Analytical evaluation of certain on-shell two-loop three-point diagrams

A. I. Davydychev\textsuperscript{a}[Mainz]|Department of Physics, University of Mainz, Staudingerweg 7, D-55099 Mainz, Germany and V. A. Smirnov\textsuperscript{b}[Moscow|Institute for Nuclear Physics, Moscow State University, 119992 Moscow, Russia\textsuperscript{b}

\textsuperscript{a}[Moscow|Institute for Nuclear Physics, Moscow State University, 119992 Moscow, Russia

\textsuperscript{b}[Schlumberger, SPC, 110 Schlumberger Dr., MD-5, Sugar Land, TX 77479, USA

An analytical approach is applied to the calculation of some dimensionally-regulated two-loop vertex diagrams with essential on-shell singularities. Such diagrams are important for the evaluation of QED corrections to the muon decay, QCD corrections to top quark decays $t \to W^+ b$, $t \to H^+ b$, etc.

1. INTRODUCTION

We consider the two-loop diagram shown in Fig. 1, using the one-loop case as an example. All external momenta are ingoing, $P + p + q = 0$, and satisfy $P^2 = M^2$, $p^2 = m^2$. We denote

$$J \equiv \int d^n k \left\{ k^2 + 2(P k) \left[ k^2 - 2(p k) \right] \right\}^{-1} \quad (1)$$

$$F \equiv \int d^n k \int d^n l \left\{ \left[ k^2 + 2(P k) \right] \left[ l^2 + 2(p l) \right] \right\}^{-1} \times \left\{ \left[ k^2 - 2(p k) \right] \left[ l^2 - 2(p l) \right] k^2(k - l)^2 \right\}^{-1} \quad (2)$$

where $n = 4 - 2\varepsilon$ is the space-time dimension.

We assume that $m^2 \ll M^2, |q^2|$ and expand the vertices in the ratio of $m$ and the large parameters. We apply the so-called strategy of expansion by regions [1–4], confining ourselves to the leading power term of the expansion (including all logarithms and the constant part).

In section 2 of [3] (the second example), this approach was applied to $F|_{q^2=0}$. Introduce $n_{1,2} = (\underline{1}, \underline{1}, \underline{0}, \underline{0})$, so that $2(n_{1,2} k) \equiv k_{\pm} = k_0 \pm k_1$. Here “underlined” means all remaining components: e.g., $k = (k_0, k_1, \underline{k})$. It is convenient to choose $P = (-M, 0, \underline{0})$ and $p = \alpha n_1 + (m^2/\alpha)n_2$, with $2M\alpha = M^2 + m^2 - q^2 + [\lambda(M^2, m^2, q^2)]^{1/2}$, where $\lambda(M^2, m^2, q^2) = [(M + m)^2 - q^2][(M - m)^2 - q^2]$.

Then, the relevant regions are

- **hard** ($h$): $k \sim M$
- **1-collinear** ($1c$): $k_+ \sim m^2/M, k_- \sim M, \underline{k} \sim m$
- **2-collinear** ($2c$): $k_+ \sim M, k_- \sim m^2/M, \underline{k} \sim m$
- **ultrasoft** ($us$): $k \sim m^2/M$

Note the change ($1c$) $\rightarrow$ ($2c$), as compared to [3].

For the one-loop diagram $J$, only the ($h$) and ($1c$) contributions are relevant in the leading order of the expansion in $m$. For the two-loop diagram $F$, the following ($k$-$l$) regions yield non-zero contributions: ($h$-$h$), ($1c$-$h$), ($1c$-$1c$) and ($us$-$1c$).

2. ONE-LOOP DIAGRAM

In the on-shell case, the three-point function $J$ reduces to a two-point function with the space-time dimension $2 - 2\varepsilon$, multiplied by $\pi/(2\varepsilon)$ (see section 3.2 of [5]). Therefere, the two-point function should be expanded in $\varepsilon$ up to the next

![Figure 1. One- and two-loop vertex diagrams](image-url)
term. In fact, any term of the \( \varepsilon \) expansion can be calculated in terms of the logarithmic functions (see in [6,5]) whose analytic continuation yields Nielsen polylogarithms \( S_{a,b}(z) \) (see in [7]).

Expanding exact result in \( \varepsilon \) and \( m^2 \), we obtain

\[
J = i\pi^2 e^{-\pi\varepsilon}(M^2)^{-1-\varepsilon}\sigma^{1+2\varepsilon}
\left\{ -\frac{1}{2}\varepsilon^{-1}L^2 - \frac{1}{4}L^2 - \text{Li}_2(u) \right\} + O(\varepsilon, m^2 L^2),
\]

where \( \sigma \equiv M^2/(M^2 - q^2), u \equiv 1 - 1/\sigma = q^2/M^2 \), \( L \equiv \ln(m^2 \sigma^2/M^2) \). Let us check whether the sum of the (h) and (1c) contributions gives the same. For the (h) contribution, \( J|_{m=0} \), we get

\[
J|_{m=0} = i\pi^2 \varepsilon \Gamma(1 + \varepsilon)(M^2)^{-1-\varepsilon}\sigma^{1+2\varepsilon}
\left\{ -\frac{1}{2}\varepsilon^{-2} + \ln^2 \sigma + \text{Li}_2(u) \right\} + O(\varepsilon).
\]

Adding to (4) the (1c) contribution (see in [3]),

\[
i\pi^2 \varepsilon \Gamma(1 + \varepsilon)(M^2)^{-1-\varepsilon}\sigma/m^2 \sigma/2(2\varepsilon)^2,
\]

and expanding in \( \varepsilon \), we reproduce Eq. (3).

3. TWO-LOOP DIAGRAM WITH \( m = 0 \)

The (h-h) region generates Taylor expansion in \( m^2 \) of the integrand of \( F \), yielding \( F|_{m=0} \) in the leading order. Here, we could not employ the methods of [8,9] which were useful when \( p^2 = 0 \). Instead, using [10] we derived a four-fold Mellin–Barnes representation for \( F|_{m=0} \),

\[
\pi^2 e^{-\pi\varepsilon}(M^2)^{-2-2\varepsilon} \cdot \frac{1}{\Gamma(1 - 2\varepsilon)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty dz \; d\tilde{z} \; dt \; dw
\times \sigma^{2+2\varepsilon + z + \tilde{z}} \left[ \Gamma(-t) \Gamma(-w) \Gamma(1 + t + w) \right]
\times \left[ \Gamma(1 + \varepsilon + t + w + z) \Gamma(1 - \varepsilon - t - w + \tilde{z}) \right]
\times \left[ \Gamma(1 + \varepsilon + t - w + z) \Gamma(1 - \varepsilon - t - w + \tilde{z}) \right]
\times \left[ \Gamma(-\varepsilon - t - z) \Gamma(-\varepsilon + t + \tilde{z}) \right]
\times \left[ \Gamma(-\varepsilon - w + z) \Gamma(-\varepsilon + w + \tilde{z}) \right].
\]

The contour integrals separate the right and left series of poles of \( \Gamma \) functions in \( z, \tilde{z}, t \) and \( w \). For small negative \( \varepsilon \), this can be satisfied by straight contours (parallel to the imaginary axes), if we take, say, \( \text{Re}z = \text{Re}\tilde{z} = \frac{1}{2}\varepsilon \), \( \text{Re}t = \varepsilon \), \( \text{Re}w = \frac{1}{2}\varepsilon \). In fact, one can integrate by parts [11] to shrink a line, but this does not simplify the calculation.

The result of a tedious calculation of \( F|_{m=0} \) is

\[
\pi^4 e^{-4\pi\varepsilon}(M^2)^{-2-2\varepsilon}\sigma^{1+4\varepsilon}
\times \left\{ \frac{1}{18}\varepsilon^{-4} + \varepsilon^{-2} \left[ \frac{1}{12}\pi^2 + \frac{1}{4}\text{Li}_2(u) \right]
+ \varepsilon^{-1} \left[ \frac{91}{360}\pi^2 + \frac{3}{2}\text{Li}_2(u) \right]
+ \frac{17\pi^4}{1440} \pi^2 + \frac{7}{12}\pi^2\text{Li}_2(u) - \left[ \text{Li}_2(u) \right]^2
+ 2S_{1,3}(u) + S_{2,2}(u) + \frac{2}{3}\text{Li}_4(u) \right\} + O(\varepsilon).
\]

4. TWO-LOOP DIAGRAM WITH \( m \neq 0 \)

The (1c-1c) and (us-1c) contributions can be trivially obtained from those for the limit with \( q^2 = 0 \) [3,4], substituting \( M^2 \rightarrow M^2 - q^2 \). To calculate the leading-order (1c-h) contribution, one can apply the technique of \( \alpha \) parameters and the Mellin–Barnes representation. In this way, the problem is reduced to a two-fold contour integral which can be evaluated by the standard technique of taking residues and shifting contours.

Collecting all contributions, we obtain for \( F \)

\[
\pi^4 e^{-2\pi\varepsilon}(M^2)^{-2-2\varepsilon}\sigma^{2+4\varepsilon}\left[ \frac{1}{18}\varepsilon^{-2} L^2
- \varepsilon^{-1} \left[ \frac{91}{120} L^3 + \frac{1}{12} \pi^2 L - \frac{7}{12} \text{Li}_2(u) + \zeta_3 \right]
+ \frac{13}{60}L^4 + \frac{5}{16} \pi^2 L^2 - \frac{1}{12} L^2 \text{Li}_2(u)
- L S_{1,2}(u) + \frac{3}{2} L \text{Li}_2(u) + \frac{1}{2} \zeta_3 L - 2 S_{2,2}(u)
+ \frac{1}{2} \left[ \text{Li}_2(u) \right]^2 + \frac{1}{12} \pi^4 \right] + O(\varepsilon, m^2 L^4).
\]

Note that the \( \varepsilon^{-4} \) and \( \varepsilon^{-3} \) terms have cancelled. Eqs. (6)–(7) were checked numerically, using [12]. At \( q^2 = 0 \), they reproduce Eqs. (15) and (21) of [3].

REFERENCES

    Fiz. 89 (1991) 56.