Hard Thermal Loops and QCD Thermodynamics *

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The conventional weak-coupling expansion for thermodynamic quantities in hot field
theories shows poor convergence unless the coupling constant is tiny. I discuss screened
perturbation theory (SPT) which is a way of reorganizing the perturbative expansion
for scalar theories and hard-thermal-loop perturbation theory (HTLPT), which is its
generalization to gauge theories. I present results for the pressure to three loops in SPT
and to two loops in HTLPT. We compare the latter with three- and four-dimensional
lattice simulations of pure-glue QCD.

1. Introduction

The heavy-ion collision experiments at RHIC and LHC give us for the first time the
possibility to study the properties of the high-temperature phase of QCD. There are many
methods that can be used to calculate the properties of a quark-gluon plasma. One of these
methods is lattice gauge theory, which gives reliable results for equilibrium properties such
as the pressure but cannot easily be applied to real-time processes. Another method is the
weak-coupling expansion, which can be applied to both static and dynamical quantities.
In the case of the pressure, the weak-coupling expansion has been carried out to order
$g^5$ for massless $\phi^4$ theory [1–3], for QED [4,5], and for nonabelian gauge theories [1,6,7].
However, it turns out that the weak-coupling expansion does not converge unless the
strong coupling constant $\alpha_s$ is tiny. For instance the $g^3$ term is smaller than the $g^2$ term
only if $\alpha_s \simeq 1/20$. This corresponds to a temperature of $10^5$ GeV, which is several orders
of magnitude larger than those relevant for experiments at RHIC and LHC.

In this talk, I will discuss recent advances in the calculation of thermodynamic quantities
in hot field theories based on SPT and HTLPT. Due to lack of time, I cannot critically
compare the approach presented here and other resummation methods that have recently
been advocated by Blaizot, Iancu and Rebhan [9] and by Peshier [10] based on the phi-
derivable approach [11] and hard thermal loops. Instead, I refer to the talk at SEWM
2002 by A. K. Rebhan [12].

2. Thermal scalar field theory

Consider a massless scalar field theory with a quartic interaction. The Euclidean La-
grangian is

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\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g^2}{24} \phi^4 + \Delta \mathcal{L},
\]
where \(\Delta \mathcal{L}\) includes the counterterms needed to remove ultraviolet divergences. At finite temperature, the naive perturbative expansion in \(g^2\) also generates infrared divergences. These infrared divergences can be removed by resummation of the higher order diagrams that generate a thermal mass. The resummed series is then an expansion in \(g\) rather than in \(g^2\). Through order \(g^5\) it reads \([1–3]\):

\[
\mathcal{P} = \mathcal{P}_{\text{ideal}} \left[ 1 - \frac{5}{4} \alpha + \frac{5\sqrt{6}}{3} \alpha^{3/2} + \frac{15}{4} \left( \log \frac{\mu}{2\pi T} + 0.40 \right) \alpha^2 - \frac{15\sqrt{6}}{2} \left( \log \frac{\mu}{2\pi T} - \frac{2}{3} \log \alpha - 0.72 \right) \alpha^{5/2} + \mathcal{O}(\alpha^3 \log \alpha) \right],
\]

where \(\mathcal{P}_{\text{ideal}} = (\pi^2/90)T^4\) is the pressure of an ideal gas of free massless bosons, \(\alpha = g^2(\mu)/16\pi^2\), and \(g(\mu)\) is the \(\overline{\text{MS}}\) coupling constant at the renormalization scale \(\mu\).

In Fig. 1, we show the successive perturbative approximations to \(\mathcal{P}/\mathcal{P}_{\text{ideal}}\) as a function of \(g(2\pi T)\). The bands are obtained by varying the renormalization scale \(\mu\) from \(\pi T\) to \(4\pi T\). In order to express \(g(\mu)\) in terms of \(g(2\pi T)\), we solve the renormalization group equation for the running coupling constant with a five-loop beta function. The poor convergence of the weak-coupling expansion is evident from Fig. 1. The successive approximations fluctuate wildly and the bands become broader with the increasing coupling \(g\). This indicates large theoretical errors.

2.1. Screened Perturbation theory

Screened perturbation theory is a way of reorganizing perturbation theory by adding and subtracting a local mass term in the Lagrangian. It was introduced by Karsch, Patkós and Petreczky \([13]\) and can be made more systematic by using a framework called “optimized perturbation theory” that was applied to a spontaneously broken scalar field theory \([14]\).

The Lagrangian is written as

\[
\mathcal{L} = \mathcal{E}_0 + \frac{1}{2} (\partial_\mu \phi)^2 + (1 - \delta)m^2 + \frac{g^2}{24} \phi^4 + \Delta \mathcal{L} + \Delta \mathcal{L}_{\text{SPT}},
\]

where \(\mathcal{E}_0\) is the vacuum energy term and \(m\) is a mass term. If we set \(\delta = 0\), we recover the original Lagrangian (1). SPT is defined by treating \(\delta\) to be of order \(g^2\) and expanding systematically in powers of \(g^2\). The reorganization of the pertubative series generates new ultraviolet divergences that are cancelled by the addiational counterterms in \(\Delta \mathcal{L}_{\text{SPT}}\).
2.2. Mass prescriptions

I would like to emphasize that the mass parameter \( m \) at this point is completely arbitrary. In order to make definite predictions within SPT, we need a prescription for it as a function of \( g \) and \( T \). The prescription of Karsch, Patkős, and Petreczky for \( m \) is the solution to the one-loop gap equation:

\[
m^2 = \frac{1}{2} \alpha(\mu) \left[ \int \frac{dp}{e^{\beta \sqrt{p^2 + m^2}} - 1} - \left( 2 \log \frac{\mu}{m} + 1 \right) m^2 \right],
\]

Their choice for the renormalization scale was \( \mu = T \). In the weak-coupling limit, the solution to (4) is \( m = g(\mu)T/\sqrt{24} \).

There are many possibilities for generalizing Eq. (4) to higher orders in the coupling constant \( g \). A thorough discussion can be found in Ref. [15].

**Screening Mass:**
The screening mass is given by the pole of the propagator at zero frequency:

\[
p^2 + \Pi(p, 0) = 0, \quad p^2 = m^2_s.
\]

**Tadpole mass:**

\[
m^2 = g^2 \langle \phi^2 \rangle.
\]

**Variational mass:**

\[
\frac{dF}{dm^2} = 0.
\]

A few remarks are in order. In scalar theory, the screening mass can be calculated to all orders in SPT, but in QCD it cannot be calculated beyond leading order due to infrared divergences arising at the nonperturbative magnetic scale \( g^2 T \). The tadpole mass cannot easily be generalized to gauge theories, since a term \( \langle A_\mu A_\mu \rangle \) is gauge variant.

The screening mass, the tadpole mass, and the variational mass all satisfy the same gap equation at leading order in \( g \) and coincides with the gap equation used by Karsch, Patkős, and Petreczky. At two loops, however, the gap equations differ. Fig. 2 shows the two- and three-loop approximations to the pressure normalized to that of an ideal gas of massless bosons. For comparison, the successive weak-coupling approximations to the pressure are also shown (curves labelled 2 to 5). From Fig. 2, it is obvious that SPT converges much better than the weak-coupling expansion. I have not shown the bands that one obtains by varying the renormalization scale \( \mu \) around the central value \( 2\pi T \) by a factor of two, but both bands lie well within the band of the \( g^5 \) approximation that is shown in Fig. 1. This indicates that the uncertainty in SPT is significantly reduced compare to the weak-coupling expansion. See also Ref. [15] for a thorough discussion.
3. Hard Thermal Loop Perturbation Theory

HTLPT is the generalization of SPT to gauge theories \[16\]. One cannot simply add and subtract a local mass term as this would violate gauge invariance \(^2\). Instead one adds and subtracts to the QCD Lagrangian an HTL improvement term:

\[
\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{HTL}} + \Delta \mathcal{L}_{\text{HTL}} ,
\]

(8)

where \(\Delta \mathcal{L}_{\text{HTL}}\) includes extra counterterms necessary. \(\mathcal{L}_{\text{HTL}}\) is proportional to \((1 - \delta)m_D^2\), where \(m_D\) is a variational mass parameter that can be identified with the Debye mass. HTLPT is then a systematic expansion in \(\alpha_s\) and \(\delta\). The QCD pressure at leading order in HTLPT can be calculated exactly by replacing the sum over Matsubara frequencies, extracting the the poles in \(\epsilon\), and then reducing the momentum integrals that were at most two-dimensional and could therefore be easily evaluated \[16,17\]. This is intractable at NLO, and we therefore evaluated the sum integrals approximately by expanding them in powers of \(m_D/T\). This was done at the three loops in SPT \[18\], and it was shown that reasonable approximations are obtained after including a few terms in this expansion.

The final results for the LO and NLO HTLPT predictions for the pressure of pure-glue QCD are plotted in Fig. 3 as a function of \(T/T_c\), where \(T_c\) is the deconfinement transition temperature \[21\]. The bands are again obtained by varying the renormalization scale \(\mu\) by a factor of two around its central value \(\mu = 2\pi T\). The two bands overlap all the way to \(T_c\) and they are very narrow compared to the corresponding bands for the weak-coupling expansion. In Fig. 3, we have also included the four-dimensional lattice results of Boyd et al \[19\] and the three-dimensional lattice results of Kajantie et al \[20\]. The LO and NLO predictions of HTLPT differ significantly from the four-dimensional lattice result in the whole temperature range where they are available. In the high-temperature limit, the HTLPT pressure approaches that of an ideal gas. This is in qualitative agreement with the three-dimensional lattice simulations of Ref. \[20\].

4. Summary

In this talk, I have briefly discussed screened perturbation theory and hard thermal loop perturbation theory which is a way of reorganizing the perturbative expansion for

\(^2\)One can, however, add a term proportional to the Polyakov loop, which would act like a mass term for the zero-frequency component of \(A_0\). I thank L.G. Yaffe for pointing this out to me.
thermal scalar field theory and thermal gauge theories, respectively. Compared to the conventional weak-coupling expansion for thermodynamic quantities such as the pressure, SPT and HTLPT show dramatically improved convergence properties. SPT and HTLPT therefore represent a consistent framework for calculating static and dynamical properties of thermal field theories. However, the resulting predictions for e.g. the pressure fails to agree with temperatures for which they are available. The failure of HTLPT for temperatures below $\sim 20T_c$ could be that a quasi-particle picture is not a good one and a description in terms of e.g. Wilson lines [22] is more appropriate.

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REFERENCES

12. A. K. Rebhan, these proceedings.