi) \), the angular velocity \( \Omega_{bh} \) and the electric potential \( \Phi_{bh} \). In order to get black hole solutions it is necessary to impose
Here, we limit our considerations to the so called non-extremal black holes for which the inequality holds. The other case is the subject of the sect. VI.² An explicit knowledge of the dependence of the “internal energy” $M$ on $S, Q, J$ is sufficient in order to give a complete specification of the black hole state in the thermodynamic configuration space. In fact, it corresponds to the fundamental equation of a standard thermodynamic system in the internal energy representation. If $U$ is the internal energy, $S$ is the entropy and $X_i$ is a further set of extensive variables, then $U = U(S, X_i)$ represents the energetic fundamental relation of standard thermodynamics, which allows a complete thermodynamic description of the system [25]. The Bekenstein–Hawking formula $S = A/4$ allows to write [24,3]

\[ M = \left( \frac{S}{4\pi} + \pi \frac{J^2}{S} + \pi \frac{Q^4}{4S} + \frac{Q^2}{2} \right)^{1/2}. \]

It is also remarkable that in the black hole case the state equations

\[
T_{\text{bh}} = T_{\text{bh}}(S, Q, J),
\]

\[
\Phi_{\text{bh}} = \Phi_{\text{bh}}(S, Q, J),
\]

\[
\Omega_{\text{bh}} = \Omega_{\text{bh}}(S, Q, J)
\]

are known and are fixed by the black hole geometry. In fact [3]

\[
T_{\text{bh}} = \left( \frac{\partial M}{\partial S} \right)_{Q,J} = \frac{S^2 - \pi^2 (Q^4 + 4J^2)}{8\pi M S^2},
\]

\[
\Phi_{\text{bh}} = \left( \frac{\partial M}{\partial Q} \right)_{S,J} = \frac{Q (S + \pi Q^2)}{2 MS},
\]

\[
\Omega_{\text{bh}} = \left( \frac{\partial M}{\partial J} \right)_{S,Q} = \frac{\pi J}{MS}.
\]

Moreover, if one chooses the entropy representation of a standard thermodynamic system, one gets

\[ S(M, Q, J) = \frac{\pi}{k} M - \frac{2\pi \Omega_{\text{bh}}}{k} J - \frac{\pi \Phi_{\text{bh}}}{k} Q. \]

$S$ is a quasi-homogeneous function of degree one and weights $(1/2, 1/2, 1)$, i.e. it satisfies $S(\lambda^{1/2} M, \lambda^{1/2} Q, \lambda J) = \lambda S(M, Q, J)$ [in literature it has been realized that $S$ is homogeneous of degree one as a function of $M^2, Q^2, J$]. One gets, obviously, (3) apart from the proportionality factor $1/4$. This representation has the particular feature that the state equations are explicitly given in terms of the standard black hole parameters $M, Q, J$:

\[
T_{\text{bh}}(M, Q, J) = \frac{1}{2\pi} \frac{\sqrt{g(M, Q, J)}}{2M^2 - Q^2 + 2M \sqrt{g(M, Q, J)}},
\]

\[
\Phi_{\text{bh}}(M, Q, J) = \frac{Q(M + \sqrt{g(M, Q, J)})}{2M^2 - Q^2 + 2M \sqrt{g(M, Q, J)}},
\]

\[
\Omega_{\text{bh}}(M, Q, J) = \frac{J}{M} \frac{1}{2M^2 - Q^2 + 2M \sqrt{g(M, Q, J)}}.
\]

It is useful to note that, for infinitesimal changes in $M, Q, J$, the following infinitesimal change in $A$ is obtained

\[
dA = \frac{2A}{\sqrt{g(M, Q, J)}} (dM - \Phi_{\text{bh}} dQ - \Omega_{\text{bh}} dJ). \tag{7}
\]

Transformations such that the term in brackets is zero are isoareal and isoentropic and are studied in Appendix A. We are interested in studying the limit $T_{\text{bh}} \to 0$ that can be obtained either for $g \to 0$ or for $M \to +\infty$. The former corresponds to approaching extremal black hole states, the latter to approaching infinitely massive black holes (cf.

²Solutions with $g(M, Q, J) < 0$ describe naked singularities.
A thermodynamic analysis for black holes in terms of differential forms has been developed in Ref. [26], and it is fundamental in order to understand the thermodynamic status of extremal black holes. The approach of Carathéodory to thermodynamics identifies the the infinitesimal heat exchanged reversibly with an integrable Pfaffian form; we have shown in Ref. [26] that it can be applied to black hole thermodynamics and that it represents a strong link with the formalism of standard thermodynamics [27]. The Pfaffian form is \( \delta Q_{\text{rev}} \equiv dM - \Omega \, dJ - \Phi \, dQ \) and it is a non-singular integrable Pfaffian form defining a foliation of the thermodynamic manifold by means of the solutions of the Pfaffian equation \( \delta Q_{\text{rev}} = 0 \). See also Ref. [26]. This one-form is smooth on the non-extremal submanifold, and it is continuous everywhere. Its integrability means that, in the inner part of the thermodynamic manifold (non-extremal states) \( \delta Q_{\text{rev}} \wedge d(\delta Q_{\text{rev}}) = 0 \) is verified [26]. Another fundamental property of the Pfaffian form \( \delta Q_{\text{rev}} \) is to be a quasi-homogeneous Pfaffian form: under the rescaling \( M \to \lambda^a M, Q \to \lambda^a Q, J \to \lambda^a J \) one finds \( \delta Q_{\text{rev}} \to \lambda^a \delta Q_{\text{rev}} \). Quasi-homogeneity means that the infinitesimal symmetry generator is the so-called Euler vector field

\[
D \equiv \alpha M \frac{\partial}{\partial M} + \alpha Q \frac{\partial}{\partial Q} + 2\alpha J \frac{\partial}{\partial J}
\]  

(to be compared with the Liouville vector field which generates a homogeneity symmetry). See [26–28] for more details. An integrating factor is then easily found, and it is given by

\[
f = \delta Q_{\text{rev}}(D) = \alpha \sqrt{M^2 - Q^2 - J^2/M^2}.
\]

This framework allows to find a strong link between standard thermodynamics and black hole thermodynamics, because, by integrating \( \delta Q_{\text{rev}}/f \) one finds a one-parameter family of thermodynamic potentials which is realized to be proportional to the logarithm of the black hole area; then, by using a purely thermodynamic argument, it is possible to find the Bekenstein-Hawking formula \( S \propto A \) (i.e., \( \alpha = 1/2 \)) within an undetermined multiplicative constant [26]. No reference to the laws of black hole mechanics is a priori done. A discussion on the status of the extremal submanifold in this picture is very important and it is developed in sect. VII.

Before concluding this section, we recall that black hole thermodynamics has some very peculiar features that make it special with respect to “standard” thermodynamics. For details see e.g. Ref. [29,30] and references therein.

V. BLACK HOLES BRANCHES AS \( T \to 0 \)

In this section we discuss black hole thermodynamic branches near \( T = 0 \). We first recall that for the entropy, as a function of \( T, M, Q \), there is a branching into two different functions. In fact, although the entropy is a continuous function of \( M, J, Q \), there are points such that the state equation \( T(M, J, Q) \) cannot be inverted in order to get \( M(T, J, Q) \). It happens that \( \partial T/\partial M = 0 \) can be satisfied on suitable submanifolds, where standard conditions for the implicit function theorem fail. As a consequence, one can invert \( T \) and obtain the desired function \( M(T, J, Q) \), to be substituted into \( S(M, J, Q) \) only away from these submanifolds, on the two branches \( \partial T/\partial M > 0 \) and \( \partial T/\partial M < 0 \), which are associated with the zero temperature limits obtained as finite mass extremal limit and the infinite mass limit respectively. In particular, for the same value of the variables \( T, J, Q \) it is possible to get two different values of \( S \), the branch \( \partial T/\partial M > 0 \) giving a different result with respect to the branch \( \partial T/\partial M < 0 \). This is a sufficient reason for a multi-branching in the \( S - T \) plane. The two disconnected branches describe two different systems. Critical submanifold points correspond to points where \( C_{Q,J} \) diverges and changes sign. It has been proposed that a second order phase transition takes place there [3].

We first consider the finite mass extremal limit and then the infinite mass limit.

A. black hole extremal limit \( M < +\infty \)

In the black hole case we get a sort of “exotic” thermodynamic also from the point of view of the third law, because (N) is violated and (U) is valid. For the sake of completeness, we sketch a possible way to prove that (N) is violated in black hole thermodynamics. Particularly, we show that (1) fails in the general case of a Kerr–Newman black hole. Let us define

\[
M_E^2 \equiv \frac{1}{2} (Q^2 + \sqrt{Q^4 + 4J^2})
\]

\[
M_N^2 \equiv \frac{1}{2} (Q^2 - \sqrt{Q^4 + 4J^2}) < 0;
\]
$M_{E}^{2}, M_{N}^{2}$ are the roots of the equation $(M^2)^2 - Q^2 M^2 - J^2 = 0$ and $M_{E}^{2}$ corresponds to the squared mass of the extremal Kerr–Newman solution having charge $Q$ and angular momentum $J$. It is useful to explicit the following relations between the above roots and the charge $Q$ and the angular momentum $J$ of the black hole: $Q^2 = M_{E}^{2} + M_{N}^{2}$, $J^2 = -M_{E}^{2} M_{N}^{2}$. Moreover, the difference $(M_{E}^{2} - M_{N}^{2})$ is related to the area of the extremal solution for given values of $Q, J$ by $A_{E} = 4 \pi (M_{E}^{2} - M_{N}^{2})$. We can rewrite

$$T = \frac{M}{2\pi} \frac{((M^2 - M_{E}^{2})(M^2 - M_{N}^{2}))^{1/2}}{(M^2 + ((M^2 - M_{E}^{2})(M^2 - M_{N}^{2}))^{1/2})^2 - M_{E}^{2} M_{N}^{2}}$$

$$S = \frac{\pi}{M^2} \left( \frac{(M^2 + ((M^2 - M_{E}^{2})(M^2 - M_{N}^{2}))^{1/2})^2 - M_{E}^{2} M_{N}^{2}}{M^2} \right).$$

It is easy to show that, near the extremal states $M^2 \sim M_{E}^{2}$, one has

$$M^2 = M_{E}^{2} + 4 \pi^2 M_{E}^2 \left( M_{E}^{2} - M_{N}^{2} \right) T^2 + \cdots$$

Then, for $T \to 0$,

$$S(T, Q, J) \sim \pi (M_{E}^{2} - M_{N}^{2}) + 4 \pi^2 M_{E} (M_{E}^{2} - M_{N}^{2}) T + \cdots$$

in such a way that

$$S(T, Q_1, J_1) - S(T, Q_2, J_2) \to 0 \bigg/ \left(A_{1E} - A_{2E} \right) = \pi \left( \sqrt{Q_1^2 + 4J_1^2} - \sqrt{Q_2^2 + 4J_2^2} \right). \quad (10)$$

As expected, the difference in entropies is proportional to the difference of the areas of the corresponding extremal solutions, which depend on the macroscopic parameters. The limit (10) is to be intended as a right limit as $T \to 0^+$. We remark that it could be physically improper to assign by continuity a value to the entropy on the boundary $T = 0$ of the thermodynamic manifold, in other words, it could be improper to assign the value $S_{E} = A_{E}/4$ to extremal states. The limiting process enhances that the entropy of a non-extremal black hole does not tend to zero or to a constant as $T \to 0^+$ independently from $Q, J$, whichever can be the actual value of $S$ for extremal states at $T = 0$, and this fact amounts to a violation of (N).

The failure of (N) implies that (U) cannot be intended as absence of adiabatic transformations reaching $T = 0$. It is interesting to notice that concavity (hypothesis a)) fails, as it is well-known, in black hole thermodynamics. In the black hole case, there exist curves approaching $T = 0$ such that $C_x > 0$ and other such that $C_y < 0$. The existence of paths with $C_x < 0$ allowing to approach $T = 0$ is evident in the Kerr case [3], where $C_{11} < 0$ and $C_{12} > 0$ near the extremal limit. In the general Kerr–Newman case, near the extremal state the heat capacity $C_{11};Q$ is positive and goes to zero as $T$ at the extremal limit [3]. Other heat capacities at constant deformation parameters can be taken into account [31], as $C_{q};Q = C_{11};Q; C_{12};Q; C_{13};Q; C_{14};Q$. Some of them can be negative near $T = 0$. Note that, however, the presence of negative heat capacity paths is in general not sufficient to ensure the violation of (N). It only allows to de-link the violation of (N) from the validity of (U), in fact the non-uniformity of the sign of heat capacities near $T = 0$ can allow adiabatic paths reaching $T = 0$. See also Appendix C.

**B. the branch $T \to 0$ for $M \to \infty$**

In black hole thermodynamics another limit of zero temperature is sometimes discussed [3,32]. It is the limit as $M \to \infty$ e.g. in the Schwarzschild case. In fact, $T \sim 0$ only near the extremal states or for very large masses. But the latter limit cannot be considered on the same footing as the limit where extremal states are approached, indeed it is physically related to an unattainability principle in a straightforward way. No infinite mass can be allowed on physical grounds, whereas no hindrance to consider e.g. $Q^2 = M^2$ in the Reissner–Nordström case is a priori involved in the physics. However, an astrophysical black hole represents from a thermodynamic point of view a system reaching temperatures even much lower than the ones involved in experiments of low temperature physics and in actual experimental validation of (N) and, moreover, it is interesting to stress that black hole thermodynamics allows to get systems having a very low temperature and a huge entropy in contrast with the low temperature behavior of standard systems. In particular, Planck’s postulate $S \to 0$ for $T \to 0$ is to some extent maximally violated\(^3\). It is

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\(^3\)We remark that the violation of (N) in presence of diverging deformation parameters can occur in the case of systems for which (N) holds at finite deformation parameters. See also [19].
also interesting to show that (N) is violated in the sense that (1) fails in the case of a charged black hole. Of course, such a violation is impossible in the case of a Schwarzschild black hole, due to its too constrained thermodynamic phase space. Instead, let us first consider the Reissner–Nordström case; in the limit \( M \to +\infty \) one can invert explicitly the relation between \( M \) and \( T \)

\[
M^2 \sim \frac{1}{64 \pi^2 T^2} - 8 \pi^2 Q^4 T^2;
\]

then

\[
S(T, Q) \sim \frac{1}{16 \pi T^2} - 2 \pi Q^2 + O(T^2) \tag{11}
\]

and

\[
S(T, Q_1) - S(T, Q_2) \to 2 \pi (Q_2^2 - Q_1^2). \tag{12}
\]

See fig. 2. Obviously \( S \to \infty \). If one considers the heat capacity \( C_Q \) corresponding to the process under consideration, one finds that \( C_Q < 0 \) so that the process involves the thermally unstable branch of black hole thermodynamics. We have just shown that (N) fails but for a self-evident reason a sort of (U) is automatically ensured. We stress again that (U) is not to be intended as the impossibility to reach \( T = 0 \) in a finite number of processes in this case, but simply as the impossibility to get an infinitely massive black hole. (U) amounts to (U4), the most general notion of unattainability of the absolute zero.

\[\text{FIG. 2. Violation of (N) in the large-} M \text{ limit. The qualitative behavior of the entropy is displayed for two different value of the black hole charge. Two isoentropic lines are qualitatively displayed.}\]

The general Kerr–Newman case can be treated analogously. The starting point is a large mass expansion of the equation of state for \( T \)

\[
T = \frac{1}{8 \pi} \frac{1}{M} - \frac{1}{128 \pi} (4J^2 + Q^4) \frac{1}{M^5} - \frac{1}{128 \pi} Q^2 (4J^2 + Q^4) \frac{1}{M^7} + O \left( \frac{1}{M^9} \right);
\]

by inverting one finds

\[
\frac{1}{M} = 8 \pi T + 2048 \pi^5 (4J^2 + Q^4) T^5 + 131072 \pi^7 Q^2 (4J^2 + Q^4) T^7 + O \left( T^9 \right)
\]

and

\[
S(T, Q_1, J_1) - S(T, Q_2, J_2) = 2 \pi \left( Q_2^2 - Q_1^2 \right) + 48 \pi^3 \left( 4 (J_2^2 - J_1^2) + (Q_2^4 - Q_1^4) \right) T^2 + O \left( T^4 \right). \tag{13}
\]

We can deduce that, if the black hole is uncharged but rotating, (N) is satisfied because \( S(T, J_1) - S(T, J_2) \sim 192 \pi^3 \left( J_2^2 - J_1^2 \right) T^2 \to 0 \) for \( T \to 0 \). So there is evidence in favor of the validity of (N) in the case of uncharged rotating black holes of the Kerr family on the thermally unstable large mass branch of black hole thermodynamics (again, \( C_J < 0 \) for \( M \to \infty \)). Only the presence of the charge is actually related to the failure of (N) on the same branch. Thermal instability is verified also in the general Kerr–Newman case

\[
C_{J,Q} \sim -\frac{1}{8 \pi T^2} - 96 \pi^3 (4J^2 + Q^4) T^2 + \cdots < 0.
\]
We find a behavior that is remarkable also from a thermodynamic point of view. In fact, the validity of (U4) can give rise both to (N) and to the failure of (N), as suggested by Ref. [14]. States at $T = 0$ on this branch of black hole thermodynamics are evidently unphysical and disconnected from the finite mass states, we have a discontinuity which is directly related to the unavailability of an infinite energy which would be necessary in this case in order to obtain a zero temperature state (and an infinite entropy state).

It is also remarkable that, if in general the entropy diverges for $T \to 0$, the axis $T = 0$ acts as a vertical asymptote for the graph of $S$ in the $S - T$ plane and the possibility to find an adiabat reaching $T = 0$ evidently is missing. The divergence of the entropy as $T \to 0^+$ suggests that (U) holds (an isoentropic at $S = +\infty$ appears to be unphysical).

It is also possible to find thermodynamic transformations joining together the extremal limit and the large mass limit to the zero temperature. One can choose e.g.

$$ J^2 = M^2 \left( M^2 - Q_0^2 \right) \tanh^2 \left( \frac{M}{M_0} \right) $$

where $M_0, Q_0$ are constants. It is evident that the extremal limit is implemented as $M \to \infty$. In general, the extremal limit can be approached only asymptotically along these transformations and (U) is preserved as above.

Note that, even in the charged case, one finds

$$ \lim_{T \to 0} \frac{S(T, Q_1)}{S(T, Q_2)} = 1. $$

(N) in some sense holds at the leading order but fails in the charged case when the difference is taken because of sub-leading terms depending on the charge $Q$, as it can be inferred from (11), (12).

**VI. EXTREMAL BLACK HOLES**

Extremal black holes belonging to the Kerr-Newman family are characterized by the extremality constraint $g = 0$, that is

$$ M^2 = Q^2 + \frac{J^2}{M^2}. \quad (14) $$

The event horizon radius $r_+$ and the Cauchy one $r_-$ coincide and get the value $r_+ = r_- = M$. The geometrical temperature $T_{bh}$ vanishes due to the fact that the surface gravity $k$ is zero for $r_+ = r_-$.

In the case of extremal black holes, the constraint equation (14) is equivalent to the fundamental relation specifying the black hole state. In fact, by solving (14) in $M$, one gets explicitly $M = M(Q, J)$. In other words, the “equivalent” of the internal energy $M$ is a function only of $Q, J$

$$ M(Q, J) = \frac{1}{\sqrt{2}} \sqrt{Q^2 + \sqrt{Q^4 + 4J^2}} = M_E. \quad (15) $$

It is simple to show that $M(Q, J)$ is a quasi-homogeneous function of degree $\frac{1}{2}$ and weights $(1/2, 1)$:

$$ \frac{1}{2} \frac{\partial M}{\partial Q} Q + \frac{\partial M}{\partial J} J = \frac{1}{2} M. $$

By differentiating (15) it is easy to show that along extremal states

$$ dM = \Omega_{bh}^{extr} dJ + \Phi_{bh}^{extr} dQ $$

where $\Omega_{bh}^{extr}, \Phi_{bh}^{extr}$ are the extremal black hole angular velocity and electric potential respectively. Moreover, one finds

$$ M = 2\Omega_{bh}^{extr} J + \Phi_{bh}^{extr} Q $$

Then, consistently, a state equation that could be deduced from Smarr’s one by imposing (14) is found.

In the following, extremal states constrained transformations and area variations are analyzed.
A. transformations along extremal states

We are interested in the nature of a transformation carried along extremal states, i.e. by keeping the extremality constraint fixed. It can also be considered a quasi-static (i.e. reversible) transformation in a thermodynamic sense, being assigned the transformation equation in the thermodynamic configuration space \( Q, J \). Along reversible transformations, the extremality constraint allows only infinitesimal mass variations involving purely work terms

\[
(dM - \Omega_{bh} dJ - \Phi_{bh} dQ)_{extr} = 0
\]

\[
dL_{extr}^{rev} = -(\Omega_{bh} dJ + \Phi_{bh} dQ)_{extr} = -dM_{extr}.
\]  

(16)

Here (16) corresponds to the standard characterization of the adiabatic work \( dL_{\text{adiabatic}} = -dU \). No heat exchanges terms can be involved in a reversible transformation along extremal states. This means that such a kind of transformation is adiabatic reversible. This result in standard thermodynamics at \( T > 0 \) allows to conclude that the given transformation is isoentropic (an “adiabat” according to Callen’s definition [25]). For \( T = 0 \) states it is well known that this conclusion is not automatic and a postulate is required.

It is interesting to rephrase (16) also as follows. The extremal submanifold is an integral manifold for the Pfaffian form \( (\delta Q)_{rev} \equiv (dM - \Omega_{bh} dJ - \Phi_{bh} dQ) \).

B. \( dA \) along extremal states

The area for an extremal black hole can be expressed as

\[
A_{extr} = 4\pi (r^2 + J^2)_{extr} = 4\pi \sqrt{Q^4 + 4J^2}.
\]  

(17)

From (17) one gets

\[
(dA)_{extr} = \frac{8\pi}{\sqrt{Q^4 + 4J^2}}(Q^3 dQ + 2J dJ) = \frac{32\pi^2}{A}(Q^3 dQ + 2J dJ).
\]  

(18)

The extremality constraint does not implies that \( (dA)_{extr} \) vanishes, and in general along extremal states it is impossible to get \( dA_{extr} = 0 \), e.g. the equation \( A_{extr} = \text{const.} \) admits only the trivial solution \( M^2 = Q^2 = \text{const.} \) and \( M^4 = J^2 = \text{const.} \) in the case of Reissner–Nordström and Kerr black holes respectively, whereas for non-extremal black holes it is possible to get nontrivial solutions for \( dA = 0 \). See also Appendix A herein.

Thermodynamic considerations based on the fundamental relation (5) suggest that the entropy for an extremal black hole should be again given by the Bekenstein–Hawking formula \( S_{extr} = A_{extr}/4 \) (indeed, a direct substitution into (5) gives exactly and consistently (15)).

VII. CARATHÉODORY’S APPROACH AND THE SURFACE \( T = 0 \).

Carathéodory’s approach to thermodynamics allows also to understand better the status of the surface \( T = 0 \) both in standard thermodynamics [18] and in black hole thermodynamics. We limit ourselves to discuss the latter aspect herein. For \( T > 0 \) the integral manifolds of the Pfaffian form \( \delta Q_{rev} \) are the surfaces \( S = \text{const.} \). Given any non-extremal state, any path solving the equation \( \delta Q_{rev} = 0 \) in the thermodynamic manifold has to lie on an isoentropic surface. Carathéodory’s approach ensures for standard thermodynamic formalism the existence of a foliation of the thermodynamics manifold into isoentropic hypersurfaces of codimension one by means of a condition which is equivalent to the integrability of \( \delta Q_{rev} \) [in the neighborhood of any thermodynamic state there exist states which cannot be reached along adiabatic paths]. In black hole thermodynamics the existence of the thermodynamic foliation for the non-extremal manifold is allowed by an integrability condition which is a consequence of the Einstein

\[\text{In our paper “quasi-static” is assumed to be synonymous of “reversible”. See also sect. XI 1 and for a discussion see e.g. Ref. [13].}\]
equations [i.e., it holds for the black hole solutions of the Einstein equations].
The nature of the extremal submanifold is instead very peculiar. In fact, the surface $T = 0$, which corresponds to
the extremal submanifold, is an integral manifold of the Pfaffian form $\delta Q_{\text{rev}}$, in the sense that it solves the equation
$\delta Q_{\text{rev}} = 0$, as we have seen in sect. VI.A. It could be considered naively as a leaf, which would imply immediately
that the Bekenstein-Hawking law fails on the extremal manifold, because the leaves of a foliation generated by an
integrable Pfaffian form cannot intersect each other. But the lack of some regularity properties of $\delta Q_{\text{rev}}$ on the extremal
submanifold has important consequences. Let us consider the Reissner-Nordström case. By posing $M^2 = x$; $Q^2 = y$
one finds

$$\delta Q_{\text{rev}} = \frac{1}{2\sqrt{x}} \, dx - \frac{1}{2(\sqrt{x} + \sqrt{x - y})} \, dy,$$

where $y \leq x$. Given a black hole state $(x_0, y_0)$, the states which are adiabatically reachable from it lie on the curves
that are solutions of the following Cauchy problem

$$\begin{align*}
\frac{dy}{dx} &= 1 + \sqrt{1 - \frac{y}{x}} \\
y(x_0) &= y_0.
\end{align*}$$

The solution of this problem exists and it is unique for any initial non-extremal state and corresponds to the standard
isoareal solution. If, instead, one considers an extremal state as initial point, the Cauchy problem

$$\begin{align*}
\frac{dy}{dx} &= 1 + \sqrt{1 - \frac{y}{x}} \\
y(x_0) &= x_0
\end{align*}$$

allows two solutions:

$$y(x) = x,$$

which means that the extremal states are adiabatically connected each other, and the solution

$$y(x) = 2 \sqrt{x_0 \sqrt{x} - x_0}$$

which holds for $x \in (x_0/4, x_0]$ and means that extremal states are also adiabatically connected to non-extremal ones.

The key-point is that on the extremal manifold, the right member of the differential equation (23) is no more smooth
(actually, it is not $C^1$ and even the weaker Lipschitz condition is not satisfied). This is a serious problem from a
thermodynamic point of view, because the adiabatic inaccessibility is jeopardized by the $T = 0$ manifold. It seems
indeed to be possible to reach adiabatically any non-extremal state from any other one by passing through extremal
states (which are non-isoareal). This would imply a failure of the second law of thermodynamics. On this topic, see
a discussion in sect. VIII. Moreover, because of the intersection of integral manifolds, even if only at $T = 0$, one
cannot conclude that there is a foliation of the whole thermodynamic manifold [extremal manifold included] but an
almost-foliation, i.e., a foliation except for a zero measure set (the integral manifold $T = 0$).

From a physical point of view, in order to avoid the above singular behavior of the thermodynamic foliation, one could
 decide that the surface $T = 0$ should be a leaf itself, that is, to exclude the set of solutions (25) in the thermodynamic
manifold. Notice that the set of solutions (25) corresponds to the isoareal solutions $dA = 0$. Obviously, the existence
of extremal black holes which have the same area as non-extremal states is not questioned; what is questioned by
refusing the set of solutions (25) is the validity of $S \propto A$ also in the case of extremal black holes. In other terms, there
exists a geometric foliation of the whole black hole manifold whose leaves are given by $A = \text{const}$. This geometric
foliation, which includes also the extremal states, should correspond to the thermodynamic foliation only in the case
of non-extremal states. In order to avoid problems with thermodynamics, one could construct a foliation of the
thermodynamic manifold whose leaves are

\[5\]Without this problem, and without solutions like (25), one could conclude that the extremal manifold is a leaf of the
thermodynamic manifold and, as a consequence, that the Bekenstein-Hawking law does not hold for extremal states. One is
instead forced to introduce a discontinuity in order to obtain a good foliation of the thermodynamic manifold.
the surfaces $S = A/4 = \text{const.}$ for non-extremal states

(26)

the surface of extremal states.

(27)

The leaves for the non-extremal manifold are the usual ones, which can be generated by means of the Pfaffian form $\delta Q_{\text{rev}}$. The exceptional integral manifold $T = 0$ is assumed to be isoentropic. The Bekenstein-Hawking law ensures that $S = A/4$ can assume arbitrarily large values and, a priori, also very small values, the only lower bound could be given by the onset of a quantum gravity regime. Without considering so small values of $S$ implying a quantum gravity regime, it is reasonable, on a purely thermodynamic footing, to assume that $S_E = 0$ for any extremal state. A further discussion is found in the following.

Then the thermodynamic foliation one obtains is given by the following discontinuous entropy $S(M, Q, J)$:

\[
S(M, Q, J) = \begin{cases} 
\frac{A}{4} & \text{for non-extremal states} \\
0 & \text{for extremal states.}
\end{cases}
\]

(28)

(29)

See also the discussion in sect. XA herein.

**VIII. THE BEKENSTEIN-HAWKING FORMULA FOR EXTREMAL STATES AND THE CARNOT-NERNST CYCLE**

It is interesting to discuss some problems that could arise in a naive consideration of black hole thermodynamics near and along the $T = 0$ states. The following hypotheses are taken into consideration:

- **α)** extremal states can be reached by means of reversible adiabatic paths by starting from non-extremal states;
- **β)** reversible transformations along extremal states discussed in sect. VI A are allowed;
- **γ)** non–isoareal transformations along extremal states exist;
- **δ)** the Bekenstein–Hawking law holds for extremal states.

We discuss now the above hypotheses, and the implications of their validity or failure. An extensive discussion of the hypothesis **δ)** is the subject of the following section.

Hypothesis **α)** is verified, in the light of the existence of integral manifolds of $\delta Q_{\text{rev}}$, which are allowed to reach $T = 0$, unless some discontinuity occurs.

Hypothesis **β)** is more critical. In the case of standard thermodynamics objections against the possibility to perform a reversible transformation at $T = 0$ have been raised [33], because of the impossibility to improve a change between the adiabatic constraint used in approaching $T = 0$ and the adiabatic constraint in performing the adiabatic isotherm $T = 0$. Actually, e.g. in the case of a Reissner-Nordström black hole, one can operatively distinguish between the adiabat approaching the extremal states, which is characterized by a law $Q^2 = 2M r_+ - (r_+)^2$, where $r_+$ is the radius of the initial black hole state, and the law which has to hold for a transformation along extremal states $Q^2 = M^2$. Rejecting a priori **β)** would mean implicitly to introduce a “discontinuity” for thermodynamics in the behavior of extremal states with respect to non-extremal ones. The impossibility to perform any transformation along the $T = 0$ isothermal surface would be surely peculiar, in the sense that it is a property which distinguishes the $T = 0$ submanifold with respect to the thermodynamic space at $T > 0$.

Hypothesis **γ)** is verified, because $dA \neq 0$ along extremal states (see sect. VI A). Note that one has to implement **β)** in order that the verification of **γ)** is associated with the possibility to implement a non-isoareal transformation along extremal states.

Let us assume that the hypotheses **α)**, **β)**, **γ)** and **δ)** are all verified. Then, the second law of thermodynamics is violated according to the standard argument of Nernst (see e.g. Ref. [13–15]). In fact, let us consider a thermal Carnot cycle in the plane $T - S$ (see. fig. 3; the cycle is clockwise), having the lower isotherm exactly at $T = 0$. We define it as Carnot–Nernst cycle.
FIG. 3. Carnot–Nernst cycle involving the isotherm at $T = 0$. Its efficiency is one, against the second principle.

If it is possible to obtain a continuous reversible non isoentropic transformation at $T = 0$, then one should be able to construct a thermal machine with efficiency exactly equal to one. In other words, it would be possible to get a perpetuum mobile of the second kind by means of a Carnot–Nernst cycle involving extremal states, which implies a violation of the second law of thermodynamics.

A possibility is to reject $\alpha$ in the frame of black hole thermodynamics, by requiring that a discontinuity does not allow to perform the adiabats and reach the extremal states (even if non-extremal states accumulate near the extremal ones along adiabats).  

It is remarkable that, without the introduction of quantum effects it is impossible to implement even at a gedanken experiment level the above cycle in classical black hole thermodynamics. Moreover, one could raise against the Carnot–Nernst cycle a further objection related with the fact that along extremal states it seems impossible to get $dA < 0$ (no problem arises instead with non–extremal states, whose area can decrease because of the Hawking effect; actually fermionic matter in the so called “superradiance” frequency range can violate the area theorem [34]). If $dA < 0$ would be impossible along the line $T = 0$, then one could consider only an irreversible counter–clockwise cycle, i.e. a refrigeration cycle, and a paradoxical result about the performance coefficient (defined as the absolute value of the ratio of the heat subtracted to the cold source over the work delivered to the refrigerator; cf. Ref. [25]) would still arise.

The possibility to perform of transformations along $T = 0$ states in standard thermodynamics has been criticized by Einstein (as quoted in Ref. [11,14]), both from the point of view of the unavoidable presence of non-negligible irreversibilities occurring near the absolute zero, and from the point of view of the actual possibility to perform an ideal transformation along $T = 0$. See also Ref. [33].

If the hypotheses $\alpha), \beta, \gamma, \delta$ are satisfied, then a violation of the second law is unavoidable.
IX. A DISCONTINUITY BETWEEN EXTREMAL STATES AND NON-EXTREMAL ONES. A THERMOSTATIC FRAME ANALYSIS AND THE FAILURE OF \( S = A/4 \)

It is also interesting to investigate if there exist thermostatic arguments in favor of a discontinuity between extremal and non-extremal states, even if “discontinuity” in thermodynamic behavior does not automatically mean unattainability of the extremal states, (it is not to be considered on the same footing as the discontinuity in the sense of Landsberg, to be discussed in the following section). The hypothesis of a discontinuity is appealing in black hole thermodynamics because a very different physics is involved in the case of extremal black holes with respect to non-extremal ones (see e.g. Ref. [7,9,35,36,6]).

A. A discontinuity in thermodynamics

We discuss in this section the hypothesis

\( \delta \) the Bekenstein-Hawking law holds for extremal states.

Hypothesis \( \delta \) is crucial in order to show that there is a discontinuity in thermodynamics between non-extremal states and extremal ones. In fact, both if the Bekenstein-Hawking law is verified and if it is not verified along extremal states, one is forced to admit that thermodynamics does not behave continuously.

If Bekenstein-Hawking law is maintained also in the extremal case, it is anyway true that a discontinuity is verified, because reversible transformations along extremal states are always adiabatic, as seen in sect. VI A, and in general non-isentropic (cf. also Appendix A), contrarily to what happens for non-extremal states, where adiabatic reversible transformations are necessarily isoentropic. This means that adiabatic reversible transformations are isoentropic for non-extremal black holes and in general non isoentropic for extremal ones. The Bekenstein-Hawking law could be obtained by means of an extension of \( S \) on the boundary of the thermodynamic manifold represented by extremal states. But the latter procedure, even if mathematically correct, is not physically appealing.

In the case one has a different law for the entropy extremal states, a thermodynamic discontinuity of extremal states with respect to non-extremal ones evidently occurs (one can suppose either \( \Delta S \neq 0 \) or \( \Delta S = 0 \) along extremal states). In fact, an evident discontinuous behavior of \( S \) near the absolute zero with respect to the limiting definition of \( S \) for extremal black holes is required if the Bekenstein-Hawking law is violated by extremal states. See fig. 4.

![T - S plane in black hole thermodynamics](image)

**FIG. 4.** \( T - S \) plane in black hole thermodynamics. The dashed line near \( T = 0 \) indicates that a discontinuity in the thermodynamic manifold has to appear both in the case that \( S = A/4 \) holds for extremal states and in the case a different law is implemented (the case \( S = 0 \) is displayed by means of a small box). Three isoentropes approaching extremal states are shown.

B. discontinuity at \( T = 0 \)

Extremal black hole entropy can be, in line of principle, valued by means of quantum mechanics: The Von Neumann entropy of extremal black holes is a priori calculable and it is the only meaningful entropy that can be associated with a state by studying it at exactly \( T = 0 \), without considering a limiting process as \( T \to 0 \). A dichotomy between the limit as \( T \to 0 \) of the thermodynamic entropy and the \( T = 0 \) entropy for extremal states appears in Ref. [37] and in a paper concerning the quantization of extremal Reissner-Nordström black holes [38]. Zero entropy is found by working separately on extremal states, a non-vanishing entropy is allowed if a limiting process starting from a quantization of non-extremal states is given rise [38]. The same dichotomy is implicit in Ref. [7,9], where \( S = 0 \) for extremal states
and \( S = A/4 \) for any non-extremal black hole \(^9\).

Doubts against limiting processes for calculating the entropy of extremal black holes are raised also in Ref. [38] (therein and in Ref. [37] interesting comments about the results obtained for BPS states in string theory approach are found too). In black hole thermodynamics, one is suggested to introduce a discontinuity of thermostatics between \( T > 0 \) and the absolute zero. From this point of view, one has also to take into account that, although a continuous behavior of some geometrical properties is verified, there are important differences in properties like the topology of the manifold.

The Euler characteristic changes and this topological difference has been related with the thermodynamic differences between extremal and non-extremal black holes, being the global thermodynamic functionals linked with the global properties of the manifold \([7,8,39,40]\).

In the following, we discuss the third law from the point of view of standard thermodynamics, and we point out some peculiar properties occurring in black hole thermodynamics.

X. THERMOSTATICS AND (U)

The violation of \((N)\) near the extremal states \( M < +\infty \) from a thermodynamic point of view does not forbid the attainment of the zero temperature state. In order to conciliate the validity of \((U)\) and the violation of \((N)\) a reasonable hypothesis is that near \( T = 0 \) in black hole thermodynamics a (possibly abrupt) change in thermodynamic properties of the system occurs. We are inspired by Landsberg’s hypothesis \(c\), relative to a possible discontinuity ensuring \((U)\) against the failure of \((N)\). It is very interesting, because in studying the implication \((U) \Rightarrow(N)\) Landsberg not only postulates the validity of \((U)\) but also he tries to retrace the possibility to get \((U)\) and not \((N)\) in a peculiar behavior of some thermodynamic functions. We learn, from the infinite mass case, that it is not strictly necessary such a behavior, because the attainment of \( T = 0 \) is forbidden in that case simply by the first law (conservation of energy). In standard thermodynamics, when \((N)\) holds, the attainment of the absolute zero is generally thought to be forbidden by the second law (impossibility of \( \Delta S < 0 \) for an adiabatic process of a closed system). For the finite mass extremal case we have shown that there is the possibility to de-link \((U)\) from \((N)\) but we think it is interesting to investigate also if there are thermodynamic arguments suggesting \((U)\) beyond the dynamic theorem of Israel \([4]\), whose thermodynamic implications are discussed in sect. XI.

In order to corroborate the hypothesis of a discontinuity in the sense of Landsberg, the most straightforward study involves the analysis of the behavior of “standard” adiabatic quantities near the extremal limit. If e.g. some adiabatic compressibility should vanish then the hypothesis would be verified, indeed it would be impossible to carry out the adiabat connecting a non-extremal state to an extremal one. Landsberg makes the example of an abrupt divergence in the elastic constants of a solid as a conceivable ideal process preventing a solid to reach a zero temperature state by means of quasi-static adiabatic volume variations (the hypothesis of Ref. [13] is compatible with the vanishing near \( T = 0 \) of the (adiabatic) compressibilities that are related with elastic constants in ordinary thermodynamics; particularly, for standard systems one can define the compressibility modulus as the inverse of the compressibility; it is proportional to the Young modulus in the case of a solid). But our analysis does not show a peculiar behavior of adiabatic derivatives and does not suggest the kind of discontinuity characterizing Landsberg’s hypothesis. Cf. also the appendix of Ref. [31], where some adiabatic derivatives are calculated. Nevertheless, in a thermostatic frame, the analysis of stability properties for Kerr–Newman black holes can suggest the \((U)\) property of extremal states and are carried out in sect. X.B.

A proper understanding of the \((U)\) property is allowed by dynamic considerations allowed by Ref. [4] \(^{10}\).

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\(^9\)In statistical mechanics, there are subtleties related with the order in which the limit as \( T \to 0 \) and the thermodynamic limit are taken in the calculation; the “ground state degeneracy”, obtained by taking first the limit as \( T \to 0 \) and then the thermodynamic limit, could also not coincide with the entropy obtained by taking the above limits in the opposite order (which corresponds to the correct procedure). This further problem seems to be not relevant for black holes, because they are, or, at least, behave as finite–size systems; see also Ref. [17].

\(^{10}\)We have been discussing the semiclassical properties of black hole thermodynamics. An important remark is that black holes are treated classically and this could be a potentially relevant point. In fact, the violation of \((N)\) for classical systems is not an exotic behavior. For example, it is well known that this violation occurs for a classical ideal gas, for which \( \lim_{T \to 0} (S(T, V) - S(T, \tilde{V})) = nR \log(V/\tilde{V}) \) and so an explicit dependence on the volume \( V \) appears. Moreover \( S \to -\infty \). This unphysical behavior near the absolute zero is corrected by quantum considerations. A priori it seems legitimate to think that the violation of \((N)\) in the black hole case should be confirmed by full quantum gravity calculations.
We start by recalling the potential relevance of (U) in relation with the Cosmic Censorship Conjecture (CCC). The unattainability of the extremal states is generally considered to be linked with the Cosmic Censorship Conjecture (see e.g. Ref. [3,41–43,32]). In fact, the possibility to reach extremal states in a finite number of physical steps is commonly assumed to leave open the possibility to carry further on the same process till a naked singularity is produced\(^{11}\). In this sense, the third law in the (U) form appears as a sort of “thermodynamic side” of the CCC. It is also remarkable that, from a classical point of view test particles cannot create naked singularities from even extremal black holes [47] and that quantum processes seem to enforce (U) in the sense that they imply a loss of charge/angular momentum from the black hole, acting as a protection against a further approaching the naked state [48].

In the following subsection, we give a thermostatic argument in favor of the unattainability of extremal states. It is based on the assumption that extremal states have zero entropy, as suggested by calculations in canonical quantum gravity.

A. (U) and extremal states with \(S_E = 0\)

It is remarkable that, if extremal states have \(S_E = 0\), then one has a natural naive argument for implementing (U). Let us consider the following hypotheses:

s1) the system is composed by a non-extremal black hole and matter; it is closed and undergoes an adiabatic transformation.

s2) Both the initial and the final state are equilibrium states (which can be a non-trivial postulate for states at \(T = 0\)).

s3) The final state is an equilibrium state between the black hole and matter.

s4) \(S_E = 0\).

Hypothesis s1) means that the principle of increase of entropy holds. Hypothesis s2) is natural, in the sense that the thermodynamic entropy is unambiguously defined only for equilibrium states. s3) and s4) are discussed below. We can consider at first the case in which all the matter can be used for making extreme the black hole. For the initial state one has contributions to the entropy from the matter and a non-extremal black hole (NE): \(S_{\text{tot}}^{\text{ini}} = S_{\text{NE}}^{\text{ini}} + S_{\text{mat}}^{\text{ini}}\); in the final state one has only an extremal black hole, so that, according to s4), \(S_{\text{tot}}^{\text{fin}} = S_E = 0\). Clearly the second law requires for the adiabatic process \(S_{\text{tot}}^{\text{fin}} \geq S_{\text{tot}}^{\text{ini}}\), which is impossible in our case.

One can also relax the hypothesis that all the matter is used for making extreme the black hole; if the matter at the end is in equilibrium with the black hole according to hypothesis s3), then the second law would require \(S_{\text{tot}}^{\text{fin}} = S_{\text{mat}}^{\text{fin}} \geq S_{\text{NE}}^{\text{fin}} + S_{\text{mat}}^{\text{fin}}\), which for ordinary matter is again impossible \((S_{\text{mat}}^{\text{fin}} \sim 0 \text{ because of (N)})\). One could then arbitrarily approach an extremal state but the “jump” onto extremality would be forbidden by the second law.

The final state could also be an equilibrium state if at least a portion of the residual matter is kept thermally insulated with respect to the black hole (violation of condition s3)) by some external mean, in which case the final state should have a contribution \(S_{\text{mat}}^{\text{fin}} > 0\). But it is hard to see how this framework could allow the attainability of extremal states without a violation of the second law. In other words, one should allow the formation of extremal states from non-extremal ones in such a way to preserve the principle of increase of the entropy for the thermodynamic universe under consideration\(^{12}\) (an highly non-trivial task in light of the fact that \(S_{\text{NE}}^{\text{fin}}\) is a huge number in general).

Hypothesis s1) is a critical hypothesis, in the sense it prescribes that a sort of adiabatic container forbids heat exchanges with the rest of the universe. In a collapse situation, there can be surroundings of the system matter+black hole whose entropy variation could play a role in the application of the second law of thermodynamics. It can be also noted that, if one postulates that the quantum-mechanical entropy the system has at \(T = 0\) is finite and lower than any value of the entropy at \(T > 0\), then (U) is automatically ensured by the second law, both in the case (N) holds and in the case (N) is violated, because no adiabatic transformation on the system can lead to \(T = 0\).

This hypothesis relating (U) to the second law is to be compared with processes that allow to get extremal states. They are apparently possible and they are not of class P(x), more in general, they don’t correspond to quasi-static

\(^{11}\)This point of view has been criticized by means of some counter-examples [44,45], but no definitive proof that a violation of \(U\) does not imply the failure of CCC exists. See also Ref. [4,46].

\(^{12}\)For “thermodynamic universe” we mean the smallest closed and thermally isolated system of interest (e.g., in the case of a black hole and matter falling into it, if thermal and matter exchanges with the surroundings are impossible, the thermodynamic universe is the system black hole + matter).
processes. Extremal black hole formation from extremal collapsing thin shells [9] and from charged thin shell collapsing on a non-extremal black hole (see Ref. [44]) are examples of these processes. Could one define their initial state as an equilibrium state? In the first example, the shell is pushed from infinity. In the second case, the shell has to be fired onto the non-extremal black hole. If the quantum gravity result $S_E = 0$ is true, then a careful analysis of the second law is required in order to ensure that, at a deeper level with respect to the naive analysis above, the second law is actually preserved. Even if this analysis should be essentially unmodified, a detailed analysis of stability could reveal that the probability of these processes is very low; in Ref. [45] is indeed underlined that the extremalization process by means of thin shells is highly unstable under perturbations (see the conclusions therein). Quantum effect could also play a relevant role in this case, as follows from Refs. [49,50]. In Ref. [51], it is conjectured $S_E = 0$ by starting from quantum gravity considerations. It is interesting to note that there is consistency with our second-law based conjecture.

1. $S_E = 0$ and the merging of two extremal black holes

There is a consistency check for the hypothesis $S_E = 0$ that should be discussed. Let us consider two black extremal black holes $(M_1, Q_1, J_1)$ and $(M_2, Q_2, J_2)$ (one variable, e.g. $M$, of course depends on the other two because of the extremal constraint). Let us allow the two extremal black holes to merge and that no heat is exchanged with the rest of the universe during the process. We suppose that the final state consists of a single black hole resulting from the adiabatic merging of the two initial extremal states. We wonder if the final state could be extremal. The point is that, if the final state could be extremal, then a violation of the second law could still occur, in fact the process is irreversible and the final entropy should be greater than the initial one but, if the final state is extremal and no heat is exchanged with the rest of the universe, one would find $S_{E1} + S_{E2} = S_{in} = 0 = S_{E12} = S_{fin}$. Let us define, as in Ref. [31],

$$a^2(M, Q, J) \equiv M^4 - M^2 Q^2 - J^2;$$

from Ref. [31] we know that, by defining $a_{12}^2 \equiv a^2(M_1 + M_2, Q_1 + Q_2, J_1 + J_2)$, $a_1^2 \equiv a^2(M_1, Q_1, J_1)$, and $a_2^2 \equiv a^2(M_2, Q_2, J_2)$ one has

$$a_{12}^2 - (a_1 + a_2)^2 = \left( \frac{M_2}{M_1} a_1 - \frac{M_1}{M_2} a_2 \right)^2 + \left( \frac{M_2}{M_1} J_1 - \frac{M_1}{M_2} J_2 \right)^2 + 2 M_1 M_2 \left[ (M_1 + M_2)^2 - (Q_1 + Q_2)^2 \right] + 2 \left( M_1 M_2 - Q_1 Q_2 \right) (M_1 + M_2)^2 .$$

(31)

We wish to see if it is possible that $a_{12} = 0$, which would imply that the final state is extremal. In our case, the above formula simplifies because $a_1 = 0 = a_2$ for the initial extremal states. Moreover, one has $M_1 = M_{1E}$ and $M_2 = M_{2E}$. Then one finds

$$a_{12}^2 = \left( \frac{M_{2E}}{M_{1E}} J_1 - \frac{M_{1E}}{M_{2E}} J_2 \right)^2 + 2 M_{1E} M_{2E} \left[ (M_{1E} + M_{2E})^2 - (Q_1 + Q_2)^2 \right] + 2 \left( M_{1E} M_{2E} - Q_1 Q_2 \right) (M_{1E} + M_{2E})^2 .$$

(32)

If $J_1 \neq 0$ and/or $J_2 \neq 0$, no matter which values one considers for $Q_1, Q_2$, then $a_{12} > 0$ and the final state is a non-extremal state. The final entropy is surely much greater than the initial one. If $J_1 = 0 = J_2$, then it is possible to find a final state which is still extremal if one merges two extremal Reissner-Nordström black holes having charges with the same sign, as it is evident from

$$a_{12}^2 = 2 M_{1E} M_{2E} \left[ (M_{1E} + M_{2E})^2 - (Q_1 + Q_2)^2 \right] + 2 \left( M_{1E} M_{2E} - Q_1 Q_2 \right) (M_{1E} + M_{2E})^2 .$$

(33)

Then the two black holes should be fired one against the other in order to win the electrostatic repulsion. One should question if it is possible to allow $S_E = 0$ and to preserve the second law in a real process of merging. Notice that even a very small angular momentum would protect the second law.

We don’t study here this problem, we limit ourselves to the above considerations. Of course, in light of the risk for violations of the second law, also the hypothesis s4), $S_E = 0$ should be questioned.

In the following section, the analysis of stability properties for Kerr–Newman black holes suggests the unattainability of extremal states.
The entropy is a non-concave function even in the case in which the thermal stability is ensured (as near the extremal states). In fact, there are principal minors of the Hessian (see e.g. Ref. [52,25]) that don’t satisfy the concavity (stability) requirement. In Ref. [31] it is shown that in the black hole case the minor $\Delta_3$ does not implement the stability requirement for any value of the physical parameters. In fact [31]

$$\Delta_3 = \frac{\pi}{8} \frac{1 + 3M^2 Q^2 - \Phi^2}{MT^3S^3}$$

and the stability requirement $\Delta_3 < 0$ is never satisfied. Moreover, the instability becomes maximal at the extremal limit (where all the minors diverge). We show explicitly this property in the Reissner–Nordström and in the Kerr cases. Let us define

$$a_{X_i} x_j \equiv \frac{\partial^2 S}{\partial X_i \partial X_j} \quad (34)$$

where $X_i$ are the extensive variables appearing in the fundamental relation in the entropy representation. In the Reissner–Nordström and in the Kerr cases we get respectively

$$a^{RN}_{MM} = -(2\pi) \frac{1}{(M^2 - Q^2)^{3/2}} (M + \sqrt{M^2 - Q^2}) (M - 2 \sqrt{M^2 - Q^2})$$

$$a^{RN}_{QQ} = -(2\pi) \frac{1}{(M^2 - Q^2)^{3/2}} (M^3 + (M^2 - Q^2)^{3/2})$$

$$a^{RN}_{MQ} = (2\pi)^2 \frac{Q^3}{(M^2 - Q^2)^{3/2}}$$

$$a^{Kerr}_{MM} = (4\pi) \frac{1}{(M^4 - J^2)^{3/2}} (M^6 - 3 M^2 J^2 + (M^4 - J^2)^{3/2})$$

$$a^{Kerr}_{JJ} = -(2\pi) \frac{M^4}{(M^4 - J^2)^{3/2}}$$

$$a^{Kerr}_{MJ} = (4\pi) \frac{J M^3}{(M^4 - J^2)^{3/2}} \quad (35)$$

and

$$(a^{RN}_{MQ})^2 - a^{RN}_{MM} a^{RN}_{QQ} = (2\pi)^2 (M^2 - Q^2)^{-3/2} \left[ M^3 + (4 M^2 - Q^2) \sqrt{M^2 - Q^2} + 3 M^2 (M^2 - Q^2) \right]$$

$$(a^{Kerr}_{MQ})^2 - a^{Kerr}_{MM} a^{Kerr}_{JJ} = 8 \pi^2 M^4 (M^4 - J^2)^{-2} (M^2 + \sqrt{M^4 - J^2}).$$

In the case of two independent thermodynamic variables $X_1, X_2$, stability requires that $a_{X_1} X_1 \leq 0, a_{X_2} X_2 \leq 0, (a_{X_1} X_1)^2 - a_{X_1} X_1 a_{X_2} X_2 \leq 0$ [25]. The third condition in both cases is always violated and moreover a divergence in the extremal limit appears. The condition that the entropy hypersurface lie everywhere below its family of tangent hyperplanes [25] is so violated maximally near the extremal states. Such an instability can give a thermodynamic “runaway” from extremal states and so also for (U).

**XI. IRREVERSIBILITY FRAME**

We now discuss the meaning of (U) as it is rigorously proved in Ref. [4]. It is important to stress that in Israel’s proof (U) holds from a dynamic point of view. In particular, it is shown that a non-extremal black hole cannot become extremal in a finite advanced time if the accreted matter stress–energy tensor satisfies the weak energy condition in a neighborhood of the outer apparent horizon and remains bounded and continuous [4]. Israel’s result shows that, along a continuous process, in a finite advanced time it is impossible to destroy the trapped surfaces, which are present in the non-extremal states and instead are missing in the extremal one. This implies that the process (NE)→(E) from a non–extremal black hole $r^{NE}_+ - r^{NE}_- > 0 \Leftrightarrow k^{NE} > 0$ to an extremal black hole $r^{E}_+ - r^{E}_- = 0 \Leftrightarrow k^{E} = 0$ as a
final state requires an infinite time\textsuperscript{13}. In the following we stress that also the subclass of thermodynamic processes is constrained by Israel’s result, at least as far as \textit{approximations of quasi-static processes are implemented by means of accretion of matter whose stress-energy tensor is bounded and continuous and satisfies the hypotheses of Ref. [4].} Approximations of quasi-static processes e.g. by means of point-like particles satisfy the aforementioned requirements if suitably corrected (they would imply a distributional stress-energy tensor that should be corrected by taking into account the finiteness of the Compton wavelength of the particles). Some dynamic restrictions in Israel’s proof don’t allow a full identification of such a result with the unattainability (U4) of the extremal states.

1. irreversible thermodynamics framework

We show that Israel’s result implies that extremal black holes cannot be considered as equilibrium states contiguous to non-extremal black hole equilibrium states. The thermodynamic manifold of equilibrium states has to present a discontinuity.

The framework of irreversible thermodynamics is the most appropriate in order to include in thermodynamics Israel’s dynamic information. In fact, irreversible thermodynamics allows the introduction of the notion of relaxation time to an equilibrium state, and gives the effective physical time-scale with respect to which a process can be considered properly as a good approximation of a quasi-static process. As far as strictly quasi-static processes are concerned, at least according to the meaning they have in our paper, in line of principle they represent an ordered succession of equilibrium states without any time information [13,25] (but note that in literature there is no unique definition of quasi-static process, which can be defined also as “infinitely slow processes” or “processes whose rate is infinitely slow”). The inaccessibility in a finite time of the extremal states from non-extremal ones by means of a continuous process can be rephrased as the impossibility to carry on an approximate quasi-static process joining non-extremal states to extremal ones, due to the divergence, which is met in approaching extremal states, of the “relaxation” time to the equilibrium state (that corresponds to the formation time of the stationary black hole state). We corroborate this point of view as follows. In Ref. [4] is found that, in order to squeeze out trapped surfaces, it is necessary an infinite time $\tau_{\text{no \ trapped}} = \infty$. Since an extremal black hole has no trapped surfaces, it follows the third principle $\tau_{(NE) \rightarrow (E)} = \infty$. In the framework of irreversible black hole thermodynamics one can conclude that $\tau_{\text{relaxation}}$ to an extremal equilibrium state coincides with $\tau_{(NE) \rightarrow (E)} = \infty$ if the latter is relative to a generic dynamical bounded and continuous process as in Ref. [4], even if a correspondence between relaxation phenomena and irreversible black hole thermodynamics still is missing. See however Ref. [53], in particular section 6.3.3 therein, where a time-scale $\tau \sim 1/T_{bh}$ is proposed for the decay of a perturbed black hole to a stationary state [53]\textsuperscript{14}. Aspects of irreversible black hole thermodynamics are also explored in Ref. [54] (where a formation time-scale again order of $1/T_{bh}$ appears) and in particular in Ref. [55,56]. We don’t develop herein an irreversible thermodynamics formalism for black holes, which should be the subject of further investigations.

Israel’s result implies that thermodynamic formalism involving quasi-static processes cannot be extended to the extremal states because, from a physical point of view, approximating ideal quasi-static processes by means of (roughly) very slow processes starts loosing sense near the extremal states due to the infinity in the relaxation time. Equilibrium thermodynamic formalism gives rise to a consistent description of very slow processes only if the relaxation times of various parameters defining the equilibrium state are much bigger or much smaller than the measurement time. In the former case the the parameters get a constant value, in the latter they get their equilibrium value [16]. The case of a measurement time of the same order as the relaxation time is critical [16]. In the case of a black hole, a measurement time suitable in order to measure the reaching of a black hole equilibrium state should be much longer than the formation time. For the case of an extremal black hole state attained by means of a continuous process, there is no satisfactory measurement time because at best an infinite measurement time should be compared with an infinite relaxation (formation) time. There is so an intrinsic inaccessibility in phase space of extremal states if they have to be reached by means of a continuous quasi-static process. Other considerations about the failure of thermodynamics near the extremal limit can be found in Ref. [17].

We can now implement a better comparison of Landsberg’s hypothesis c) with the actual behavior of black holes near the extremal limit as dictated by Ref. [4]. The infinity of the time required in order to get an extremal state suggests that the dynamic process meets some hindrance near the extremal state to be carried further on in a finite

\textsuperscript{13}The process (NE)\textsubscript{1} \rightarrow (NE)\textsubscript{2} between non extremal states (NE)\textsubscript{1} and (NE)\textsubscript{2} can occur in a finite time.

\textsuperscript{14}Note that also the “mining process” time-scale of Ref. [43] is of the same order.
time, if the process is “continuous”. Qualitatively, this is e.g. suggested by the fact that adding charge to a non-extremal Reissner–Nordström black hole becomes more and more difficult due to the increase in the electrostatic repulsion. Cf. also Appendix A. The impossibility to increase the black hole charge/angular momentum in a finite time till the extremality condition is implemented just resembles Landsberg’s suggestion [13] of a discontinuity near the absolute zero (violation also of the hypothesis c)), but it involves an intrinsically dynamical information (as the infinite time required in order to implement the process). The peculiar characteristic of black hole thermodynamics, from the point of view of the third principle, is that “infinity of processes” of the standard formulation for (U) is substituted with “infinity of time”. Actually the latter formulation seems more general than the former, in the sense that “infinity of processes” can easily imply “infinity of time” (but obviously not vice versa). We note also that, in the black hole case, the failure of thermodynamic formalism at the extremal limit is also to be attributed in a very subtle way to dynamic reasons and not only maybe to finite size effects. In a rephrasing that goes parallel with respect to Einstein suggestion for the case of standard thermodynamics [11], black hole thermodynamics near extremal states becomes irreducibly irreversible.

A final comment can be made concerning (U) according to Israel and (U) as required by the second law in X A. The two versions are in some sense equivalent if one assumes the following conjecture: The gravitational entropy of black holes is non-zero and given by Bekenstein–Hawking law only in presence of trapped surfaces. Then Israel’s proof would be a dynamic implementation of the second law-based hindrance to reach extremal states shown in X A.

XII. THE FAILURE OF (N) IN BLACK HOLE THERMODYNAMICS AND DIAGNOSTIC OF THE DEGREES OF FREEDOM INVOLVED IN BLACK HOLE ENTROPY

One can also suggest the conjecture that black hole entropy is associated with the presence of trapped surfaces. This hypothesis, together with the above thermodynamic analysis, enforces the conjecture interpreting black hole entropy as “entanglement entropy” [57]. In fact, the “entanglement entropy”, which is obtained by tracing the von Neumann entropy over unavailable degrees of freedom, can be associated with a trapped surface (the trace should be taken over the quantum field modes contained in the trapped surface). Moreover, for an “entanglement entropy” satisfying the area law there is no reason why it should approach a constant (zero) value as the extremal states are approached, because the area of the trapping surface depends on geometric parameters and it does not vanish near the extremal states, no matter how near the extremal boundary the black hole could be. Even the discontinuity of $S$ at the extremal boundary could be justified (no trapped surface would mean zero entanglement entropy).

We don’t want to claim that “entanglement entropy” is mandatory, we simply limit ourselves to note that it seems to have chances to match both the failure of (N) and the validity of (U) in black hole thermodynamics. The alternative view to consider the failure of (N) as due to “frozen non-equilibrium states” in black hole thermodynamics does not seem to be plausible. See a discussion in Appendix B.

It is also more and more evident that black holes represent, in many fundamental respects, systems completely exceptional from a thermodynamic point of view, so that the violation of (N) they give rise cannot be considered a counter-example to the Nernst heat theorem of standard thermodynamics.

CONCLUSIONS

We have analyzed thermodynamic arguments supporting the idea of a discontinuity in thermodynamics occurring between the manifold of non-extremal black holes and its extremal black hole boundary. An analysis of the failure of the implication $(U) \Rightarrow (N)$ in black hole thermodynamics has revealed a special status to $(U)$ in black hole thermodynamics. The analysis of the limit $T \to 0$ for large black hole masses, where $(U)$ is necessarily implemented, has shown that $(N)$ is satisfied if the black hole is uncharged. In the branch $M \to \infty$ the failure of $(N)$ vs. the validity of $(U)$ in the charged case has to be considered as a particular property of black hole thermodynamics.

In the framework of Carathéodory’s approach to thermodynamics, we have point out which kind of problems can arise if the Bekenstein-Hawking law holds also for extremal states. We have shown that, by requiring that the entropy of extremal black holes is zero, one can support $(U)$ from a thermostatic point of view. A remark is that finding e.g. $S = 0$ for extremal states does not mean that $(N)$ is valid, unless such a result is corroborated by a limit approach as $T \to 0^+$. In fact $(N)$ gets, as a matter of facts, its real

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15For the value of the constant see also Ref. [28].
meaning and its real experimental verifications in standard thermodynamics if it is intended not as the behavior of the entropy at exactly $T = 0$, but as the limit of entropy differences as $T \to 0^+$. The result of Ref. [4] (the appropriate frame is a dynamic one and an irreversible thermodynamic one) has been interpreted as a strong corroboration for the picture in which extremal states are separated by a discontinuity with respect to non-extremal ones.

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**APPENDIX A: ADIABATIC TRANSFORMATIONS IN BLACK HOLE THERMODYNAMICS**

We determine the equations for reversible adiabatic transformations in black hole thermodynamics. In the non-extremal cases they correspond to isoareal transformations.

a. Reissner–Nordström case

In order to get the equation for adiabatic transformations for non-extremal black holes it is sufficient to impose the obvious constraint

$$ A = A_0 = \text{const.} $$

and to solve it with respect to e.g. the variable $Q$ in the Reissner–Nordström case. This is equivalent to imposing

$$ dA = \left( 4 \pi \frac{r_+}{k \ M} \right) d(M^2) - \left( 4 \pi \frac{\Phi}{k \ Q} \right) d(Q^2) = 0 $$

that is $dS = 0$. By defining of the parameters of an arbitrary initial state $(M_0, Q_0)$ with $M^2_0 > Q^2_0$ and its event horizon radius $r^0_+$, the equation for isoentropic transformations of a Reissner–Nordström non-extremal black hole can be written as follows:

$$ Q^2 = 2 \ M \ r_+^0 - (r^0_+)^2. $$

Note that the condition $Q^2 < M^2$ is automatically implemented. The above equation for adiabatic transformations in the Reissner–Nordström case holds for $M \in (r^0_+ / 2, r^0_+)$ and it is equivalent to the one given in Ref. [29]. Cf. figure 3(a) in Ref. [3]. Note also that the equation of the transformation holds for $Q^2 \geq M^2$ (attainment of the extremal state). By means of an adiabat one can arbitrarily approach an extremal state with $Q^2 = M^2$ and reach it unless some discontinuity occurs. The condition to be satisfied is easily found. One has to impose $r^0_+ = A_1 = | Q_1 | = r^0_+$, that is, the extremal black hole has the same area as the starting non-extremal one. Moreover, it is simple to show that $M_1 > M_0; | Q_1 | > | Q_0 |$, that is, the mass and the absolute value of the charge have to increase. In order to reach the extremal black hole state having the same area as the initial black hole case, one has to add smoothly charge and mass in the black hole and so qualitatively it becomes more and more difficult to approach the extremal state because of the increased electric repulsion, that is, it becomes more and more difficult to implement the isoareal transformation. Cf. also Ref. [47].

We can conclude that in the space of parameters $M^2, Q^2$ the adiabatic transformations for non-extremal black holes are represented by curves that can arbitrarily approach extremal states and different non-extremal black holes (e.g. with different values of $r^0_+$) approach different extremal states with different values for $A$. It is also interesting to note that the above isoareal curves are tangent to the extremal line $Q^2 = M^2$. The intersection is in some sense “mild”. In fact, by posing $M^2 = x; Q^2 = y$ one can easily verify that

$$ y = 2 \ r_+^0 \ \sqrt{x} - (r_+^0)^2 $$

are tangent for $x = (r_+^0)^2$. It appears that, without some kind of discontinuity, one cannot implement (U) because by means of an isoentropic process one can reach the extremal states.
Concluding this section, we note that there are transformations which intersect the extremal manifold in a less smooth way. The $Q^2 = \text{const.}$ transformations approach the extremal states as the mass decreases (they involve “pure evaporation” of the black hole, without quanta emission related with the Klein paradox [54,48,58]); there are also transformations which, in the thermodynamic manifold, are orthogonal to the extremal manifold; they have equation

$$Q^2 = - M^2 + 2 M_0^2,$$

where $M^2 \in [M_0^2, 2M_0^2)$. The extremal state is reached for $M = M_0$. The charge increases as the mass decreases to the extremal limit.

### b. Kerr case

It is completely analogous to the Reissner-Nordström case. By means of the geometric parameters $(M_0, J_0)$ and the event horizon radius $r_+^0$ of an arbitrary initial state we find

$$J^2 = M_0 r_+^0 (2 M^2 - M_0 r_+^0).$$

with $M^2 \in (M_0 r_+^0/2, M_0 r_+^0)$. Cf. also figure 4(a) in Ref. [3]. Also in this case it is possible to approach arbitrarily extremal states. In fact, the only condition to be implemented is that the horizon radius of the extremal state is related to the one of the initial state by

$$r_1 = M_0 r_+^0.$$

This condition ensures the two states to be isoareal. Obviously the same considerations as for the Reissner-Nordström case hold. Again, the extremal line and the above isoareal line are tangent at $M^2 = M_0 r_+^0$.

### c. Kerr-Newman case

Adiabatic transformations for non-extremal black holes satisfy the equation

$$A = A_0 = \text{const.}$$

that in the general case of a Kerr-Newman black hole becomes

$$r_+^2 + \frac{J^2}{M^2} = C$$

where $C = A_0/(4 \pi)$ is a positive constant. If one defines $x \equiv M^2; y \equiv Q^2; z \equiv J^2$ then the above equation is equivalent to the following one:

$$2 x - y + 2 \sqrt{x} \sqrt{x - \frac{z}{x} - y} = C.$$

One can solve e.g. for $z$ and find

$$z = C x - \frac{1}{C} (y + C)^2.$$

It is easy to show that $x \in (x_L^2/C, x_L)$, where $x_L \equiv (y + C)/2$. Also in the general case it is possible to approach extremal Kerr-Newman states. The extremal sub-manifold is defined by

$$z_E = x^2 - x y$$

and its intersection with the above isoareal surface takes place at

$$x - \frac{1}{2} (y + C) = 0 \iff x = x_L.$$  \hspace{1cm} (A1)
Also in the general case the extremal state surface and the isoareal surface are tangent. In fact, their tangent planes coincide along (A1).

We can conclude that there are two classes of adiabatic transformations: 1) standard adiabatic transformations for non–extremal black holes; they are isoareal and isoentropic; 2) “extremal” adiabatic transformations that in general are not isoareal. In the Kerr-Newman extremal case it is possible to allow also for isoareal extremal transformations that can be obtained by imposing $dA_E = 0$. Their equation is

$$Q^4 + 4J^2 = \left(\frac{A_0}{4\pi}\right)^2$$

They represent a nontrivial sub-manifold of the general extremal case. If $J = 0$ or $Q = 0$ one gets that this manifold becomes a single point.

**APPENDIX B: GLASSY SYSTEMS, FROZEN EQUILIBRIUM AND (N)**

It is well known that a wide discussion about the validity of (N) was given rise by some physical systems that seemed to violate Nernst’s postulate of isoentropy of the zero–temperature states. Actually, it has been shown that these peculiar systems (e.g. CO and glassy substances) don’t satisfy the condition of internal equilibrium, that is, near $T = 0$ some degrees of freedom remain frozen in a non-equilibrium meta-stable configuration [11,23,59]. Elements of configurational disorder can remain unchanged during the cooling down of the system towards a low temperature. The relaxation time to a condition of inner equilibrium is much bigger than the measurement time and it can be effectively infinite. A residual molar entropy is then allowed at $T = 0$. Long-time measurements have shown the convergence to (N) of the calorimetric entropy for some substances violating (N).

Thus, a further hypothesis has been added in order to ensure the validity of (N), which is the condition of internal equilibrium [11,59]. There is no definitive agreement about this hypothesis, indeed some authors reject it, see e.g. Ref. [22]. A strongly different position appears in Ref. [60], where the entropic version of Nernst’s theorem has been raised to the role of a fundamental operational tool for the definition of thermodynamic equilibrium and has so a key conceptual part in the approach to thermodynamics developed therein.

In the black hole case, it is still difficult to find out a definitive notion of “internal states” and of micro–states. “Meta-stability of non–equilibrium states” and “frozen-in disorder” can hardly justify the violation of (N) in the case of black holes. The difference with respect to the case of the apparent violation of (N) in “glassy systems” having finite relaxation times is evident, indeed by increasing the measurement time no convergence to the implementation of (N) can be expected for black holes. Anyway, see also Ref. [61].

**APPENDIX C: FAILURE OF CONCAVITY AND (U)$\Leftrightarrow$(N)**

In the following, we analyze what happens if condition a) is relaxed. In fact, in black hole thermodynamic a) is not satisfied. For simplicity of notation, we substitute $S_{T,x}$ for $S(T,x^1,\ldots,x^n)$ and $C_x$ for $C_{x^1,\ldots,x^n}$.

1. relaxing condition a) against (U)$\Rightarrow$(N)

The presence of heat capacities with opposite sign can invalidate the proof of the double implication (U)$\Leftrightarrow$(N). Let us assume then that (U) is satisfied and that there exist isometric curves (i.e. isometric transformations) reaching $T = 0$ such that some have $C_x > 0$ and other $C_y < 0$. Note that the presence of heat capacities $C_y < 0$ at constant deformation parameters means the failure of the standard concavity properties of the entropy. Then, this non–uniformity of the sign of the heat capacities along transformation reaching the absolute zero allows to violate the implication (U)$\Rightarrow$(N), and, moreover, it seriously jeopardizes the identification of (U) with the absence of isoentropic reaching $T = 0$. See fig. 5. Let us first consider the case where along different isometric transformations with opposite signs of the corresponding heat capacities and starting from $T = 0$ it is possible to reach the same isoentropic surface. Let us define

$$S_{T_1,x} = S_{0,x} + \int_0^{T_1} \frac{C_x}{T}dT > S_{0,x}$$

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\[ S_{T_2,y} = S_{0,y} - \int_0^{T_2} \frac{|C_y|}{T} dT < S_{0,y}. \]

If \( S_{T_1,x} = S_{T_2,y} \) and if \( T_1, T_2 > 0 \), the equality
\[ S_0 + \int_0^{T_1} \frac{C_x}{T} dT = S_0 - \int_0^{T_2} \frac{|C_y|}{T} dT \]
is obviously impossible. The only possibility is that \( T_1 = 0 = T_2 \), but, then, (U) cannot be implemented in general as “absence of isoentropic transformation reaching \( T = 0 \)”, except for a \( S - T \) diagram of the type sketched in fig. 6.

Note that this reasoning concerning the non-uniformity of signs of heat capacities for transformations connected to \( T = 0 \) can be easily extended to the case where generic curves \( \gamma^0 \) which arrive at \( T = 0 \) and having \( C_x > 0 \) are allowed.

FIG. 5. Isoentropes approaching \( T = 0 \) at finite parameters exist and (U) is ensured by a discontinuity in the sense of Landsberg. (N) is allowed only in case (a).

If, instead, no intersection to the same isoentrope is possible, then, again a violation of (N) can occur. In the following figures, some possible diagrams are sketched. They imply a violation of b) and/or of c).

As we have shown, (N) can be allowed for only if the starting isoentropic coincides with the zero–temperature one (cf. fig. below), but, if \( S \geq 0 \) is assumed, then at least Planck’s restatement of (N) has to be violated.

FIG. 6. A multi–branches structure is allowed. In all cases, (U) is ensured without invoking discontinuities near the absolute zero. In (a), in spite of the presence of \( C_y < 0 \) connected to \( T = 0 \), (N) holds. In (b) no isoentrope is intersected both by curves reaching \( T = 0 \) and having \( C_x > 0 \) and by curves reaching \( T = 0 \) with \( C_y < 0 \). In (c), such an isoentrope exists but no isoentrope reaching the absolute zero is allowed. In (b) and in (c) the entropic version (N) is violated.

2. relaxing condition a) against (N)⇒(U)

It is also easy to deduce that the existence of paths with opposite sign near \( T = 0 \) could be reconciled with (N) without implying (U). In fact, if \( C_x > 0, C_y < 0 \) then
\[ S_0 + \int_0^{T_1} \frac{C_x}{T} dT > S_0 \]
\[ S_0 - \int_0^{T_2} \frac{|C_y|}{T} dT < S_0 \]
and, in absence of suitable multi-branching (hypothesis b)), the isentropic $S = S_0$ allows to get $T = 0$. In J.C.Wheeler’s papers [62] another counter-example to (N)$\Rightarrow$(U) is shown. See fig. 7. The system studied therein displays a particular behavior, in the sense that $S = 0$ is attained at $T_1 > 0$ and $C_p = 0$ for $0 \leq T \leq T_1$ is allowed. Condition a) is then violated. According to standard proofs, $S = S_0$ cannot be attained at $T > 0$.

![FIG. 7. Two examples of violation for the implication (N)$\Rightarrow$(U). (a) Paths having $C < 0$ and reaching $T = 0$ are allowed; (b) A behavior like the one of Wheeler’s counter-example is displayed (the original counter-example requires $S_0 = 0$).](image)
