Constraint structure of O(3) nonlinear sigma model revisited

Soon-Tae Hong\textsuperscript{1}, Yong-Wan Kim\textsuperscript{2}, Young-Jai Park\textsuperscript{2} and Klaus D. Rothe\textsuperscript{3}

\textsuperscript{1}Department of Science Education  
Ewha Womans University, Seoul 120-750, Korea

\textsuperscript{2}Department of Physics and Basic Science Research Institute,  
Sogang University, C.P.O. Box 1142, Seoul 100-611, Korea

\textsuperscript{3}Institut für Theoretische Physik,  
Universität Heidelberg, Philosophenweg 16, D-69120 Heidelberg, Germany

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ABSTRACT

We study the constraint structure of the O(3) nonlinear sigma model in the framework of the Lagrangian, symplectic, Hamilton-Jacobi as well as the Batalin-Fradkin-Tyutin embedding procedure.

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1 Introduction

Since the (2+1) dimensional O(3) nonlinear sigma model (NLSM) was first discussed [1, 2], there have been many attempts to improve this soliton model associated with the homotopy group $\pi_2(S^2) = \mathbb{Z}$. Moreover, the O(3) NLSM with the Hopf term [3], as well as the $CP^1$ model associated with the O(3) NLSM via the Hopf map projection from $S^3$ to $S^2$, were canonically quantized [4, 5, 6] in the standard Dirac formalism [7]. Furthermore, the geometrical constraints involved in the soliton models such as the Skyrmion models [8, 9], O(3) NLSM [10] and $CP^1$ model [11] have recently been analyzed in the framework of the Batalin-Fradkin-Tyutin (BFT) scheme [12, 13, 14], to obtain the corresponding first-class Hamiltonians and BRST [15] invariant effective Lagrangians [16, 17].

On the other hand, an alternative symplectic scheme [18] has been proposed for treating first-order Lagrangians. This scheme has been applied to various models [19, 20, 21, 22] and has recently been extended to include BRST invariant embeddings [23]. Moreover, since the Hamilton-Jacobi (HJ) scheme [24] has first been applied to constrained systems [25], there have been further developments in this direction to quantize singular systems [26, 27, 28]. In particular, the HJ description of second-class constrained systems, which posed some problems in previous work [29], has been shown to be complete after addition of suitable integrability conditions, corresponding to the Dirac consistency conditions, requiring time independence of the constraints [28]. Recently the O(3) NLSM was also considered in an extended HJ scheme [27] based on the extension of the second-class fields to the first-class fields previously introduced in ref. [10]. However, this extension of the HJ scheme continues to involve some subtleties still to be resolved.

In this paper, we will revisit the O(3) NLSM and examine the relation among the Lagrangian, symplectic, HJ and BFT embedding schemes for this model. In section 2 we will analyze the constraint structure of this model from the point of view of the Lagrangian and symplectic algorithm. In section 3, we will investigate the HJ quantization of the O(3) NLSM taking account of the integrability condition. In section 4 we then discuss the BFT quantization scheme for this model, and conclude in section 5.
2 NLSM in Lagrangian and symplectic schemes

In this section, we consider the Lagrangian and symplectic algorithms to generate the constraint structure of the O(3) NLSM, defined by the Lagrangian

\[ L = \frac{1}{2f} (\partial_\mu n^a)(\partial^\mu n^a) + n^0(n^a n^a - 1), \] (2.1)

where \( n^a \) (\( a = 1, 2, 3 \)) is a multiplet of three real scalar fields which parameterize an internal space \( S^2 \), and \( n^0 \) is a Lagrange multiplier field implementing the geometrical constraint \( n^a n^a - 1 = 0 \). From the Lagrangian (2.1) the canonical momenta conjugate to the field \( n^0 \) and the real scalar fields \( n^a \) are given by

\[ \pi^0 = 0, \quad \pi^a = \frac{1}{f} \partial_\mu n^a, \] (2.2)

to yield the canonical Hamiltonian

\[ H^{(0)} = \frac{f}{2} \pi^a \pi^a + \frac{1}{2f} (\vec{\nabla} n^a)^2 - n^0(n^a n^a - 1). \] (2.3)

Following ref. [18], we rewrite the second-order Lagrangian (2.1) in first-ordered form as

\[ L^{(0)} = \pi^a \partial_\mu n^a - H^{(0)}, \] (2.4)

where \( L^{(0)} \) is to be treated as a function of the Lagrangian variables \( (n^0, n^a, \pi^a) \), and \( H^{(0)} \) is the canonical Hamiltonian (2.3). The original second-order formulation is recovered on shell, upon using the equation of motion for \( \pi^a \). Note that since \( H^{(0)} \) depends on \( \pi^a \) and \( n^a \), but not on their time derivatives, it can be regarded as the (level zero) symplectic potential. The Lagrangian is of the form

\[ L^{(0)} = a_\alpha \partial_\beta \xi^\alpha - H^{(0)}, \] (2.5)

where we have denoted the set of the symplectic variables \( (n^a, \pi^a, n^0) \) collectively by \( \xi^\alpha \) and the corresponding conjugate momenta by \( a_\alpha \). The dynamics of the model is then governed by the symplectic two-form matrix:

\[ F^{(0)}_{\alpha \beta}(x, y) = \frac{\partial a_\beta(y)}{\partial \xi^\alpha(x)} - \frac{\partial a_\alpha(x)}{\partial \xi^\beta(y)}, \] (2.6)
via the equations of motion

\[ \int d^2y \ F^{(0)}(x, y) \partial_0 \xi(y) = K^{(0)}(x), \]  

(2.7)

where

\[ K^{(0)}(x) = \frac{\delta}{\delta \xi^\alpha(x)} \int d^2y \mathcal{H}_0(y) = \begin{pmatrix} -2n^0(x)n^a(x) - \frac{1}{4} \nabla^2 n^a(x) \\ f \pi^a(x) \\ -n^a(x)n^a(x) + 1 \end{pmatrix}. \]  

(2.8)

In the O(3) NLSM the symplectic two-form matrix is given by

\[ F^{(0)}(x, y) = \begin{pmatrix} O & -I & \tilde{0} \\ I & O & \tilde{0} \\ \tilde{0}^T & \tilde{0}^T & 0 \end{pmatrix} \delta^2(x - y), \]  

(2.9)

where \( O \) and \( I \) stand for \( 3 \times 3 \) null and identity matrices, respectively, and the superscript \( T \) denotes “transpose”. The zero-level symplectic two-form has a zero mode \( \nu^{(0)}(x) = (\tilde{0}, \tilde{0}, 1) \delta^2(x - y) \), which generates the constraint

\[ \int d^2y \ \nu^{(0)}(y) K^{(0)}(y) = -\Omega_1(x) = 0 \]  

(2.10)

with

\[ \Omega_1 = n^a n^a - 1. \]  

(2.11)

From here on we may proceed in two ways.

i) **Lagrangian algorithm**

In the Lagrangian algorithm, we add the time derivative of the constraint \( \Omega_1 = 0 \) to the Lagrange equations of motion (2.7). This leads to the following enlarged set of equations (for a more detailed discussion see ref. [21]),

\[ \int d^2y \ W^{(1)}(x, y) \partial_0 \xi(y) = K^{(1)}(x) \]  

(2.12)

\[ ^1 \text{For notational convenience we will omit component indices for vectors and matrices except cases where confusion may arise.} \]

\[ ^2 \text{The superscript} T \text{ on the vectors is implied.} \]
where $W^{(1)}(x, y)$ are now the elements of a rectangular matrix

$$W^{(1)}(x, y) = \begin{pmatrix} F^{(0)}(x, y) \\ M_1(x, y) \end{pmatrix}$$

with

$$M_{1\alpha}(x, y) = \frac{\partial \Omega_1(x)}{\partial \xi^\alpha(y)} = (2\bar{n}(x), \bar{0}, 0)\delta^2(x - y)$$

and

$$K^{(1)}(x) = \begin{pmatrix} K^{(0)}(x) \\ 0 \end{pmatrix}.$$  \hspace{1cm} (2.15)

The matrix $W^{(1)}$ has two left zero modes, of which one just reproduces the previous constraint, while the other

$$\nu^{(1)}_y(x) = (\bar{0}, 2\bar{n}(x), 0, -1)\delta^2(x - y)$$

yields the new constraint

$$\Omega_2 = n^a\pi^a = 0.$$  \hspace{1cm} (2.17)

We proceed in this fashion, and add the time derivative of this constraint to the previous equations of motion, which now read

$$\int d^2y \ W^{(2)}(x, y)\partial_0\xi(y) = K^{(2)}(x)$$  \hspace{1cm} (2.18)

where $W^{(2)}(x, y)$ are now the elements of a rectangular matrix

$$W^{(2)}(x, y) = \begin{pmatrix} F^{(0)}(x, y) \\ M_1(x, y) \\ M_2(x, y) \end{pmatrix},$$

with

$$M_{2\alpha}(x, y) = \frac{\partial \Omega_2(x)}{\partial \xi^\alpha(y)} = (\bar{\pi}(x), \bar{n}(x), 0)\delta^2(x - y)$$

and

$$K^{(2)}(x) = \begin{pmatrix} K^{(0)}(x) \\ 0 \\ 0 \end{pmatrix}.$$  \hspace{1cm} (2.21)
Now, $W^{(2)}$ is found to have three zero modes, of which two reproduce the previous constraints, while the zero mode
\[\nu_{y}^{(2)}(x) = (\vec{n}(x), -\vec{\pi}(x), 0, 0, 1)\delta^{2}(x - y)\] (2.22)
leads to a further constraint:
\[\Omega_{3} = 2n^{0}n^{a}\pi^{a} + f\pi^{a}\pi^{a} + \frac{1}{f}n^{a}\vec{\nabla}^{2}n^{a} = 0.\] (2.23)

Extending the matrix $W^{(2)}$ to $W^{(3)}$ in the manner described above, we find that there are no new zero modes, and the iterative algorithm terminates at this point. We have thus four constraints, of which two constraints $\Omega_{i} = 0$ ($i = 1, 2$) represent constraints among the variables $n^{a}$ and $\pi^{a}$ alone.

As one readily checks, the constraints are found to completely agree with the constraints generated by the conventional Dirac algorithm, departing either from the original second-order Lagrangian (2.1) or from the first-order Lagrangian (2.4).

ii) Symplectic algorithm

In the symplectic algorithm of ref. [19] we add the constraint $\Omega_{1}$ to the canonical sector of the Lagrangian (2.4) in the form $-\Omega_{1}\partial_{0}\rho$, thereby enlarging the symplectic phase space via the addition of a dynamical field $\rho$, which enforces the stability of the constraint $\Omega_{1}$ in time. The once iterated first-level Lagrangian is then given as follows
\[L^{(1)} = \pi^{a}\partial_{0}n^{a} - \Omega_{1}\partial_{0}\rho - \mathcal{H}^{(0)}.\] (2.24)

In the spirit of ref. [19] we absorb the Lagrange multiplier field $n^{0}$ in $\mathcal{H}^{(0)}$ into the new dynamical variable $\rho$ by making the substitution $\partial_{0}\rho - n^{0} \rightarrow \partial_{0}\rho$. This amounts to the replacement
\[L^{(1)} \rightarrow L^{(1)} = \pi^{a}\partial_{0}n^{a} - \Omega_{1}\partial_{0}\rho - \mathcal{H}^{(1)}.\] (2.25)

where $\mathcal{H}^{(0)}$ in Eq. (2.24) has been replaced by
\[\mathcal{H}^{(1)} = \frac{f}{2}\pi^{a}\pi^{a} + \frac{1}{2f} (\vec{\nabla}n^{a})^{2}.\] (2.26)

\[\text{The minus sign is included in order to align our results with those of the Lagrangian algorithm.}\]
Note that at this stage the field $n^0$ has completely disappeared, and any information concerning it has been lost. In fact, the model that we are discussing at this point is the model considered in ref. [27], and hence is no longer the O(3) NLSM in that sense.

The new equations of motion now read

$$\int d^2 y \ F^{(1)}(x, y) \partial_0 \xi^{(1)}(y) = K^{(1)}(x) = \begin{pmatrix} -\frac{1}{f} \nabla^2 n^a(x) \\ f \pi^a(x) \\ 0 \end{pmatrix}$$

(2.27)

where

$$\xi^{(1)}(x) = (\vec{n}(x), \vec{\pi}(x), \rho(x))$$

(2.28)

and the first-iterated symplectic two-form is now given by

$$F^{(1)}(x, y) = \begin{pmatrix} O & -I & -2\vec{n}(x) \\
I & O & \vec{0} \\
2\vec{n}(x)^T & \vec{0}^T & 0 \end{pmatrix} \delta^2(x - y).$$

(2.29)

Note that the symplectic two-form matrix still has a zero mode $\nu^{(1)}_y(x) = (\vec{0}, 2\vec{n}(x), 1) \delta^2(x - y)$, which generates the constraint $\Omega_2$ in the context of the symplectic formalism [18] as follows

$$\int d^2 y \ \nu^{(1)}_y(x) K^{(1)}(y) = 2f \Omega_2(x) = 0,$$

(2.30)

Following ref. [19] we further enlarge the symplectic phase space to include the constraint $\Omega_2$ via an additional dynamical field $\sigma$, as follows,

$$L^{(2)} = \pi^a \partial_0 n^a - \Omega_1 \partial_0 \rho - \Omega_2 \partial_0 \sigma - H^{(2)},$$

(2.31)

with $H^{(2)} = H^{(1)}$. The symplectic variables $\xi^{(2)}$ are now given by

$$\xi^{(2)}(x) = (n^a(x), \pi^a(x), \rho(x), \sigma(x)),$$

(2.32)

and the second-iterated symplectic two-form matrix is given by

$$F^{(2)}(x, y) = \begin{pmatrix} O & -I & -2\vec{n}(x) & -\vec{\pi}(x) \\
I & O & \vec{0} & -\vec{n}(x) \\
2\vec{n}(x)^T & \vec{0}^T & 0 & 0 \\
\vec{\pi}(x)^T & \vec{n}(x)^T & 0 & 0 \end{pmatrix} \delta^2(x - y).$$

(2.33)
We easily show that this symplectic matrix is invertible, i.e. no zero modes exist. Hence the algorithm ends at this point.

Although the constraints $\Omega_i = 0$ ($i = 1, 2$) are identical with those generated by the Lagrangian algorithm, a complete analysis of the NLSM should have treated the Lagrange multiplier field $n^0$ as an additional degree of freedom, by not performing the shift in variable $\partial_0 \rho - n^0 \rightarrow \partial_0 \rho$. As shown in ref. [21] the symplectic algorithm described above does not generate in this case the complete set of constraints as given by the Lagrangian algorithm, which rests on a solid foundation. The reason for this failure has been examined in ref. [21] for the corresponding quantum mechanical model, as well as in general terms.

### 3 NLSM in Hamilton-Jacobi scheme

In this section we revisit the O(3) NLSM in the HJ scheme, for which the generalized HJ partial differential equations are given by

$$
\mathcal{H}'_0 = p^0 + \mathcal{H}_0 = 0,
$$

$$
\mathcal{H}'_1 = \pi^0 + \mathcal{H}_1 = 0,
$$

(3.1)

where $\mathcal{H}_0$ is a canonical Hamiltonian density $\mathcal{H}^{(0)}(n^0, n^a, \pi^a)$ in Eq. (2.3) and $\mathcal{H}_1$ is the (only) primary constraint in the Dirac terminology [7] ($\mathcal{H}_1$ is actually zero for the case of the O(3) NLSM, as seen from Eq. (2.2)).

Following the generalized HJ scheme [23, 28], one obtains from Eq. (3.1),

$$
der_q^\underline{\alpha} = \frac{\partial \mathcal{H}'_\underline{\alpha}}{\partial p^\underline{\alpha}} dt_\underline{\alpha},
$$

$$
der_p^\underline{\alpha} = -\frac{\partial \mathcal{H}'_\underline{\alpha}}{\partial q^\underline{\alpha}} dt_\underline{\alpha},
$$

(3.2)

where $q^\underline{\alpha} = (t, n^0, n^a)$, $p^\underline{\alpha} = (p^0, \pi^0, \pi^a)$ and $t_\underline{\alpha} = (t, n^0)$ with $\underline{\alpha} = (0, 1)$. Explicitly

$$
\partial_0 n^0 = \partial_0 n^0, \quad \partial_0 n^a = f \pi^a,
$$

$$
\partial_0 \pi^0 = n^a n^a - 1, \quad \partial_0 \pi^a = \frac{1}{f} \vec{\nabla}^2 n^a + 2 n^0 n^a.
$$

(3.3)

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4This scheme was treated incompletely in ref. [27]. With the identification $\pi^a(x) = \frac{\delta S}{\delta n^a(x)}$ and $p^0 = \frac{\delta S}{\delta t}$, these equations are the (incomplete set of) Hamilton-Jacobi partial differential equations for the Hamilton principal function $S$. 


Since the equation for $n^0$ is trivial, one cannot obtain any information about the variable $n^0$ at this level, and the set of equations is not integrable at this stage.

In order to fix this deficit of information, we supplement Eq. (3.1) with the generalized integrability conditions [23, 28],

$$\partial_0 H'_{\bar{\alpha}} = \{H'_{\bar{\alpha}}, H'_0\} + \{H'_{\bar{\alpha}}, H'_\beta\} \partial_0 q^\beta = 0, \quad (3.4)$$

where, unlike in the usual case, the Poisson bracket is defined in terms of the extended index $a$ corresponding to $q^a = (t, n^0, n^a)$ as follows

$$\{A, B\} = \frac{\partial A}{\partial q^a} \frac{\partial B}{\partial p^a} - \frac{\partial B}{\partial q^a} \frac{\partial A}{\partial p^a}. \quad (3.5)$$

The index $\beta$ in (3.4) labels the primary constraints, whose number is only one for the case in question ($\partial_0 q^\beta \rightarrow \partial_0 n^0$). The index $\bar{\alpha}$, on the other hand, labels the primary constraints as well as the secondary constraints which emerge iteratively from the integrability condition (3.4). Thus for the case in question we have three secondary constraints, $H'_2 = 0$, $H'_3 = 0$ and $H'_4 = 0$ corresponding to Eqs. (2.11), (2.17) and (2.23), respectively. They emerge iteratively from Eq. (3.4) as follows:

$$\partial_0 H'_0 = -H'_2 \partial_0 n^0, \quad \partial_0 H'_1 = H'_2, \quad \partial_0 H'_2 = 2fH'_3, \quad \partial_0 H'_3 = H'_4,$$

with

$$H'_2 = n^a n^a - 1, \quad H'_3 = n^a \pi^a, \quad H'_4 = 2n^0 n^a n^a + f n^a \vec{\nabla}_2 n^a + \frac{1}{f} n^a \vec{\nabla} n^a. \quad (3.6)$$

From $\partial_0 H'_4 = 0$ we have, using $H'_2 = H'_3 = 0$ as well as (3.3),

$$\partial_0 n^0 = -\pi^a \vec{\nabla}_2 n^a + \vec{\nabla} n^a \cdot \vec{\nabla} \pi^a \quad (3.7)$$

which replaces the identity $\partial_0 n^0 = \partial_0 n^0$ in Eq. (3.3), and thus completes the set of Hamilton-Jacobi equations to an integrable system.
Finally note that Eqs. (3.2) and (3.6) imply for the Hamilton principal function

\[
dS = \int d^2x \left( -\mathcal{H}_0 + \pi^a \frac{\partial \mathcal{H}'}{\partial \pi^a} \right) dt,
\]

\[= dt \int d^2x \left( -\mathcal{H}_0 + \pi^0 \frac{\partial n^0}{\partial t} + \pi^a \frac{\partial n^a}{\partial t} \right). \quad (3.8)
\]

Since we have now from Eqs. (3.3) and (3.7) a complete set of equations of motion for \(n^a\) and \(n^0\), Eq. (3.8) can be integrated in time to yield the standard action

\[
S = \int d^3x \mathcal{L}^{(0)} \quad (3.9)
\]

where \(\mathcal{L}^{(0)}\) is the first-order Lagrangian (2.4). Note that it was only after taking account of the secondary constraints generated iteratively by the integrability condition (3.4), that one could construct the action (3.9) for the second-class system in question.

### 4 BFT Hamiltonian embedding

In this section we reconsider the BFT quantization for the O(3) NLSM with the Lagrangian (2.1) and the canonical Hamiltonian (2.3). The primary constraint reads

\[
\Omega_0 = \pi^0 \approx 0, \quad (4.1)
\]

and correspondingly we have for the total Hamiltonian,

\[
\mathcal{H}_T = \mathcal{H}_0 + v\Omega_0, \quad (4.2)
\]

where \(\mathcal{H}_0\) is the canonical Hamiltonian \(\mathcal{H}^{(0)}\) in Eq. (2.3). The usual Dirac algorithm is readily shown to lead recursively to the constraints \(\Omega_i (i = 0, 1, 2, 3)\) as follows

\[
\{\Omega_0(x), \mathcal{H}_0(y)\} = \mathcal{H}_1(x) \delta^3(x - y),
\]

\[
5\text{In previous work [10] we have carried out the BFT embedding for the O(3) NLSM without explicitly including the geometrical constraint } n^a n^a - 1 = 0 \text{ with the Lagrangian multiplier field } n^0 \text{ in the starting Lagrangian.}
\]

\[
6\text{Note that these constraints correspond to } \mathcal{H}_i = 0 (i = 1, 2, 3, 4) \text{ in the HJ scheme, respectively.}
\]
\[
\{\Omega_1(x), H_0(y)\} = 2f\Omega_2(x)\delta^2(x - y), \\
\{\Omega_2(x), H_0(y)\} = \Omega_3(x)\delta^2(x - y).
\] (4.3)

The requirement of time-independence of \(\Omega_3\) then finally fixes the Lagrange multiplier field in Eq. (4.2) to \(v = 0\). Notice that for the case of the motion of a particle on a sphere, the constraint \(\Omega_3 = 0\) just fixes the “string tension” of the central force governing its motion.

The complete system of constraints is thus second-class. The subset \(\Omega_1 = 0\) and \(\Omega_2 = 0\) only links the fields \(n^a\) and \(\pi^a\), while the remaining constraints \(\Omega_0 = 0\) and \(\Omega_3 = 0\) link these variables to the rest of the variables. We may thus “gauge” the \((n^a, \pi^a)\) sector a la BFT, while leaving a strong implementation of the constraints \(\Omega_0 = 0\) and \(\Omega_3 = 0\) to the end. This is possible since the corresponding Dirac brackets, constructed from the inverse of the matrix

\[
\{\Omega_1(x), \Omega_2(y)\} = \begin{pmatrix} 0 & 2n^a(x)n^a(x) \\ -2n^a(x)n^a(x) & 0 \end{pmatrix} \delta^2(x - y),
\] (4.4)

reduce to ordinary Poisson brackets for functionals of \(n^a\) and \(\pi^a\) alone.

Following the BFT scheme [12, 13, 14], we systematically convert the second-class constraints \(\Omega_i = 0\) \((i = 1, 2)\) into first-class ones by introducing two canonically conjugate auxiliary fields \((\theta, \pi_\theta)\) with Poisson brackets

\[
\{\theta(x), \pi_\theta(y)\} = \delta^2(x - y).
\] (4.5)

The strongly involutive first-class constraints \(\tilde{\Omega}_i\) are then constructed as a power series of the auxiliary fields [10],

\[
\tilde{\Omega}_1 = \Omega_1 + 2\theta, \\
\tilde{\Omega}_2 = \Omega_2 - n^a n^a \pi_\theta,
\] (4.6)

which satisfy the closed algebra \(\{\tilde{\Omega}_1, \tilde{\Omega}_2\} = 0\).

We next construct the first-class fields \(\tilde{\mathcal{F}} = (\tilde{n}^a, \tilde{\pi}^a)\), corresponding to the original fields defined by \(\mathcal{F} = (n^a, \pi^a)\) in the extended phase space. They are obtained as a power series in the auxiliary fields \((\theta, \pi_\theta)\) by demanding that they be in strong involution with the first-class constraints (4.6), that is \(\{\tilde{\Omega}_i, \tilde{\mathcal{F}}\} = 0\). After some tedious algebra, we obtain for the first-class physical
fields
\[ \tilde{n}^a = n^a \left( \frac{n^a n^a + 2\theta}{n^a n^a} \right)^{1/2}, \]
\[ \tilde{\pi}^a = (\pi^a - n^a \pi_\theta) \left( \frac{n^a n^a}{n^a n^a + 2\theta} \right)^{1/2}. \] (4.7)

They are found to satisfy the Poisson algebra
\[ \{ \tilde{n}^a(x), \tilde{\pi}^b(y) \} = (\delta^{ab} - \frac{\tilde{n}^a \tilde{n}^b}{\tilde{n}^c \tilde{n}^c}) \delta^2(x - y), \]
\[ \{ \tilde{\pi}^a(x), \tilde{\pi}^b(y) \} = \frac{1}{\tilde{n}^c \tilde{n}^c} (\tilde{n}^b \tilde{\pi}^a - \tilde{n}^a \tilde{\pi}^b) \delta^2(x - y). \] (4.8)

To our knowledge, this is the first time that the first-class fields in the O(3) NLSM have been given in this compact form. \(^7\) Note that in terms of the first-class fields (4.7) the constraints (4.6) take the natural form
\[ \tilde{\Omega}_1 = \tilde{n}^a \tilde{n}^a - 1, \quad \tilde{\Omega}_2 = \tilde{n}^a \tilde{\pi}^a. \] (4.9)

Eq. (4.9) illustrates that any functional \( K(\tilde{F}) \) of the first-class fields \( \tilde{F} \) is also first-class. We correspondingly construct the first-class Hamiltonian in terms of the above first-class physical variables by making the replacements \( n^a \rightarrow \tilde{n}^a, \pi^a \rightarrow \tilde{\pi}^a \) in the canonical Hamiltonian \( H_0 \),
\[ \tilde{H}_0 = \frac{f}{2} \tilde{\pi}^a \tilde{\pi}^a + \frac{1}{2f} (\tilde{\nabla} \tilde{n}^a)^2 - n^0(\tilde{n}^a \tilde{n}^a - 1). \] (4.10)

Note that the first-class Hamiltonian (4.10) is manifestly strongly involutive with the first-class constraints, \( \{ \tilde{\Omega}_i, \tilde{H}_0 \} = 0 \). This need not be so. Indeed, this Hamiltonian is not unique, as we may always add to it terms proportional to the first class constraints without altering the dynamics of the first-class fields. Thus we could ask the gauged total Hamiltonian to generate the first-class constraints in an analogous way to Eq. (4.3)
\[ \{ \tilde{\Omega}_0(x), \tilde{H}(y) \} = \tilde{\Omega}_1(x) \delta^2(x - y), \]
\[ \{ \tilde{\Omega}_1(x), \tilde{H}(y) \} = 2f \tilde{\Omega}_2(x) \delta^2(x - y), \]
\[ \{ \tilde{\Omega}_2(x), \tilde{H}(y) \} = \Omega_3(x) \delta^2(x - y). \] (4.11)

\(^7\)Note that the first-class physical fields were obtained in terms of the infinite power series of the auxiliary fields \( (\theta, \pi_\theta) \) in the previous work [10].
This may be achieved via the replacement of (4.10) by

\[ \tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + f \pi_\theta \tilde{\Omega}_2 - \frac{1}{2} \ln(\tilde{n}^a \tilde{n}^a - 2\theta) \Omega_3. \] (4.12)

Since the second class constraint \( \Omega_3 = 0 \) should eventually be implemented strongly in our scheme, we shall set it equal to zero. As for \( \tilde{\Omega}_2 \), it is first class, so that it will not participate in the dynamics of gauge invariant functionals in the \((n^0, \pi^0, n^a, \pi^a, \theta, \pi_\theta)\) space. This allows us to set the last two terms in Eq. (4.12) equal to zero. Rewriting the Hamiltonian (4.10) in terms of the original fields in the \((n^a, \pi^a)\) sector, and the auxiliary ones in the \((\theta, \pi_\theta)\) sector, we have

\[ \tilde{\mathcal{H}}_0 = \frac{f}{2} \pi^a \pi_\theta (\pi^a - n^a \pi_\theta) \frac{n^c n^c}{n^c n^c + 2\theta} + \frac{1}{2f} (\nabla n^a)^2 \frac{n^c n^c + 2\theta}{n^c n^c} \]

\[ -n^0 (n^a n^a - 1 + 2\theta), \] (4.13)

where we have used the conformal map condition, \( n^a \nabla n^a = 0 \), which states that the radial vector is perpendicular to the tangent on the \( S^2 \) sphere in the extended phase space of the O(3) NLSM [10].

In the framework of the BFV formalism [16, 17], we now construct the nilpotent BRST charge \( Q \), the fermionic gauge fixing function \( \Psi \) and the BRST invariant minimal Hamiltonian \( H_m \) by introducing two canonical sets of ghost and anti-ghost fields, together with auxiliary fields \((C^i, \bar{P}_i), (\bar{P}^i, \bar{C}_i)\) and \((N^i, B_i), (i = 1, 2)\),

\[ Q = \int d^2 x \left( C^i \bar{\bar{\Omega}}_i + \bar{P}^i B_i \right), \]

\[ \Psi = \int d^2 x \left( \bar{C}_i \chi^i + \bar{\bar{P}}_i N^i \right), \]

\[ H_m = \int d^2 x \left( \tilde{\mathcal{H}}_0 - 2f C^1 \bar{P}_2 \right), \] (4.14)

with the properties \( Q^2 = \{Q, Q\} = 0 \) and \( \{\Psi, Q\}, Q\} = 0 \). The nilpotent charge \( Q \) is the generator of the following infinitesimal transformations,

\[
\begin{align*}
\delta_Q n^0 &= 0, & \delta_Q n^a &= -C^2 n^a, & \delta_Q \theta &= C^2 n^a n^a, \\
\delta_Q \pi^0 &= 0, & \delta_Q \pi^a &= 2C^1 n^a + C^2 (\pi^a - 2n^a \pi_\theta), & \delta_Q \pi_\theta &= 2C^1, \\
\delta_Q \bar{C}_i &= B_i, & \delta_Q C^i &= 0, & \delta_Q \bar{\bar{P}}_i &= 0, \\
\delta_Q \bar{P}^i &= 0, & \delta_Q \bar{P}_i &= \bar{\Omega}_i, & \delta_Q N^i &= -\bar{P}^i, \\
\end{align*}
\] (4.15)
which in turn imply \( \{Q, H_m\} = 0 \), that is, \( H_m \) in Eq. (4.14) is the BRST invariant minimal Hamiltonian.

After some algebra, we arrive at the effective quantum Lagrangian of the manifestly covariant form

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \mathcal{L}^{WZ} + \mathcal{L}^{\text{ghost}}
\]

where \( \mathcal{L}^{(0)} \) is given by Eq. (2.1) and

\[
\begin{align*}
\mathcal{L}^{WZ} &= \frac{1}{f n^c n^c} (\partial_{\mu} n^a) (\partial^{\mu} n^a) \theta - \frac{1}{2 f (n^c n^c)^2} \partial_{\mu} \theta \partial^{\mu} \theta + 2n^0 \theta, \\
\mathcal{L}^{\text{ghost}} &= -\frac{1}{2f} (n^a n^a)^2 (B + 2 \bar{C} C)^2 - \frac{1}{n^c n^c} \partial_{\mu} \theta \partial^{\mu} B + \partial_{\mu} \bar{C} \partial^{\mu} C.
\end{align*}
\]

This Lagrangian is invariant under the BRST transformation

\[
\begin{align*}
\delta_\epsilon n^0 &= 0, & \delta_\epsilon n^a &= \epsilon n^a C, & \delta_\epsilon \theta &= -\epsilon n^a n^a C, \\
\delta_\epsilon \bar{C} &= -\epsilon B, & \delta_\epsilon C &= 0, & \delta_\epsilon B &= 0,
\end{align*}
\]

where \( \epsilon \) is an infinitesimal Grassmann valued parameter. Note that the Wess-Zumino Lagrangian in (4.17) involves \( n^0 \), which originates from the geometrical constraint in the starting Lagrangian.

## 5 Conclusion

We have investigated the constraint structure of the O(3) NLSM, in the Lagrangian, symplectic, Hamilton-Jacobi (HJ) and Batalin-Fradkin-Tyutin (BFT) quantization schemes. In particular we showed that the symplectic algorithm led to an incomplete constraint structure for this model. In fact, the missing constraint (2.23) was shown to play an essential role as a part of the integrability conditions in the Hamilton-Jacobi formulation, necessary to recover the action for this second class system.

We further considered the gauge-embedding of the NLSM following the BFT quantization scheme for constructing the first-class fields and first-class Hamiltonian including the Wess-Zumino terms. A similar study, based on the “first-class field approach” was done in ref. [27] in the framework of the Hamilton-Jacobi scheme. Even though the authors of [27] exploited the first-class Hamiltonian in a form similar to \( \mathcal{H}_0 \) in Eq. (4.10), their final Lagrangian
is not covariant, in contrast to the covariant Lagrangian (4.16) involving the Wess-Zumino and ghost fields. Moreover the “integrability conditions” were not taken into account. The attempt of achieving such an embedding for the NLSM in the symplectic approach based on a biased “educated guess” for the general Lorentz covariant form of the first-class gauged Lagrangian, as was done in refs. [23] for the Proca and self-dual models, seems very problematic as seen from Eq. (4.17), which involves inverse powers of $(n^a n^a)$. We thus believe that for the case of the O(3) NLSM, the BFT embedding procedure is the most tractable one as compared to the Lagrangian, symplectic and HJ schemes.

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