where $|\psi_{i}\rangle$ denotes a Fock state with $n$ photons in mode $i$ of the $S'f$ beam, and $|c_{i}(t)|^2$ is the probability amplitude for finding a photon in mode $i$. Moreover, the output entangled state is given by

$$|\psi_{out}\rangle = \prod_{i=1}^{n} |c_{i}(t)|^2 |\psi_{i}\rangle = \sum_{\{n_{i}\}} |\psi_{\{n_{i}\}}\rangle |\psi_{\{n_{i}\}}\rangle,$$

where $\{n_{i}\}$ labels the number of photons in each mode. The transformation $U(t)$, as defined in Eq. (2), maps the input state into the output state as follows:

$$U(t)|\psi_{in}\rangle = |\psi_{out}\rangle.$$

The entanglement properties of the output state are characterized by the entanglement entropy $S(\rho) = -\sum_{i} \rho_{ii} \log \rho_{ii}$, where $\rho_{ii}$ is the density matrix of the $i$th mode. Note that for a two-mode system, the entanglement entropy is given by $S(\rho) = -\rho_{11} \log \rho_{11} - \rho_{22} \log \rho_{22} - \rho_{12} \log \rho_{12}$. The entanglement entropy can be used to quantify the amount of entanglement in the output state, and it is known that the maximum entanglement is achieved when the two modes are in a maximally entangled state.

In the case of the field $\phi$, the entanglement entropy is given by

$$S(\phi) = -\int \rho_{\phi\phi} \log \rho_{\phi\phi} d\phi,$$

where $\rho_{\phi\phi}$ is the density matrix of the field. The entanglement entropy is known to be zero for a field in a thermal state, which is a classical state, and it is known to be maximal for a squeezed state, which is a quantum state.

In summary, the output state $|\psi_{out}\rangle$ is a maximally entangled state of the two modes, characterized by a high entanglement entropy $S(\rho)$, which indicates the presence of a large amount of entanglement. The entanglement properties of the output state can be used to characterize the amount of entanglement in the input state and to quantify the degree of entanglement in the output state.
where \( \langle n(\vec{q}) \rangle \) is the average number of photons in mode \( \vec{q} \). Almost all down-conversion literature is limited to the case \( \langle n(\vec{q}) \rangle \ll 1 \), in which state (2) reduces to

\[
|\psi\rangle \approx \prod_{\vec{q}} c_0(\vec{q}) |0, \vec{q}\rangle s|0, -\vec{q}\rangle_t + \sum_{\vec{q}} \left\{ c_1(\vec{q}) |1, \vec{q}\rangle s|1, -\vec{q}\rangle_t \prod_{\vec{q}_i \neq \vec{q}} c_0(\vec{q}_1) |0, \vec{q}_1\rangle s|0, -\vec{q}_1\rangle_t \right\} \quad (4)
\]

In this case, which we refer to as the microscopic case, one detects coincidences of single photon pairs; in application to imaging, the image is reconstructed from a statistics over a large number of coincidences. In this paper we focus, instead, on the case in which the average photon number per mode is not negligible, so that all the terms of the expansion (2) are relevant (we call it the macroscopic case). In this case the entanglement is with respect to photon number, and this model predicts ideal perfect correlations in \( S/I \) photon number detected in two symmetric modes \( \vec{q} \) and \( -\vec{q} \) [10].

An interesting analytical limit is that of a short crystal (where diffraction and walk-off of the \( S/I \) fields along the crystal become negligible), where the output state can be written in the form

\[
|\psi\rangle = \prod_{\vec{x}} \left\{ \sum_{n=0} c_n(\vec{x} = 0) |n, \vec{x}\rangle s|n, \vec{x}\rangle_t \right\} \quad (5)
\]

\( \vec{x} \) denotes position in the transverse plane at the crystal exit ("near-field"), and \( |n, \vec{x}\rangle \) is the Fock state with \( n \) photons at point \( \vec{x} \). In this limit one has ideally a perfect correlation in the number of \( S/I \) photons detected at the same near field position. We incidentally note that if only one beam of the two is considered, its reduced density matrix is diagonal in the Fock state basis, and corresponds to a thermal statistics with average of photons given by (3).

In the more sophisticated numerical model, the finite size of the pump has the effect that, if one idler photon is emitted in direction \( \vec{q} \), its twin photon will travel in the symmetric direction \( -\vec{q} \), within an uncertainty \( \delta q \approx 1 / w_p \), which hence represents the uncertainty in the signal transverse momentum when determined from a measurement of the idler transverse momentum. On the other hand, due to the finite length of the crystal, twin photons created in a single down-conversion process at the same position, are separated by diffraction along the crystal. Hence the uncertainty in the position of a signal photon conditioned to the detection of an idler photon at position \( \vec{x} \) is given by a coherence length \( l_{coh} = 1 / q_0 \approx (\lambda c / 2 \pi)^{1/2} \). Really important for the purpose of imaging is the number of pixels that can be resolved in an imaging scheme based on correlation measurements. This number is assessed, both in the near and in the far field, by the ratio \( (w_p / l_{coh})^2 = (q_0 / \delta q)^2 \). The simultaneous presence of entanglement both in momentum and in position is a fundamental property of the down-converted photons, which, as we will see, plays a crucial role in the imaging process. This property persists for a large photon number, a case in which the entanglement assures a photon number spatial correlation at the quantum level both in the near and the far field.

Figure 1 illustrates a compact imaging scheme. The \( S/I \) beams are separated by the polarizing beam splitter PBS (we assume for simplicity that the distance from the crystal exit to PBS is negligible). In the path of the \( S \) beam there is an object, which is imaged by a lens on the far-field plane, where it is detected by a single point-like detector \( D_S \). An identical lens images the \( I \) beam on its detection plane, where it is observed by an array of detectors \( D_I \). The distance \( z \) between the PBS and the lens, and between the lens and \( D_I \), can be varied; we focus on the cases \( z = f \) and \( z = 2f \), in which we will see that the diffraction pattern \( (z = f) \) and the image \( (z = 2f) \) of the object can be reconstructed by correlation measurements. For definiteness, we discuss the case in which the object is a double slit, with \( a \) being the width of the two slits and \( d \) their distance. For \( z = f \), Fig.1 corresponds to the setup of some of the experiments in [6].

The object is imaged by the path of the \( S \) beam, which is observed by a point-like detector, and no information about it can be obtained by direct detection. As a straightforward generalization from the coincidence measurements of the microscopic case[5, 6, 7], we consider the spatial correlation of the \( S/I \) detected intensities. Precisely, we denote with \( I_S(\vec{x}_S) \) and \( I_I(\vec{x}_I) \) the intensities detected by \( D_S \) and by the array \( D_I \) averaged over a detection time \( \tau_D \) (in typical pulsed experiments \( \tau_D \approx \tau_P \)), and we introduce the spatial correlation function

\[
\langle I_S(\vec{x}_I) I_S(\vec{x}_S) \rangle = \langle I_S(\vec{x}_I) \rangle \langle I_S(\vec{x}_S) \rangle + \langle \delta I_S(\vec{x}_I) \delta I_S(\vec{x}_S) \rangle.
\]

The object information is contained in the correlation of intensity fluctuations \( G(\vec{x}_I, \vec{x}_S) = \langle \delta I_I(\vec{x}_I) \delta I_S(\vec{x}_S) \rangle \) as a function of \( \vec{x}_I \) for fixed \( \vec{x}_S \). Together with the scheme (a) of Fig.1, we consider the alternative scheme (b) in which, conversely, the \( S \) beam is detected by an array, and \( I \) by a point-like detector. Such a scheme was analysed in [1] in the microscopic case. In the macroscopic case the image is provided by \( G(\vec{x}_I, \vec{x}_S) \) as a function of \( \vec{x}_S \) for fixed \( \vec{x}_I \).
FIG. 1: (a) Imaging scheme. $P$ = pump beam, $\chi^{(2)}$ = type II crystal, $S$ = signal field, $I$ = idler field, $D_S$, $D_I$ = detectors, $L$ = lens with focal length $f$, $NF$ = near field, $FF$ = far field, $PBS$ = polarizing beam splitter. (b) Scheme for the discussion of fundamental aspects.

Let us first consider the case $z = f$, in scheme (b). If the $I$ field is not detected, there is no possibility of observing the interference fringes by direct measurement of field $S$ alone, unless the object is contained in a coherence area, i.e. $d \ll l_{coh}$. In the microscopic case it was argued [1] that in principle one could detect the $I$ photon, and obtain "which-path" information on the $S$ photon, and this is enough to cancel the fringes. More in general, we argue that, since the $S$ beam alone is in an incoherent thermal mixture, the interference fringes are not visible due to the lack of coherence. However, in order to make fringes visible, it is enough to condition the $S$ beam measurement to a measurement of the $I$ beam by a single point-like detector. In the microscopic case the fringes are observed via coincidence measurements, as explained in [1], because detection of the $I$ photon in the far field determines the $S$ photon momentum before the double slit, due to momentum entanglement, providing a quantum erasure [13] of any which-path information. In the general case, the basic mechanism is the $S/I$ far field intensity correlation, and calculations performed with the analytical model (1) show that

$$G(\vec{x}_S, \vec{x}_I) \propto \left| \tilde{T} \left( \vec{q} = \frac{2\pi}{\lambda f} (\vec{x}_S + \vec{x}_I) \right) \right|^2 \left| U_S \left( -\frac{2\pi \vec{x}_I}{\lambda f} \right) V_I \left( \frac{2\pi \vec{x}_I}{\lambda f} \right) \right|^2,$$

where $\tilde{T}(\vec{q})$ is the Fourier transform of the transmission function $T(\vec{x})$ describing the object. Under the conditions $a \gtrsim l_{coh}$, $d < w_p$, the entire interference-diffraction pattern is visible with good resolution. The result (7) is symmetric with respect to $\vec{x}_S$ and $\vec{x}_I$, hence the same pattern appears in the imaging scheme (a). Fig. 2b shows the result of a 1-D numerical simulation of the pattern reconstruction via intensity correlation function. A statistical average over a reasonable number of pump shots was enough, because $\tau_p$ was on the same order of magnitude as the amplifier coherence time $\tau_{coh} = 1/\Omega_I$. Our calculations show that when $\tau_D \gg \tau_{coh}$, the first term at rhs of Eq. (6) becomes much larger than the second term, that contains all the information about the object, at the expenses of the visibility. Consider now scheme (b) in the $z = 2f$ case, in which the $D_I$ detector lies in the image plane with respect to the object, and the measurement exploits the $S/I$ spatial correlation in the near-field. In the microscopic case fringes are not visible because the detection of the $I$ photon in the near field, due to position entanglement, provides perfect...
FIG. 2: Numerical simulation of the experiment in Fig. 2a with $z = f$. Parameters are those of a 4 mm BB0 crystal, with $w_1 = 332 \mu m$, $\gamma = 1.5 \text{ps}$ ($\tau_{coh} = 0.87 \text{ps}$, $L_{coh} = 9.6 \mu m$), $a = 17 \mu m$, $d = 104 \mu m$. (a) Mean intensity of the $S/I$ beams after 10000 pulses; (b) Solid line: correlation $G(x_I, x_S)$ as a function of $x_I$ after 10000 pulses; dashed line: plane-wave result of Eq.(7), $x_0 = \lambda f q_0/(2\pi)$.

FIG. 3: Numerical simulation of the experiment in Fig. 2a with $z = 2f$. Parameters as in figure 2 The figure shows $G(x_I, x_S)$ as a function of $x_I$ after 10000 pulses

"which path" information about the $S$ photon [1]. Our general result is that, again, for $a > l_{coh}$, $d < w_p$

$$G(\vec{x}_S, \vec{x}_I) \propto |T(\vec{x}_I)|^2 U_S \left( \frac{2 \pi \vec{x}_S}{\lambda f} \right) V_I \left( -\frac{2 \pi \vec{x}_I}{\lambda f} \right).$$

(8)

In scheme (b), where $\vec{x}_I$ is fixed, there is no information about the object. However, in scheme (a) ($\vec{x}_S$ fixed) the object image can be reconstructed via the correlation measurement. Hence, by only changing the optical setup in the path of the idler, which does not go through the object, one is able to pass from the diffraction pattern to the image of an object. This result is confirmed by our numerical simulation shown in Fig. 3.

In order to assess the quantum nature of the phenomena observed in the imaging scheme (a), the key question is whether these results can be reproduced by using a classical mixture, instead of the pure entangled state (2,5). It is natural to focus on the two mixtures

$$W = \prod_q \{ \sum_{n=0}^{N_q} |\phi_n(q)|^2 |n, q\rangle \langle n, -q| \}$$

(9)
\[ W^\alpha = \prod_{\beta} \left\{ \sum_{n=0}^{\infty} |c_{n}(0)|^2 |n, \bar{z}\rangle s_{n} \langle n, \bar{z}| h_{j} \langle \bar{x}, \bar{y}, \bar{z}| \langle \bar{x}, \bar{y}| \right\} \]  

(10)

Mixture (9) preserves the local \( S / I \) spatial intensity correlation in the far field, while the intensity correlation function is completely delocalized in the near field. By following the same notation of [7], we indicate by \( h_{j}(\bar{x}, \bar{z}) \) the linear kernel describing propagation through the imaging setup of beam \( j = 1, S \); we introduce their Fourier transforms \( \hat{h}_{j}(\bar{x}, \bar{q}) \) describing how a \( \bar{q} \) component of the \( j \) beam at the crystal exit face is transformed into the field at point \( \bar{x}_{j} \) at the detection plane. With mixture (9) we obtain

\[
G(\bar{x}, \bar{y}) = \int d\bar{q} |\hat{h}_{S}(\bar{x}, \bar{q})|^2 |\hat{h}_{I}(\bar{x}, -\bar{q})|^2 \left| U_{S}(\bar{q}) V_{I}(-\bar{q}) \right|^2.
\]

(11)

The optical setup in the \( S \) beam arm is fixed, and

\[
\hat{h}_{S}(\bar{x}, \bar{q}) = \frac{1}{i\lambda f} \left( \frac{2\pi}{\lambda f} \bar{x} - \bar{q} \right)
\]

(12)

In the \( z = f \) configuration of Fig 2a, \( |\hat{h}_{I}(\bar{x}, -\bar{q})|^2 \propto \delta(\frac{2\pi}{\lambda f} \bar{x} + \bar{q}) \) and we obtain the same result of Eq.(7). Hence fringes are visible with the classical mixture (9) in the same way as with the pure EPR state (2). However, for \( z = 2f \), \( |\hat{h}_{I}(\bar{x}, -\bar{q})|^2 \propto \exp(i\bar{x} \cdot \bar{q}) \), and the correlation function is constant with \( \bar{x}_{I} \); thus in this case the scheme gives no information at all about the object. Conversely, the mixture (10) preserves the \( S / I \) local intensity correlation only in the near field. Not surprisingly, in this case the \( z = 2f \) scheme (a) provides the image of the object, as with the pure state, but in the \( z = f \) case the fringes are not visible. The key point is that only the pure EPR state (2.5) displays \( S / I \) spatial correlation both in the near and in the far field. This analysis is not in contrast with the basic conclusion of Ref.[11], that no single single experiment in EPI can be reproduced by a classically correlated source. Here we argue that only in the presence of quantum entanglement the whole set of results illustrated in Fig 2b and 3 can be obtained by using a single source, and by keeping the optical setup in the signal beam arm fixed.

In conclusion, we formulated a theory that encompasses both the microscopic (single photon pair detection) and the macroscopic (multi-photon detection) case. Our results show that the imaging and wave-particle duality phenomena, observed in the microscopic case, persist in the macroscopic domain, and indicate a possible experiment that is able to discriminate between the presence of quantum entanglement or classical correlation in the two beams. Clearly, there is a practical limit in the macroscopic level that can be attained preserving such phenomena. In order to increase the number of down-converted photons, usually either the pump beam is more focused (\( w_{p} \) is decreased), or the crystal length \( l_{e} \) is increased. However, when the condition \( w_{p} \approx l_{e} \approx \sqrt{M_{e}} \) is reached, the resolution of the spatial entanglement in the near and far field, as well as in the entangled imaging, is completely lost.

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