Solution of polynomiality and positivity constraints on generalized parton distributions

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Abstract
An integral representation for generalized parton distributions is suggested which satisfies both positivity and polynomiality constraints.

1 Introduction
Generalized parton distributions (GPDs) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] appear in the context of the QCD factorization in various hard exclusive phenomena including deeply virtual Compton scattering and hard exclusive meson production. Among several general constraints on GPDs an important role is played by the polynomiality of the Mellin moments [5] and by the positivity bounds [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. These properties will be briefly described in section 2. In this paper we suggest a representation for GPDs which automatically satisfies both positivity and polynomiality constraints.

For simplicity only the case of spin zero hadrons will be considered (but various types of partons will be covered). We use the following definition of GPDs which is not quite standard but allows us to study different cases with a universal formalism:

$$H^{(N)}(x, \xi, t) = \int \frac{d\lambda}{2\pi} \exp(i\lambda x) \langle P_2 | O^{(N)}(\lambda, n) | P_1 \rangle.$$  

(1)

Here $|P_k\rangle$ is the hadron state with momentum $P_k$. The light-like vector $n$

$$n^2 = 0$$

is normalized by the condition

$$n(P_1 + P_2) = 2.$$  

(2)

We use the standard notations of Ji [14] for parameters $\Delta$, $t$ and $\xi$

$$\Delta = P_2 - P_1, \quad \xi = \frac{1}{2}(n\Delta), \quad t = \Delta^2.$$  

(3)
The definitions of light-ray operators $O^{(N)}(\lambda, n)$ for various types of partons are listed in the table

<table>
<thead>
<tr>
<th>parton</th>
<th>$O^{(N)}(\lambda, n)$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalar</td>
<td>$\phi^* \left( -\frac{\lambda n}{2} \right) \phi \left( \frac{\lambda n}{2} \right)$</td>
<td>0</td>
</tr>
<tr>
<td>quark</td>
<td>$\bar{\psi} \left( -\frac{\lambda n}{2} \right) (n \cdot \gamma) \psi \left( \frac{\lambda n}{2} \right)$</td>
<td>1</td>
</tr>
<tr>
<td>gluon</td>
<td>$n^\mu G^a_{\mu\nu} \left( -\frac{\lambda n}{2} \right) n_\rho G^{a,\nu\rho} \left( \frac{\lambda n}{2} \right)$</td>
<td>2</td>
</tr>
</tbody>
</table>

We have included the scalar field $\phi$ into this table since the positivity bounds are more general than their applications in QCD. The last column of this table contains the number $N$ of factors $n^\mu$ appearing in the light-ray operator $O(\lambda, n)$. This number $N$ plays an important role in the formulation of positivity bounds and of the polynomiality conditions and we included $N$ in notation of GPD $H^{(N)}(x, \xi, t)$.

## 2 Polynomiality and positivity

Whatever limited our knowledge about GPDs is there are two basic constraints: polynomiality and positivity. The polynomiality means that Mellin moments in $x$ of GPD $H^{(N)}(x, \xi, t)$

$$\int_{-1}^{1} dx x^m H^{(N)}(x, \xi, t) = P_{m+N}(\xi, t)$$

must be polynomials in $\xi$ of degree $m + N$.

Positivity bounds on GPDs have a simple form in the impact parameter representation. Let us define

$$\tilde{F}^{(N)}(x, \xi, b^\perp) = \int \frac{d^2 \Delta^\perp}{(2\pi)^2} \exp \left[ i(\Delta^\perp \cdot b^\perp) \right] H^{(N)} \left( x, \xi, -\frac{|\Delta^\perp|^2}{1 - \xi^2} + \frac{4\xi^2 M^2}{1 - \xi^2} \right)$$

We use notation $\tilde{F}^{(N)}$ for GPDs in the impact parameter representation in order to avoid confusion with the nucleon GPD $H$ and to keep the compatibility with the notations of ref. [22] where the following inequality was derived

$$\int_{-1}^{1} d\xi \int_{|\xi|}^{1} dx (1-x)^{-N-4} p^* \left( \frac{1-x}{1-\xi} \right) p \left( \frac{1-x}{1+\xi} \right)$$

$$\times \tilde{F}^{(N)} \left( x, \xi, \frac{1-x}{1-\xi^2} b^\perp \right) \geq 0.$$
This inequality was obtained in ref. [22] for the case \( N = 1 \) but the generalization to arbitrary \( N \) is straightforward.

Inequality (7) should hold for any function \( p(z) \). Therefore we actually deal with an infinite set of positivity bounds on the GPD and inequalities (7) cover various inequalities suggested for GPDs [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] as particular cases with some special choice of functions \( p(z) \).

It is well known that the double distribution representation [1, 2, 7] with the \( D \) term [23]

\[
H(x, \xi, t) = \int_{|\alpha|+|\beta|\leq 1} d\alpha d\beta \delta(x - \xi\alpha - \beta) F_D(\alpha, \beta, t) + \theta(|\xi| - x) D \left( \frac{x}{\xi} \right) \text{sign}(\xi)
\]

(8)
guarantees the polynomiality property (5). Another interesting parametrization for GPDs supporting the polynomiality was suggested in ref. [24].

On the other hand, the positivity bound on GPDs (7) is equivalent to the following representation for GPDs in the impact parameter representation (see Appendix A)

\[
\tilde{F}(x, \xi, b^\perp) = (1 - x)^{1+N} \times \sum_n Q_n \left( \frac{1-x}{1+\xi}, (1 - \xi)b^\perp \right) Q_n^* \left( \frac{1-x}{1-\xi}, (1 + \xi)b^\perp \right)
\]

(9)

with arbitrary functions \( Q_n \). Instead of the discrete summation over \( n \) one can use the integration over continuous parameters.

Although both polynomiality and positivity are basic properties that must hold in any reasonable model of GPDs usually the model building community meets a dilemma: one can use the double distribution representation (8) but then there are problems how to obey an infinite set of inequalities (7). Alternatively one can build models based on the representation (9) or on the so called overlap representation [15] which also automatically supports positivity bounds but then one meets problems with the polynomiality. In this paper a rather general representation for GPDs is suggested which guarantees both positivity and polynomiality.

3 Modified double distribution representation

For the construction of GPDs \( H^{(N)}(x, \xi, t) \) obeying both polynomiality and positivity constraints we use the double distribution representation which differs from the standard one (8) by an extra factor \((1 - x)^N\)

\[
H^{(N)}(x, \xi, t) = (1 - x)^N \int_{|\alpha|+|\beta|\leq 1} d\alpha d\beta \delta(x - \xi\alpha - \beta) F_D'(\alpha, \beta, t)
\]

(10)
Here $N$ depends on the type of the parton distribution according to table (4).

Representation (10) obviously satisfies the polynomiality condition (5). Indeed,

$$\int_{-1}^{1} dx x^n \int_{|\alpha|+|\beta|\leq 1} d\alpha d\beta \delta(x - \xi \alpha - \beta) F_D'(\alpha, \beta, t) = P_n(\xi, t)$$  \hspace{1cm} (11)$$

where $P_n(\xi, t)$ is a polynomial of degree $n$. Therefore

$$\int_{-1}^{1} dx x^m H(N)(x, \xi, t) = \int_{-1}^{1} dx \int_{|\alpha|+|\beta|\leq 1} d\alpha d\beta x^m (1-x)^N$$

$$\times \delta(x - \xi \alpha - \beta) F_D'(\alpha, \beta, t) = Q_{N+m}(\xi, t)$$  \hspace{1cm} (12)$$

is a polynomial of degree $N + m$ in agreement with (5).

For our aims it is much more convenient to use parameters

$$\alpha_1 = \frac{1}{2}(1 - \beta - \alpha), \hspace{0.5cm} \alpha_2 = \frac{1}{2}(1 - \beta + \alpha).$$  \hspace{1cm} (13)$$

instead of $\alpha, \beta$. Actually it is $\alpha_1$ and $\alpha_2$ ($x, y$ in notation of refs. [2, 7]) that appear as $\alpha$ parameters in the perturbative diagrammatic justification of the double distribution representation. The modified double distribution expressed in terms of parameters $\alpha_1, \alpha_2$ will be denoted as follows:

$$F_D(\alpha_1, \alpha_2, t) \equiv F_D'(\alpha, \beta, t).$$  \hspace{1cm} (14)$$

After these changes the modified double distribution representation (10) takes the following form

$$H(N)(x, \xi, t) = 2(1-x)^N \int_{0}^{1} d\alpha_1 \int_{0}^{1-\alpha_1} d\alpha_2 F_D(\alpha_1, \alpha_2, t)$$

$$\times \delta \left[ x - \xi (\alpha_2 - \alpha_1) - (1 - \alpha_1 - \alpha_2) \right].$$  \hspace{1cm} (15)$$

Here we use the triangle integration region in the $\alpha_1, \alpha_2$ plane which corresponds to the constraint $\beta > 0$ in terms of variables $\alpha, \beta$. Hence our GPD vanishes in the “antiquark” region (for brevity we use the word “quark” for any type of partons):

$$H(N)(x, \xi, t) = 0 \hspace{0.5cm} \text{if} \hspace{0.5cm} x < -|\xi|. \hspace{1cm} (16)$$

Therefore we must take care about the positivity constraints only in the “quark” region $x > |\xi|$. Once this pure quark GPD is constructed we can use the transformation $x \to -x$ to build GPDs with appropriate properties in both quark and antiquark regions.
The presence of the factor \((1 - x)^N\) in the modified double distribution representation (15) restricts the class of GPDs which are covered by this representation. These restrictions are partly compensated by the possibility to add \(D\)-like terms

\[
H^{(N)}(x, \xi, t) \rightarrow H^{(N)}(x, \xi, t) + \sum_{k=0}^{N-1} x^k D_k \left( \frac{x}{\xi} \right) \theta \left( 1 - \frac{|x|}{|\xi|} \right) \text{sign}(\xi).
\]  

(17)

These \(D\) terms are localized in the region \(|x| < |\xi|\) and therefore do not appear in the positivity condition (7). On the other hand, the polynomiality is obvious for these terms. Therefore the polynomiality and positivity do not constraint the form of the \(D\) terms.

4 Ansatz for double distributions

Now the problem is to find double distributions \(F_D(\alpha_1, \alpha_2, t)\) which lead to GPDs \(H^{(N)}(x, \xi, t)\) obeying the positivity constraint. We use the following ansatz for the modified double distributions (15)

\[
F_D(\alpha_1, \alpha_2, t) = \int_0^\infty d\lambda \int d\nu \left( \frac{1}{\lambda \alpha_1 \alpha_2} - t \right)^{-\nu-1} L_\nu(\lambda \alpha_1, \lambda \alpha_2).
\]  

(18)

Our double distribution is parametrized by an infinite set of functions \(L_\nu(w_1, w_2)\) defined for \(w_1, w_2 \geq 0\) and depending on parameter \(\nu\). We assume that for any \(\nu\) function \(L_\nu(w_1, w_2)\) corresponds to a positive definite quadratic form in \(w_1, w_2\), i.e. for any function \(\phi(w)\)

\[
\int_0^\infty dw_1 \int_0^\infty dw_2 L_\nu(w_1, w_2) \phi(w_1) \phi^*(w_2) \geq 0.
\]  

(19)

This is equivalent to the existence of the following representation for \(L_\nu(w_1, w_2)\)

\[
L_\nu(w_1, w_2) = \int d\rho F_\nu(w_1, \rho) F_\nu^*(w_2, \rho)
\]  

(20)

or to its discrete series analog.

The lower limit of the integral over \(\nu\) in the rhs of eq. (18) determines the asymptotics of \(F_D(\alpha_1, \alpha_2, t)\) at large \(|t|\). If one integrates over positive \(\nu\) then \(F_D \sim |t|^{-1}\). Functions \(L_\nu\), appearing in eq. (18) have the \(\nu\) dependent dimension which is a bit ugly but simplifies equations.

Below it will be shown that for any set of positive definite functions \(L_\nu(w_1, w_2)\) under the assumption that the integrals in the rhs of (18) are convergent, the

\[^1\text{In our notation the gluon distribution corresponds to the case } N = 2. \text{ In the standard approach the double distribution representation for gluons is written not for our function } F^{(2)}(x) \text{ but for } F^{(2)}(x)/x \text{ so that in the standard approach there is no analog of our term } D_1.\]
resulting double distribution $F_D(\alpha_1, \alpha_2, t)$ (18) leads to the GPD $H^{(N)}(x, \xi, t)$ (15) which satisfies the positivity bound (7). This check of positivity will be done in section 7 but first we prefer to derive some useful relations.

5 Expression for GPDs

Let us derive the expressions for GPDs $H^{(N)}(x, \xi, t)$ corresponding to the double distribution (18). First we insert ansatz (18) for the double distribution $F_D(\alpha_1, \alpha_2, t)$ into representation (15) for GPD $H^{(N)}(x, \xi, t)$.

\[ H^{(N)}(x, \xi, t) = 2(1-x)^N \int_0^1 \frac{1 - \alpha_1}{\alpha_1} \int_0^1 \frac{1 - \alpha_2}{\alpha_2} (x - \xi(\alpha_2 - \alpha_1) - (1 - \alpha_1 - \alpha_2)] \]

\[ \times \int_0^\infty d\lambda \lambda \int_0^\nu d\nu \left( \frac{1}{\lambda \alpha_1 \alpha_2} - t \right)^{\nu-1} L_\nu(\lambda \alpha_1, \lambda \alpha_2). \]  

(21)

We can rewrite this as follows

\[ H^{(N)}(x, \xi, t) = 2(1-x)^N \int_0^\infty d\lambda \lambda \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \]

\[ \times \theta(1 - \alpha_1 - \alpha_2) \delta [x - \xi(\alpha_2 - \alpha_1) - (1 - \alpha_1 - \alpha_2)] \]

\[ \times \int d\nu \left( \frac{1}{\lambda \alpha_1 \alpha_2} - t \right)^{\nu-1} L_\nu(\lambda \alpha_1, \lambda \alpha_2). \]  

(22)

Let us introduce new integration variables

\[ w_k = \lambda \alpha_k \]  

(23)

instead of $\alpha_k$ and integrate over $\lambda$ using the delta function

\[ H^{(N)}(x, \xi, t) = 2(1-x)^N \int_0^{\infty} dw_1 \int_0^{\infty} dw_2 \theta \left( x - \xi \frac{w_2 - w_1}{w_1 + w_2} \right) \]

\[ \times \int d\nu \left( \frac{1}{w_1} + \frac{1}{w_2} - t \right)^{\nu-1} L_\nu(w_1, w_2). \]  

(24)

In the region $x > |\xi|$ the step function does not vanish anywhere so that the above expression simplifies to

\[ H^{(N)}(x, \xi, t) \bigg|_{x > |\xi|} = 2(1-x)^N \int_0^\infty dw_1 \int_0^\infty dw_2 \]

\[ \times \left( \frac{1}{w_1} + \frac{1}{w_2} - t \right)^{\nu-1} L_\nu(w_1, w_2). \]  

(25)
\[ \times \int d\nu \left( \frac{1}{w_1} \frac{1 + \xi}{1 - x} + \frac{1}{w_2} \frac{1 - \xi}{1 - x} - t \right)^{\nu - 1} L_\nu(w_1, w_2). \] 

This representation can be rewritten in the following form

\[ H^{(N)}(x, \xi, t) \bigg|_{x>|\xi|} = 2(1-x)^{N-1} \int_0^\infty dw_1 \int_0^\infty dw_2 \]

\[ \times \int_0^\infty d\gamma e^{\tau\gamma} \int d\nu \frac{\gamma^\nu}{\Gamma(\nu + 1)} L_\nu(w_1, w_2) \exp \left[ -\gamma \left( \frac{1}{w_1} \frac{1 + \xi}{1 - x} + \frac{1}{w_2} \frac{1 - \xi}{1 - x} \right) \right]. \] 

6 Forward distribution

In the forward limit \( \xi \to 0, t \to 0 \) we obtain from (25)

\[ q(x) = H^{(N)}(x, 0, 0) = 2(1-x)^{N-1} \]

\[ \times \int_0^\infty dw_1 \int_0^\infty dw_2 \int d\nu \left( \frac{w_1 w_2 (1-x)}{w_1 + w_2} \right)^{\nu + 1} L_\nu(w_1, w_2). \] 

We remind that the positivity of forward parton distributions is a consequence of the general positivity bounds on GPDs which will be established in the next section. On the other hand, we can see the positivity of the forward parton distribution directly from (27):

\[ \int_0^\infty dw_1 \int_0^\infty dw_2 \frac{1}{\Gamma(\nu + 1)} \int_0^\infty d\tau \tau^\nu \exp \left( -\tau \frac{w_1 + w_2}{w_1 w_2} \right) L_\nu(w_1, w_2) \]

\[ = \int_0^\infty dw_1 \int_0^\infty dw_2 \frac{1}{\Gamma(\nu + 1)} \int_0^\infty d\tau \tau^\nu \exp \left( -\tau \frac{w_1 + w_2}{w_1 w_2} \right) L_\nu(w_1, w_2) \]

\[ = \frac{1}{\Gamma(\nu + 1)} \int_0^\infty d\tau \tau^\nu \int_0^\infty dw_1 \int_0^\infty dw_2 L_\nu(w_1, w_2) \exp \left( -\tau \frac{w_1}{w_2} \right) \exp \left( -\tau \frac{w_1}{w_2} \right) \geq 0. \] 

The rhs is positive

\[ \int_0^\infty dw_1 \int_0^\infty dw_2 L_\nu(w_1, w_2) \exp \left( -\tau \frac{w_1}{w_2} \right) \exp \left( -\tau \frac{w_1}{w_2} \right) \geq 0 \] 

due to the positivity (19) of the quadratic form \( L_\nu(w_1, w_2) \).
7 Proof of positivity

Now we want to show that the modified double distribution (18) with positive
definite functions $L$ (19) generates GPD $H^{(N)}(x, \xi, t)$ which satisfies the posi-
tivity bounds (7). For the positivity bounds we need the GPD in the impact
parameter representation (6)

$$
\tilde{F}(N)(x, \xi, \frac{1-x}{1-\xi^2} b^+) = \int \frac{d^2 \Delta^+}{(2\pi)^2} \exp \left[ i \frac{1-x}{1-\xi^2}(\Delta^+ b^+) \right] 
\times H^{(N)}(x, \xi, -\frac{|\Delta^+|^2 + 4\xi^2 M^2}{1-\xi^2}).
$$

(30)

Using representation (26) for the GPDs $H^{(N)}(x, \xi, t)$ we obtain

$$
\tilde{F}(N)(x, \xi, \frac{1-x}{1-\xi^2} b^+) = \int \frac{d^2 \Delta^+}{(2\pi)^2} \exp \left[ i \frac{1-x}{1-\xi^2}(\Delta^+ b^+) \right] 
\times 2(1-x)^{N-1} \int_0^\infty dw_1 \int_0^\infty dw_2 \int_0^\infty d\gamma \exp \left( -\gamma \frac{|\Delta^+|^2 + 4\xi^2 M^2}{1-\xi^2} \right) 
\times \int \frac{d\nu}{\Gamma(\nu + 1)} \exp \left[ -\gamma \left( \frac{1 + \xi}{w_2} + \frac{1 - \xi}{w_1} \right) \right] L_\nu(w_1, w_2).
$$

(31)

Integrating over $\Delta^+$, introducing compact notations

$$
r_1 = \frac{1-x}{1+\xi}, \quad r_2 = \frac{1-x}{1-\xi}.
$$

(32)

(see Appendix A) and rescaling the integration variables $w_k \to w_k r_k$ we find

$$
\tilde{F}(N)(x, \xi, \frac{1-x}{1-\xi^2} b^+) = \frac{1}{2\pi} \left( \frac{2r_1 r_2}{r_1 + r_2} \right)^{N+1}
\times \int_0^\infty dw_1 \int_0^\infty dw_2 \int_0^\infty d\gamma \exp \left[ -\gamma \left( \frac{1-x}{w_2} - \frac{1}{w_1} \right) \right] L_\nu(r_1 w_1, r_2 w_2).
$$

(33)

(34)

Next we change the integration variable $\gamma \to \gamma r_1 r_2$

$$
\tilde{F}(N)(x, \xi, \frac{1-x}{1-\xi^2} b^+) = \frac{1}{2\pi} \left( \frac{2r_1 r_2}{r_1 + r_2} \right)^{N+1}
\times \int_0^\infty dw_1 \int_0^\infty dw_2 \int_0^\infty d\gamma \exp \left[ -\gamma \left( \frac{1-x}{w_2} - \frac{1}{w_1} \right) \right] L_\nu(r_1 w_1, r_2 w_2).
$$

(35)
\[
\times \int d\nu \left( \frac{\gamma r_1 r_2}{\Gamma(\nu + 1)} \right) \exp \left[ -\frac{\gamma}{\nu} \frac{(w_1 + w_2)}{w_1 w_2} \right] L_\nu (r_1 w_1, r_2 w_2) \tag{36}
\]

and use the representation

\[
\exp \left[ -\gamma (r_1 - r_2)^2 M^2 \right] = \frac{1}{2M \sqrt{\pi \gamma}} \int_{-\infty}^{\infty} ds \exp \left[ -\frac{s^2}{4\gamma M^2} + is(r_2 - r_1) \right]. \tag{37}
\]

Then

\[
\hat{F}(N) \left( x, \xi, \frac{1-x}{1-\xi^2} b^\perp \right) = \frac{1}{4M\pi^{3/2}} \left( \frac{2r_1 r_2}{r_1 + r_2} \right)^{N+1} \int_{0}^{\infty} d\gamma \exp \left( -\frac{1}{4\gamma} |b^\perp|^2 \right)
\]

\[
\times \int_{-\infty}^{\infty} ds \exp \left( -\frac{s^2}{4\gamma M^2} \right) \exp \left[ is(r_2 - r_1) - \frac{\gamma (w_1 + w_2)}{w_1 w_2} \right]
\]

\[
\times \int_{0}^{\infty} dw_1 \int_{0}^{\infty} dw_2 \int_{0}^{\infty} d\nu \nu^{-3/2} \left( \frac{r_1 r_2}{\Gamma(\nu + 1)} \right) L_\nu (w_1 r_1, w_2 r_2). \tag{38}
\]

Now we turn to the positivity bound (7) written in the form of the integral over \( r_1, r_2 \) — see equation (49) in Appendix A. The lhs of this inequality is

\[
\int_{0}^{\infty} dr_1 \int_{0}^{\infty} dr_2 (r_1 + r_2)^{N+1} \left( p^* (r_2) p (r_1) \hat{F}(N) \left( x, \xi, \frac{1-x}{1-\xi^2} b^\perp \right) \right)
\]

\[
= \frac{2^{N-1}}{M\pi^{3/2}} \int_{0}^{\infty} d\nu \int_{0}^{\infty} d\gamma \nu^{-3/2} \left( \frac{2r_1 r_2}{r_1 + r_2} \right)^{N+1} \int_{-\infty}^{\infty} ds \exp \left( -\frac{1}{4\gamma} |b^\perp|^2 \right)
\]

\[
\times \int_{0}^{1} dr_1 \int_{0}^{\infty} dw_1 \int_{0}^{\infty} dr_2 \int_{0}^{\infty} dw_2 L_\nu (w_1 r_1, w_2 r_2)
\]

\[
\times \left[ p (r_1) r_1^{N+\nu+1} \exp \left( -is r_1 - \frac{\gamma}{w_1} \right) \right] \left[ p (r_2) r_2^{N+\nu+1} \exp \left( -is r_2 - \frac{\gamma}{w_2} \right) \right]^* . \tag{39}
\]

Here we can rescale integration variables \( w_k \rightarrow w_k/r_k \). Then

\[
\int_{0}^{1} dr_1 \int_{0}^{\infty} dw_1 \int_{0}^{\infty} dr_2 \int_{0}^{\infty} dw_2 L_\nu (w_1 r_1, w_2 r_2)
\]

\[
\times \left[ p (r_1) r_1^{N+\nu+1} \exp \left( -is r_1 - \frac{\gamma}{w_1} \right) \right] \left[ p (r_2) r_2^{N+\nu+1} \exp \left( -is r_2 - \frac{\gamma}{w_2} \right) \right]^* .
\]
\[ = \int_0^\infty \int_0^\infty \sum_{\sigma_1, \sigma_2} L_{\nu, \sigma_1 \sigma_2}^\sigma (w_1, w_2) \]
\[ \times \left[ \int_0^1 dr_1 p(r_1) r_1^{N+\nu} \exp \left( -isr_1 - \frac{\gamma r_1}{w_1} \right) \right] \]
\[ \times \left[ \int_0^1 dr_2 p(r_2) r_2^{N+\nu} \exp \left( -isr_2 - \frac{\gamma r_1}{w_2} \right) \right]^* \]
\[ = \int_0^\infty \int_0^\infty d\nu (w_1, w_2) \phi_{\nu}(w_1) \phi^* (w_2) \geq 0 \quad (40) \]

where
\[ \phi_{\nu}(w) = \int_0^1 dr p(r) r^{N+\nu} \exp \left( -isr - \frac{\gamma r}{w} \right). \quad (41) \]

The rhs of (40) is positive since \( L_{\nu} (w_1, w_2) \) is positive definite. Combining (39) and (40) we complete the proof of the positivity bound (49) for the GPD generated by the double distribution (18).

### 8 Conclusions

In this paper we have shown that representation (18) for the double distributions [understood in the sense of eq. (15)] generates GPDs (24) satisfying both polynomiality and positivity constraints. Our representation (18) for double distributions involves arbitrary positive definite quadratic forms \( L_{\nu}(w_1, w_2) \). Functions \( L_{\nu}(w_1, w_2) \) parametrizing GPDs depend on the same amount of variables \((w_1, w_2, \nu)\) as GPDs themselves \((x, \xi, t)\) which means that this representation is rather general (although probably not covering all possible solutions of the positivity and polynomiality constraints).

The parametrization of GPDs suggested here seems to be constructive for the model building: the positive definite functions \( L_{\nu}(w_1, w_2) \) can be easily generated by using eq. (20). One should not forget about the possibility to add the generalized \( D \) terms (17) which are not restricted by the polynomiality and positivity.

Certainly apart from the positivity and polynomiality there are other theoretical and phenomenological constraints on GPDs and it would be interesting whether representation (18) allows to construct viable models of GPDs.

**Acknowledgement.** I am grateful to M.V. Polyakov for useful discussions.
Appendix

A Variables $r_1, r_2$

Let us define variables $r_1, r_2$ which can be used instead of $x, \xi$

\[
    r_1 = \frac{1 - x}{1 + \xi}, \quad r_2 = \frac{1 - x}{1 - \xi}, \quad (42)
\]

\[
    \xi = \frac{r_2 - r_1}{r_2 + r_1}, \quad x = 1 - \frac{2r_1r_2}{r_1 + r_2}, \quad (43)
\]

\[
    \frac{2dx d\xi}{(1 - x)^3} = \frac{dr_1 dr_2}{r_1^2 r_2^2}, \quad (44)
\]

The region covered by the positivity bounds (7)

\[
    x > |\xi| \quad (45)
\]

is mapped to the square in the $r_1, r_2$ plane

\[
    0 < r_1, r_2 < 1 . \quad (46)
\]

Inequality (7) takes the following form in terms of integration variables $r_1, r_2$

we keep notations $x, \xi$ in GPDs implying that they are functions of $r_1, r_2$

\[
    \int_0^1 \frac{dr_1}{r_1^2} \int_0^1 \frac{dr_2}{r_2^2} \left( \frac{r_1 + r_2}{r_1 r_2} \right)^{N+1} p^*(r_1) p(r_2) \times \tilde{F}^{(N)} \left( x, \xi, \frac{1 - x}{1 - \xi^2} b^\perp \right) \geq 0 . \quad (47)
\]

Since function $p$ is arbitrary we can replace it

\[
    p(r_1) \rightarrow r_1^{N+3} p(r_1) \quad (48)
\]

which leads us to the equivalent form of inequality (47)

\[
    \sum_{r_1, r_2} \int_0^1 \frac{dr_1}{r_1^2} \int_0^1 \frac{dr_2}{r_2^2} (r_1 + r_2)^{N+1} p^*(r_2) p(r_1) \tilde{F}^{(N)} \left( x, \xi, \frac{1 - x}{1 - \xi^2} b^\perp \right) \geq 0 . \quad (49)
\]

Inequality (47) means that function

\[
    \left( \frac{r_1 + r_2}{r_1 r_2} \right)^{N+1} \tilde{F}^{(N)} \left( x, \xi, \frac{1 - x}{1 - \xi^2} b^\perp \right) \quad (50)
\]

must be a positive definite quadratic form, i.e. it has the following representation

\[
    \left( \frac{r_1 + r_2}{2r_1 r_2} \right)^{N+1} \tilde{F}^{(N)} \left( x, \xi, \frac{1 - x}{1 - \xi^2} b^\perp \right) = \sum_n R_n(r_1, b^\perp) R_n^*(r_2, b^\perp) \quad (51)
\]
with some functions $R_n$. Turning back to the variables $x, \xi$ we find

$$\hat{F}^{(N)}(x, \xi, b_{\perp}) = (1 - x)^{N+1} \times \sum_{n} R_n \left( \frac{1 - x}{1 + \xi}, \frac{1 - \xi^2}{1 - x} b_{\perp} \right) R_n^* \left( \frac{1 - x}{1 - \xi}, \frac{1 - \xi^2}{1 - x} b_{\perp} \right).$$

(52)

Introducing functions

$$Q_n(r, b_{\perp}) = R_n \left( r \frac{1}{r} b_{\perp} \right)$$

(53)

we obtain representation (9) for $\hat{F}^{(N)}(x, \xi, b_{\perp})$.

References


