1. Introduction and Summary

The renormalization procedure in Quantum Field Theory heavily introduces a renormalization group framework. This framework allows the calculation of perturbative series, which is essential for understanding the behavior of quantum fields at short distances. The renormalization group equations describe how the effective coupling constants change with energy scale, and they play a crucial role in connecting the UV and IR physics.

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TLLW-90-190
perturbative series leads to two uncoupled sets of differential equations rather than one, reflecting the particular difficulties that thermal perturbation theory present.

In the Appendix, we briefly discuss the difference between the strictly perturbative solution for the running coupling \( \alpha_s \) to two-loop order and the exact solution at two-loop order, which can be given in closed form in terms of Lambert's \( W \) function.

II. RENORMALIZATION GROUP SUMMATION IN HOT QCD

In QCD with \( n_f \) flavors of quarks, the thermodynamic free energy at high temperature has been computed to be \([3, 4, 5]\)

\[
\mathcal{F} = \frac{-8\pi^2}{45} T^4 \left\{ \left(1 + \frac{21}{32} n_f \right) + \frac{-15}{4} \left(1 + \frac{5}{12} n_f \right) \frac{\alpha_s}{\pi} + 30 \left[(1 + \frac{n_f}{6}) \left(\frac{\alpha_s}{\pi}\right)^{3/2} \right] \right. \\
+ \left. \left\{ 237.2 + 15.97 n_f - 0.413 n_f^2 + \frac{135}{2} \left(1 + \frac{n_f}{6}\right) \ln \left[\frac{\mu^2}{2\pi T}\right] \left(\frac{\alpha_s}{\pi}\right)^2 \right\} \right. \\
+ \left. \left(1 + \frac{n_f}{6}\right)^{1/2} \left[ -799.2 - 21.96 n_f - 1.926 n_f^2 \right] \right. \\
+ \left. \frac{495}{2} \left(1 + \frac{n_f}{6}\right) \left(1 - \frac{2}{33} n_f\right) \ln \left(\frac{\mu^2}{2\pi T}\right) \left(\frac{\alpha_s}{\pi}\right)^{5/2} + \mathcal{O}(\alpha_s^3 \ln \alpha_s) \right\}
\]

(1)

where \( \mu \) is the renormalization scale parameter of the \( \overline{\text{MS}} \) scheme and \( \alpha_s(\mu) \) is the running coupling in this scheme whose form to three-loop order is given in \([7]\), though we shall restrict ourselves to its two-loop version in the following (see the Appendix for more discussion).

Changing the renormalization scale parameter \( \mu \) in principle should not alter the value of \( F \), as changes in the explicit \( \mu \) are compensated for by changes of \( \alpha_s(\mu) \), and indeed, the result (1) is independent of \( \mu \) to order \( \alpha_s^2 \). However, numerically the dependence on \( \mu \) is large, in fact larger than that of the result to order \( \alpha_s^4 \) unless \( \mu \), which has to be of the same order as \( T \) to avoid large logarithms, is much larger than even the electromagn weak scale.

Extrapolating from Eq. (1), we assume the complete all-order result of \( \mathcal{F} \) can be represented by a series of the form

\[
\mathcal{F}/\mathcal{F}_0 = 1 + \sum_{n=0}^{\infty} \left( R_n(u) x^{n+1} + S_n(u) x^n + T_n(u) x^{n+2} \ln x \right)
\]

(2)

where \( \mathcal{F}_0 \) is the ideal-gas value, \( x = \alpha_s(\mu)/\pi, \ u = xL, \ L = \ln(\mu^2/(2\pi T)^2) \) and

\[
R_n(u) = \sum_{m=0}^{n} A_{n+m,m} u^m, \quad S_n(u) = \sum_{m=0}^{n} B_{n+m,m} u^m, \quad T_n(u) = \sum_{m=0}^{n} C_{n+m,m} u^m,
\]

(3)

although only the coefficients with \( n \leq 1 \) are accessible by thermal perturbation theory. In fact, all of the perturbatively accessible coefficients with \( m = 0 \) have been calculated already, with the exception of \( C_{1,0} \), which is forthcoming\(^1\).

In Ref. [6] it has been shown how to sum all RG-accessible logarithms when the lowest-order coefficients to the sums in Eq. (3), \( A_{n,0}, B_{n,0}, \) and \( C_{n,0} \), and the \( \beta \)-function coefficients (see Eq. (9) below) are known. In this paper we extend this to a perturbative series of the form (2).

\(^1\) Y. Schröder, private communication
While the perturbative expression (1) is in powers of $x^{1/2}$, successive $R \Sigma$-perturbative expressions are given by

\begin{align}
 F_R^{(1)} / F_0 &= 1 + x R_0(x L), \\
 F_R^{(2)} / F_0 &= 1 + x R_0(x L) + x^{3/2} S_0(x L), \\
 F_R^{(3)} / F_0 &= 1 + x R_0(x L) + x^{3/2} S_0(x L) + x^2 (R_1(x L) + T_0(x L) \ln x), \\
 F_R^{(4)} / F_0 &= 1 + x R_0(x L) + x^{3/2} S_0(x L) + x^2 (R_1(x L) + T_0(x L) \ln x) + x^{5/2} S_1(x L)
\end{align}

which all are perturbatively accessible in hot QCD, as we shall see.

The explicit dependence of $\mathcal{F}$ on $\bar{p}^2$ and its implicit dependence through $x(\bar{p}^2)$ are such that

\[ \bar{p}^2 \frac{d \mathcal{F}}{d \bar{p}^2} = 0 = \left( \bar{p}^2 \frac{\partial}{\partial \bar{p}^2} + \beta(x) \frac{\partial}{\partial x} \right) \mathcal{F} \]

where

\[ \beta(x) = \bar{p}^2 \frac{\partial x}{\partial \bar{p}^2} = (b_2 x^2 + b_3 x^3 + b_4 x^4 + \ldots). \]

Substitution of (2) into (8) yields

\[ 0 = \sum_{n=0}^{\infty} \left\{ R'_n x^{n+2} + (b_2 x^2 + \ldots)(u R'_n + (n + 1) R_n + x T_n) x^n \right\} \\
 + \left\{ S'_n x^{n+\frac{3}{2}} + (b_2 x^2 + \ldots)(u S'_n + \left[ n + \frac{3}{2} \right] S_n) x^{n+\frac{3}{2}} \right\} \\
 + \left\{ T'_n x^{n+3} + (b_2 x^2 + \ldots)(u T'_n + (n + 2) T_n) x^{n+1} \right\} \ln x \]

Lowest order terms of the form $x^n$, $x^{n+\frac{3}{2}}$ and $x^n \ln x$ give rise to differential equations for $R_0(u)$, $S_0(u)$ and $T_0(u)$, respectively. The boundary conditions on these equations are the computed values of $A_{0,0}$, $B_{0,0}$ and $C_{0,0}$ respectively, which can be read off Eq. (1). This gives

\[ R_0(u) = A_{0,0} w^{-1}, \quad S_0(u) = B_{0,0} w^{-3/2}, \quad T_0(u) = C_{0,0} w^{-2}, \quad w \equiv (1 + b_2 u). \]

These functions incorporate the LL contributions to $\mathcal{F}$ to all orders.

To next order in the coupling $x$, we find differential equations for $R_1$, $S_1$, and $T_1$ (which rely on knowing the above solutions for $R_0$, $S_0$, and $T_0$). Solving these equations gives

\begin{align}
 R_1 &= w^{-2} \left[ A_{1,0} - \left( \frac{b_3}{b_2} A_{0,0} + C_{0,0} \right) \ln w \right] \\
 S_1 &= w^{-5/2} \left[ B_{1,0} - \frac{3 b_3}{2 b_2} B_{0,0} \ln w \right] \\
 T_1 &= w^{-3} \left[ C_{1,0} - \frac{2 b_3}{b_2} C_{0,0} \ln w \right]
\end{align}

which incorporate the NLL contributions to $\mathcal{F}$ to all orders.

Continuing in this way, we obtain

\[ R_2 = w^{-3} \left[ A_{2,0} - \left( \frac{2 b_3}{b_2} A_{1,0} + \frac{b_3}{b_2} A_{0,0} + C_{1,0} + \frac{b_3}{b_2} C_{0,0} \right) \ln w \right] \]
\[
\frac{\mathcal{F}}{\mathcal{F}_0} \bigg|_{T/T_c = 3} = \lambda
\]

FIG. 1: Comparison of the renormalization-scale dependence of the \(\text{RG}_\Sigma\) result for \(\mathcal{F}/\mathcal{F}_0\) at \(T = 3T_c\) for \(n_f = 0\) (solid line) and the perturbative result to order \(\alpha_s^{3/2}\) (long-dashed line) when varying the renormalization scale \(\mu\) around a central value of \(2\pi T\) by a factor \(e^\lambda\). The two dots on the vertical axis give two recent lattice results from Refs. [12, 13]. The short-dashed lines forming a big Z show the dependence of the \(\text{RG}_\Sigma\) result on varying the arbitrary parameters \(\kappa\) and \(\nu\) by a factor of \(e^\lambda\) around 1.

\[
S_2 = w^{-7/2} \left[ B_{2,0} - \frac{b_3}{b_2} \left( \frac{5}{2} B_{1,0} + \frac{3 b_3}{2 b_2} \right) \ln w + \frac{15 b_3^2}{8 b_2^2} B_{0,0} \ln^2 w + \frac{3}{2} \left( \frac{b_3}{b_2} - b_4 \right) B_{0,0} w \right]
\]

\[
T_2 = w^{-4} \left[ C_{1,0} - \frac{b_3}{b_2} \left( 3 C_{1,0} + 2 \frac{b_3}{b_2} C_{0,0} \right) \ln w + \frac{1}{2} \left( \frac{b_3}{b_2} - b_4 \right) C_{0,0} w \right]
\]

these give the NLL contributions to \(\mathcal{F}\).

The coefficients \(A_{0,0}, B_{0,0}, C_{0,0}, A_{1,0},\) and \(B_{1,0}\) are determined in the \(\overline{\text{MS}}\) scheme by Eq. (1) and this allows us to construct the \(\text{RG}_\Sigma\) approximation \(\mathcal{F}_{\text{RG}_\Sigma}^{(4)}\) given in Eq. (7). The next approximation would also involve \(T_1, R_2, S_2,\) and \(T_2\). For these, one would need results for \(C_{1,0}\) and \(A_{2,0}, B_{2,0}, C_{2,0}\), but only \(C_{1,0}\) is computable in perturbation theory because the order \(\alpha_s^3\)-contribution to \(\mathcal{F}\) is inherently nonperturbative [8, 9]. The remaining coefficients would have to be derived from a nonperturbative framework such as lattice gauge theory, or they might be estimated by nonperturbative resummation techniques such as Padé approximations [10, 11]. Given that, the above formulae would allow the construction of the next \(\text{RG}_\Sigma\) approximation \(\mathcal{F}_{\text{RG}_\Sigma}^{(5)}\).

III. DISCUSSION

In Fig. 1 the renormalization scale dependence of the purely perturbative result (1) for \(\mathcal{F}/\mathcal{F}_0\) at \(T = 3T_c\) for \(n_f = 0\) is displayed using a two-loop running coupling constant \(\alpha_s(\mu^2) = e^\lambda(2\pi T)\) with \(\lambda\) varying between -1 and 1 (long-dashed line). The two dots on the vertical axis give two recent lattice results from Refs. [12, 13], and the perturbative result is seen to agree well with these for \(\mu = 2\pi T\) (i.e. \(\lambda = 0\)), but deviates strongly for different choices of the renormalization scale.

The \(\text{RG}_\Sigma\) result as obtained above is given by the solid line, and it shows a rather weak dependence on the renormalization scale. The systematic RG summation as described above is
thus able to absorb almost all of the scale dependence.

However, this does not mean that the $\text{RG}_{\Sigma}$ approach predicts the perturbative result at $\bar{\mu} = 2\pi T$. Rather, the latter has been used as the initial condition for the differential equations determining the functions $R_0$, $S_0$, $T_0$, $R_1$, and $S_1$. Organizing the RG summation in terms of a variable $L_\nu = \ln[\bar{\mu}^2/(\kappa 2\pi T)^2]$ instead of the variable $L$ introduced in Eq. (2) would have led to a different definition of the constants $A_{0,0}$, $B_{0,0}$, $C_{0,0}$ that provide the initial conditions. Moreover, the series involving half-integer powers of $x$ in Eq. (2) leads to differential equations that decouple from those responsible for the series involving integer powers and logarithms of $x$. The former series could therefore have been derived by introducing a different variable $L_\nu = \ln[\bar{\mu}^2/(\nu 2\pi T)^2]$ for $S_\nu$.

In identifying the ambiguity in $L_\kappa$ and $L_\nu$ (as parametrized by $\kappa$ and $\nu$ respectively) we are keeping the value $\bar{\mu}$ of $x$ unaltered—both $\bar{\mu}$ is held fixed and the form of $x(\bar{\mu})$ dictated by having chosen to work in the $\overline{\text{MS}}$ scheme is not changed. Rather, we note that the invariance of the perturbative result (1) under the change

$$A_{1,0} + A_{1,1} \ln \frac{\bar{\mu}}{2\pi T} \rightarrow (A_{1,0} + A_{1,1} \ln \kappa) + A_{1,1} \ln \frac{\bar{\mu}}{\kappa 2\pi T} \equiv A_{1,0}^{(\kappa)} + A_{1,1} L_\kappa \quad (18)$$

(with analogous equations for $B_{1,0}^{(\nu)}$ and $C_{1,0}^{(\nu)}$) is lost when one deals with the $\text{RG}_{\Sigma}$ expressions $R_1$, $S_1$, and $T_1$. The boundary condition for the differential equations for $R_1$, $S_1$, and $T_1$, and the logarithm in the solution to these equations have a dependence on $\kappa$ (or $\nu$) that no longer automatically compensates as in the perturbative result.

In Fig. 1 the short-dashed lines forming a big $Z$ show the result of varying $\kappa$ and $\nu$ around the value 1 by a factor $e^\lambda$ with $\lambda$ between -1 and 1. The upper bar in the big $Z$ is formed by $\nu = e^\lambda$, $\kappa = e^\lambda$, the diagonal by $\nu = \kappa = e^\lambda$, and the lower bar by $\nu = e^{-1}, \kappa = e^{\lambda}$. Evidently, when $\kappa$ and $\nu$ are identified, the ambiguity is a little bit smaller than the one given by the renormalization scale dependence of the purely perturbative result, but varying $\nu$ independently of $\kappa$ leads to even larger variations, at their extremes even exceeding the ideal gas result for $[\mathcal{F}]$. Unfortunately, when varying $\kappa$ and $\nu$, either together or independently, there is no saddle point for $\mathcal{F}^{(4)}_{\text{RG}_{\Sigma}}$ that would allow one to eliminate these ambiguities by a principle of minimal sensitivity [1].

The renormalization group equation (8) states that the thermodynamic potential is independent of the renormalization scale $\bar{\mu}$. When one uses this equation to incorporate all the logarithmic contributions coming from higher order perturbation theory whose form is implied by this equation, one indeed finds the dependence of a perturbative approximation to $\mathcal{F}$ on $\bar{\mu}$ being diminished, as expected. However, we have identified another source of ambiguity (characterized by $\kappa$ and $\nu$) which leads to large variations of the perturbative RGE result for $\mathcal{F}$. This highlights the numerically strong scheme dependence of the perturbative result which still remains after having eliminated the strong dependence on the renormalization scale $\bar{\mu}$ by the RG$_\Sigma$ method. (There are, of course, other ambiguities in the RGE result that could arise due to changing the renormalization scheme from $\overline{\text{MS}}$; these we have not addressed.)

As can be seen from Fig. 1, the extra dependence on the parameter $\nu$ associated with the part of the perturbative series involving half-integer powers in $\alpha_s$ is in fact the one which dominates the uncertainties of the perturbative result (the short-dashed lines are rather flat when only $\kappa$ is varied, but depend strongly on $\nu$). This part of the perturbative series is exclusively associated with “soft” collective phenomena such as screening and Landau damping and calls for a more complete treatment than conventional perturbation theory is able to achieve. Recent attempts in this direction that have been put forward include separate Padé approximations to soft and hard contributions [14], optimization of perturbation theory using the hard-thermal-loop effective action [15], and a approximately self-consistent propagator resummation in the so-called $\Phi$-derivable approach [16].
Acknowledgments

Financial support has been provided by NSERC. D. G. C. McK. would like to thank the Technical University of Vienna where this work was performed. R. and D. MacKenzie provided helpful suggestions. Y. Schröder and G. Moore pointed out typographical errors in the first version of this paper.

APPENDIX

The beta function to two-loop order,

\[ \mu^2 \frac{dx}{d\mu^2} = \frac{dx}{dL} = b_2 x^2 + b_3 x^3, \quad (A.1) \]

can be integrated in closed form

\[ \int_{L_0}^{L} dL' = L - L_0 = \int_{x_0}^{x} \frac{dx'}{b_2 x'^2 + b_3 x'^3} = \frac{1}{b_2} \left[ \rho \ln \frac{1 + \rho x}{x} - \frac{1}{x} \right]_{x_0}^{x} \quad (\rho = b_3/b_2) \quad (A.2) \]

which becomes, if \( x_0 \to \infty \) as \( L_0 \to \ln \Lambda^2 \),

\[ \bar{L} \equiv \ln \frac{\mu^2}{\Lambda^2} = \frac{\rho}{b_2} \ln \left[ W e^W (-\rho) \right] - \frac{\rho}{b_2} \ln \rho \quad (A.3) \]

where

\[ \frac{dW}{dx} = -1 - \frac{1}{\rho x}, \quad \text{or} \quad \ln x = -[\rho (W + 1)]^{-1}. \quad (A.4) \]

Hence, the solution to Eq. (A.1) can be written [17] as a Lambert W function [18] with

\[ W(z) e^{W(z)} = z = -\frac{1}{e} \left( \frac{\mu^2}{\Lambda^2} \right)^{b_2/\rho}, \quad (A.5) \]

where the real branch \( W_{-1} \) with \( W < -1 \) has to be taken.

![Comparison of the two-loop running coupling \( \alpha_s/\pi \) resulting from the “exact” result (A.5) (solid line) with the strictly perturbative one from Eq. (A.6) (dashed line) for the case of QCD with \( n_f = 0 \).](image)
On the other hand, the standard perturbative two-loop result is usually given as

\[ x = \frac{-1}{b_2 L} \left( 1 + \frac{\rho \ln L}{b_2} \right) \]  
(A.6)

which assumes small \( x \) and correspondingly large \( L \).

In Fig. 2 we compare the "exact" two-loop result (A.5) for \( x \equiv \alpha_s/\pi \) (solid line) with the strictly perturbative one from Eq. (A.6) (dashed line) for the case of QCD with \( n_f = 0 \) \((b_2 = -\pi / 4, \rho = 51/22)\) as a function of \( L \). The divergence at \( L = 0 \) makes itself felt significantly earlier (as \( L \) approaches zero) in the standard perturbative result (A.6) than in the exact two-loop result (A.5). However, for \( L \gtrsim 4 \), the difference between the two coupling is less than 1.5%. In the perturbative treatment of hot QCD, if one chooses a renormalization scale \( \mu = 2\pi T \) this is indeed the case for all \( T > T_c \approx \Lambda_{\text{MS}} \) (in pure glue QCD a typical value, which we have adopted in this paper, is \( T_c = 1.14 \Lambda_{\text{MS}} \)); a noticeable difference thus arises only for smaller choices of \( \mu / T \) in combination with \( T \) sufficiently close to \( T_c \).