Hamiltonian lattice quantum chromodynamics at finite density with Wilson fermions

Yi-Zhong Fang, and Xiang-Qian Luo*

Department of Physics, Zhongshan (Sun Yat-Sen) University, Guangzhou 510275, China

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Abstract

Quantum chromodynamics (QCD) at sufficiently high density is expected to undergo a chiral phase transition. In Lagrangian SU(3) lattice gauge theory, the standard approach breaks down at large chemical potential $\mu$, due to the complex action problem. (QCD at large $\mu$ is of particular importance for neutron star or quark star physics). The Hamiltonian formulation of lattice QCD doesn’t encounter such a problem. In a previous work, we developed a Hamiltonian approach at finite chemical potential $\mu$ and obtained reasonable results in the strong coupling regime. In this paper, we extend the previous work to Wilson fermions. We study the chiral behavior and calculate the vacuum energy, chiral condensate and quark number density, as well as the masses of light hadrons. There is a first order chiral phase transition at zero temperature.

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*Corresponding author. Email address: stslxq@zsu.edu.cn
1 Introduction

Quantum Chromodynamics (QCD) is the fundamental theory of strong interactions. It is a SU(3) gauge theory of quarks and gluons. Precise determination of the QCD phase diagram on temperature $T$ and chemical potential $\mu$ plane will provide valuable information for the experimental search for quark-gluon plasma (QGP). The ultimate goal of machines like the Relativistic Heavy Ion Collider (RHIC) at BNL and the Large Hadron Collider (LHC) at CERN is to create the QGP phase, and replay the birth and evolution of the Universe. Such a new state of matter may also exist in the core of neutron stars or quark stars at low temperature $T$ and large chemical potential $\mu$. Lattice gauge theory (LGT), proposed by Wilson[1] is a first principle non-perturbative method for QCD. Although it is the most reliable technique for investigating phase transitions in QCD, it is not free of problems: complex action at finite chemical potential and species doubling with naive fermions.

In Lagrangian formulation of LGT at finite chemical potential, the success is limited to SU(2) gauge theory[2, 3], while in the physical SU(3) case, complex action[4, 5] spoils numerical simulations with importance sampling. Even though much effort[6, 7, 8] has recently been made for SU(3) LGT, and some very interesting information on the phase diagram at large $T$ and small $\mu$ has been obtained, it is still extremely difficult to do simulations at large chemical potential. QCD at large $\mu$ is of particular importance for neutron star or quark star physics. Hamiltonian formulation of LGT doesn’t encounter the notorious “complex action problem”. Recently, we proposed a Hamiltonian approach to LGT at finite chemical potential[9, 10] and solve it in the strong coupling regime. We predicted that at zero temperature, there is a first order chiral phase transition at critical chemical potential $\mu_C = m^{(0)}_{\text{dyn}} = M^{(0)}_N / 3$, with $m^{(0)}_{\text{dyn}}$ and $M^{(0)}_N$ being the dynamical mass of quark and nucleon mass at $\mu = 0$ respectively. (We expect this is also true for Kogut-Susskind fermions.) By solving the gap and Bethe-Salpeter equations, the authors of Ref. [11]obtained the critical point the same as ours; but they concluded that the chiral transition is of second order, different from ours. (Our order of transition is consistent with other lattice simulation results[12]).

Wilson’s approach to lattice fermions[1] has been extensively used in hadron spectrum calculations as well as in QCD at finite temperature. It avoids the species doubling and preserves the flavor symmetry, but it explicitly breaks the chiral symmetry[13, 14, 15, 16], one of the most important symmetries of the original theory. Non-perturbative fine-tuning of the bare fermion mass has to be done, in order to define the chiral limit[17, 18].

In this paper, we study Hamiltonian lattice QCD with Wilson fermions at finite chemical potential. We derive the effective Hamiltonian in the strong coupling regime and diagonalize it by Bogoliubov transformation. The vacuum energy, chiral condensate, and masses of pseudo-scalar, vector meson and nucleons are computed. In the non-perturbatively defined chiral limit, we obtain reasonable results for the critical point and other physical quantities under the mean-field approximation.

To our knowledge, the only existing literature about the same system ($r \neq 0$ and $\mu \neq 0$) is Ref.[19], where the author used a very different approach: the solution to the gap equation. In contrary to the conventional predictions[17], the author found that that even at $\mu = 0$, there is a critical value for the effective four fermion coupling $K$, below which dynamical mass of quark
vanishes. He introduced the concept of total chemical potential and found the transition order depends on the input parameters $K$ and $r$ as well as the momentum. In contrast, we find that in the chiral limit, dynamical mass of quark doesn’t vanish for all values of $K$ if $\mu < \mu_C$; at and our order of chiral phase transition doesn’t depend on the input parameter.

The rest of the paper is organized as follows. In Sec.2, we derive the effective Hamiltonian at finite chemical potential. In Sec.3, we present the results for vacuum energy, chiral condensate, and the hadron masses are presented. In Sec.4, we estimate the critical chemical potential at zero temperature. The results are summarized in Sec.5.

2 Effective Hamiltonian in the strong coupling regime

2.1 The $\mu = 0$ case

We begin with QCD Hamiltonian[18] with Wilson fermions at chemical potential $\mu = 0$ on 1 continuum time and 3 spatial dimensional discretized lattice,

\[
H = M \sum_x \bar{\psi}(x)\psi(x) + \frac{1}{2a} \sum_x \sum_{k=\pm 1}^{\pm d} \bar{\psi}(x)\gamma_k U(x,k)\psi(x + \hat{k})
\]

\[
- \frac{r}{2a} \sum_x \sum_{k=\pm 1}^{\pm d} \bar{\psi}(x)U(x,k)\psi(x + \hat{k}) + \frac{g^2}{2a} \sum_x \sum_{j=1}^d E^\alpha_j(x)E^\alpha_j(x)
\]

\[
- \frac{1}{ag^2} \sum_p \text{Tr} \left( U_p + U_p^+ - 2 \right),
\]

where

\[
M = m + \frac{rd}{a},
\]

d = 3 is the spatial dimension and $m$, $a$, $r$ and $g$ are respectively the bare fermion mass, spatial lattice spacing, Wilson parameter, and bare coupling constant. $U(x,k)$ is the gauge link variable at site $x$ and direction $k$, and $\psi$ is the fermion field and its color, flavor and Dirac indices are summed in the Hamiltonian. The convention $\gamma_{-k} = -\gamma_k$ is used. $E^\alpha_j(x)$ is the color-electric field at site $x$ and direction $j$, and summation over $\alpha = 1, 2, ..., 8$ is implied. $U_p$ is the product of gauge link variables around an elementary spatial plaquette, and it represents the color magnetic interactions. Here we use the Lurie metric, and the author of Ref. [17] used a different metric. Both give the equivalent Hamiltonians and lead to the same results for the physical quantities. In the continuum limit $a \to 0$, Eq.(1) approaches to the continuum QCD Hamiltonian in the temporal gauge $A_4 = 0$.

In Ref.[18], we derived an effective Hamiltonian using the strong coupling expansion up to the second order

\[
H_{eff} = M \sum_x \bar{\psi}(x)\psi(x) - \frac{K(r^2 + 1)d}{a} \sum_x \bar{\psi}(x)\psi(x)
\]

2
\[ + \frac{K}{8aN_c} \sum_x \sum_{k=\pm j} \left[ (r^2 + 1) \bar{\psi}_{f_1}(x) \psi_{f_2}(x) \psi_{f_2}^\dagger(x + \hat{k}) \psi_{f_1}(x + \hat{k}) \right. \\
\left. + (r^2 - 1) \bar{\psi}_{f_1}(x) \gamma_4 \psi_{f_2}(x) \psi_{f_2}^\dagger(x + \hat{k}) \gamma_4 \psi_{f_1}(x + \hat{k}) \right. \\
\left. - (r^2 - 1) \bar{\psi}_{f_1}(x) \gamma_5 \psi_{f_2}(x) \psi_{f_2}^\dagger(x + \hat{k}) \gamma_5 \psi_{f_1}(x + \hat{k}) \right. \\
\left. + (r^2 + 1) \bar{\psi}_{f_1}(x) \gamma_4 \gamma_5 \psi_{f_2}(x) \psi_{f_2}^\dagger(x + \hat{k}) \gamma_4 \gamma_5 \psi_{f_1}(x + \hat{k}) \right. \\
\left. + \left( r^2 + (1 - 2\delta_{k,j}) \right) \bar{\psi}_{f_1}(x) \gamma_4 \gamma_j \psi_{f_2}(x) \psi_{f_2}^\dagger(x + \hat{k}) \gamma_4 \gamma_j \psi_{f_1}(x + \hat{k}) \right. \\
\left. - \left( r^2 - (1 - 2\delta_{k,j}) \right) \bar{\psi}_{f_1}(x) \gamma_j \psi_{f_2}(x) \psi_{f_2}^\dagger(x + \hat{k}) \gamma_j \psi_{f_1}(x + \hat{k}) \right. \\
\left. - \left( (1 - 2\delta_{k,j}) + r^2 \right) \bar{\psi}_{f_1}(x) \gamma_4 \sigma_j \psi_{f_2}(x) \psi_{f_2}^\dagger(x + \hat{k}) \gamma_4 \sigma_j \psi_{f_1}(x + \hat{k}) \right. \\
\left. + \left( (1 - 2\delta_{k,j}) - r^2 \right) \bar{\psi}_{f_1}(x) \sigma_j \psi_{f_2}(x) \psi_{f_2}^\dagger(x + \hat{k}) \sigma_j \psi_{f_1}(x + \hat{k}) \right], \tag{3} \]

where \( \sigma_j = \epsilon_{j_1 j_2} \gamma_{j_1} \gamma_{j_2} \). Each fermion field carries spin (Dirac), color and flavor indices, and in Eq.(3), the flavor indices are explicitly written. This effective Hamiltonian describes the nearest-neighbor four-fermion interactions, with

\[ K = \frac{1}{g^2 C_N} \tag{4} \]

being the effective coupling constant. Here \( C_N = (N_c^2 - 1)/(2N_c) \) is the Casimir invariant of the SU\( (N_c) \) gauge group.

To simplify the four-fermion operators in Eq.(3), we rewrite the bilinear operators of fermions as

\[ \bar{\psi}_{f_1}(x) \psi_{f_2}(x) = \bar{v} \delta_{f_1 f_2} + (\bar{\psi}_{f_1}(x) \psi_{f_2}(x) - \bar{v} \delta_{f_1 f_2}), \]
\[ \psi_{f_1}^\dagger(x) \psi_{f_2}(x) = v^\dagger \delta_{f_1 f_2} + (\psi_{f_1}^\dagger(x) \psi_{f_2}(x) - v^\dagger \delta_{f_1 f_2}), \tag{5} \]

where \( \bar{v} \) and \( v^\dagger \) denote respectively the expectation value of \( \bar{\psi} \psi \) and \( \psi^\dagger \psi \) in the vacuum state \( |\Omega^{eff}\rangle \) of \( H_{eff} \), i.e.,

\[ \bar{v} = \frac{1}{N_f N_s} \langle \Omega^{eff} | \sum_x \bar{\psi} \psi | \Omega^{eff} \rangle, \]
\[ v^\dagger = \frac{1}{N_f N_s} \langle \Omega^{eff} | \sum_x \psi^\dagger \psi | \Omega^{eff} \rangle. \tag{6} \]

Here \( N_s \) is the total number of lattice sites and \( N_f \) the number of flavors. Substituting them into Eq.(3), the effective Hamiltonian reduces to a diagonalized one

\[ H_{eff} = A \sum_x \bar{\psi}(x) \psi(x) + B \sum_x \psi^\dagger(x) \psi(x) + C, \tag{7} \]

where

\[ A = M - \frac{Kd}{2aN_c} (1 - r^2) \bar{v}, \]
\[ B = \frac{Kd(1 + r^2)}{a} \left( \frac{v^\dagger}{2N_c} - 1 \right), \]
\[ C = -\frac{Kd}{4aN_c} \left[ (1 + r^2)v^2 - (1 - r^2)\bar{v}^2 \right] N_s N_f. \]  

In deriving Eq.(7), we have used the mean-field approach: i.e., an approximation to ignore terms like \((\bar{\psi}(x)\psi(x) - \bar{v})^2\) and \((\bar{v}^\dagger(x)\psi(x) - v^\dagger)^2\). In doing so, the contribution of the meson operators to the Hamiltonian is neglected (we will reconsider these contributions in next section), and only the first four terms of Eq.(3) don’t vanish.

In Eq.(7), the fermion field \(\psi\) can now be expressed as
\[ \psi(x) = \begin{pmatrix} \xi(x) \\ \eta^+(x) \end{pmatrix}. \]  

The 2-spinors \(\xi\) and \(\eta^\dagger\) are the annihilation operator of positive energy fermion and creation operator of negative energy fermion respectively. In Eq.(7), \(A\) plays the role of dynamical mass of quark.

### 2.2 The \(\mu \neq 0\) case

In the continuum, the grand canonical partition function of QCD at finite temperature \(T\) and chemical potential \(\mu\) is
\[ Z = \text{Tr} \ e^{-\beta(H - \mu N)}, \quad \beta = (k_BT)^{-1}, \]  
where \(k_B\) is the Boltzmann constant and \(N\) is particle number operator
\[ N = \sum_x \bar{\psi}(x)\psi(x). \]  

According to Eq.(10) and following the procedure in Sec.2.1, the role of the Hamiltonian at strong coupling is now played by
\[ H^\text{eff}_\mu = H^\text{eff} - \mu N, \]  
where \(H^\text{eff}\) is given by Eq.(3) or in the mean-field approximation by Eq.(7). In this Hamiltonian, there are three input parameters: \(r, m\) and \(\mu\). Suppose we study the phase structure of the system in the chiral limit. Such a limit can be reach by fine-tuning the bare quark mass \(m\) so that the pion becomes massless. In such a case, there are only two input parameters: \(r\) and \(\mu\).

Let us define the state \(|n_p, \bar{n}_p\rangle\) in the momentum space by
\[ \xi_p|0_p, \bar{n}_p\rangle = 0, \quad \xi^\dagger_p|0_p, \bar{n}_p\rangle = |1_p, \bar{n}_p\rangle, \quad \xi_p|1_p, \bar{n}_p\rangle = |0_p, \bar{n}_p\rangle, \quad \xi^\dagger_p|1_p, \bar{n}_p\rangle = 0, \]
\[ \eta_p|n_p, 0_p\rangle = 0, \quad \eta^\dagger_p|n_p, 0_p\rangle = |n_p, 1_p\rangle, \quad \eta_p|n_p, 1_p\rangle = |n_p, 0_p\rangle, \quad \eta^\dagger_p|n_p, 1_p\rangle = 0. \]  

The numbers \(n_p\) and \(\bar{n}_p\) take the values 0 or 1 due to the Pauli principle. By definition, the up and down components of the fermion field are decoupled. For the ground state of \(H^\text{eff}_\mu\), we make an ansatz
\[ |\Omega^\text{eff}_\mu\rangle = \sum_p f_{n_p, \bar{n}_p} |n_p, \bar{n}_p\rangle. \]
The vacuum energy is the expectation of $H_\mu$ in its ground state $|\Omega\rangle$, and also the expectation value of $H_\mu^{\text{eff}}$ in its ground state $|\Omega^{\text{eff}}\rangle$, i.e.

$$E_\Omega = \langle \Omega | H_\mu | \Omega \rangle = \langle \Omega^{\text{eff}} | H_\mu^{\text{eff}} | \Omega^{\text{eff}} \rangle = \sum_{p', p} f_{n', \bar{n}} f_{n, \bar{n}} \langle n_p, \bar{n}_p | H_\mu^{\text{eff}} | n_{p'}, \bar{n}_{p'} \rangle$$

$$= \sum_p C_{n_p, \bar{n}_p} (n_p, \bar{n}_p | H_\mu^{\text{eff}} - \mu N | n_{p'}, \bar{n}_{p'}) = \sum_{p'} f_{n', \bar{n}} f_{n, \bar{n}} \langle n_p, \bar{n}_p | H_\mu^{\text{eff}} | n_{p'}, \bar{n}_{p'} \rangle,$$

where we have introduced the notation $C_{n_p, \bar{n}_p} = f_{n_p, \bar{n}_p}^2$.

Under the mean field approximation, the chiral condensate and quark number density are given by

$$\frac{\langle \bar{\psi} \psi \rangle}{N_f N_s} = \bar{v} = \frac{1}{N_f N_s} \sum_p C_{n_p, \bar{n}_p} \langle n_p, \bar{n}_p | \bar{\psi} \psi | n_p, \bar{n}_p \rangle = 2N_c (n + \bar{n} - 1),$$

$$n_q = \frac{\langle \bar{\psi} \psi \rangle}{2N_c N_f N_s} - 1 = \frac{\bar{v}}{2N_c} - 1 = \frac{1}{2N_c N_f N_s} \sum_p C_{n_p, \bar{n}_p} \langle n_p, \bar{n}_p | \bar{\psi} \psi | n_p, \bar{n}_p \rangle - 1 = n - \bar{n},$$

which are constrained in the range of $[0, 1]$ and determined by minimizing the vacuum expectation value of $H_\mu^{\text{eff}}$.

### 3 Physical quantities at $\mu \neq 0$ and $T = 0$

In order to calculate the masses of mesons as well as the contributions of the meson operators to the vacuum energy, we identify the annihilation and creation operators for pseudo-scalar and vector (polarized in the $l$th direction) mesons as

$$\Pi_{f_1 f_2} (x) = \frac{1}{2\sqrt{-v}} \bar{\psi}_{f_1} (x) (1 - \gamma_4) \gamma_5 \psi_{f_2} (x),$$

$$\Pi_{f_2 f_1}^\dagger (x) = \frac{1}{2\sqrt{-v}} \bar{\psi}_{f_2} (x) (1 + \gamma_4) \gamma_5 \psi_{f_1} (x),$$

$$V_{l f_1 f_2} (x) = \frac{1}{2\sqrt{-v}} \bar{\psi}_{f_1} (x) (1 - \gamma_4) \gamma_l \psi_{f_2} (x),$$

$$V_{l f_2 f_1}^\dagger (x) = \frac{1}{2\sqrt{-v}} \bar{\psi}_{f_2} (x) (1 + \gamma_4) \gamma_l \psi_{f_1} (x).$$

(18)
The factor $1/(2\sqrt{\sigma})$ is introduced so that $\langle \Pi(x)\Pi^\dagger(x) \rangle = \langle V(x)V^\dagger(x) \rangle = 1$. Then effective Hamiltonian $H_{\mu}^{eff}$ in Eq.(12) can now be expressed in terms of pseudo-scalar and vector particle operators in the following way

$$H_{\mu}^{eff} = E_{\Omega}^{(0)} + H_{\Pi} + H_{V},$$

$$E_{\Omega}^{(0)} = N_f N_s \left( M\bar{\nu} - Kd \left( \frac{1 + r^2}{a} \right) + \mu \right) \bar{v}^\dagger + \frac{Kdr^2}{4aN_c} \left( \bar{v}^\dagger \bar{v} + \bar{v} \bar{v} \right) - \frac{Kd}{4aN_c} \left( \bar{v} - \bar{v}^\dagger \right),$$

$$H_{\Pi} = \left( 2M - \frac{Kd(1 - r^2)}{aN_c} \right) \bar{v}^\dagger \sum_{x,f_1,f_2} \Pi_{f_2f_1}^\dagger(x) \Pi_{f_1f_2}(x)$$

$$+ \frac{Kr^2}{4aN_c} \bar{v}^\dagger \sum_{x,f_1,f_2,k} \left[ \Pi_{f_1f_2}^\dagger \Pi_{f_2f_1}(x + \hat{k}) + \Pi_{f_2f_1}^\dagger \Pi_{f_1f_2}(x + \hat{k}) \right]$$

$$- \frac{K}{4aN_c} \bar{v} \sum_{x,f_1,f_2,k} \left[ \Pi_{f_1f_2}^\dagger \Pi_{f_2f_1}^\dagger (x + \hat{k}) + \Pi_{f_2f_1} \Pi_{f_1f_2}(x + \hat{k}) \right],$$

$$H_{V} = \left( 2M - \frac{Kd(1 - r^2)}{aN_c} \right) \bar{v} \sum_{x,f_1,f_2,l} V_{f_2f_1}^\dagger(x) V_{f_1f_2}(x)$$

$$+ \frac{Kr^2}{4aN_c} \bar{v} \sum_{x,f_1,f_2,k,l} \left[ V_{f_1f_2}^\dagger(x) V_{f_2f_1}^\dagger (x + \hat{k}) + V_{f_2f_1}(x) V_{f_1f_2}^\dagger (x + \hat{k}) \right]$$

$$- \frac{K}{4aN_c} \bar{v} \sum_{x,f_1,f_2,k,j} \left[ V_{f_1f_2}^\dagger(x) V_{f_2f_1}^\dagger (x + \hat{k}) + V_{f_1f_2}(x) V_{f_2f_1}^\dagger (x + \hat{k}) \right] \left( 1 - 2\delta_{k,j} \right).$$

$v_2^\dagger$ and $\bar{v}_2$ are the expectation value of the four-fermion operators

$$v_2^\dagger = \frac{1}{N_f N_s} \sum_p C_{n_p,n_p} \langle n_p, \bar{n}_p | \psi_{f_1,p}^\dagger \psi_{f_2,p}^\dagger \psi_{f_2,p} \psi_{f_1,p} | n_p, \bar{n}_p \rangle$$

$$= \frac{(2N_c)^2}{N_s} \sum_p C_{n_p,n_p} (n_p - \bar{n}_p + 1)^2,$$

$$\bar{v}_2 = \frac{1}{N_f N_s} \sum_p C_{n_p,n_p} \langle n_p, \bar{n}_p | \psi_{f_1,p}^\dagger \psi_{f_2,p}^\dagger \psi_{f_2,p} \psi_{f_1,p} | n_p, \bar{n}_p \rangle$$

$$= \frac{(2N_c)^2}{N_s} \sum_p C_{n_p,n_p} (n_p + \bar{n}_p - 1)^2. \hspace{1cm} (20)$$

In Eq.(19), we have ignored the non-meson terms which give no contribution to the energy. After a Fourier transformation

$$\Pi_{f_1f_2}(x) = \sum_p e^{ipx} \tilde{\Pi}_{f_1f_2}(p),$$

$$H_{\Pi} \text{ in Eq.(19) becomes}$$

$$H_{\Pi} = \left( 2M - \frac{Kd}{aN_c}(1 - r^2) \right) \sum_{p,f_1,f_2} \tilde{\Pi}_{f_1f_2}^\dagger(p) \tilde{\Pi}_{f_2f_1}(p)$$
\[ + \frac{K r^2}{2 a N_c} \bar{v} \sum_{f_1, f_2} \sum_p \left( \tilde{\Pi}_{f_1 f_2}(p) \tilde{\Pi}_{f_2 f_1}(p) + \tilde{\Pi}(p) \tilde{\Pi}_{f_1 f_2}(p) \right) \sum_j \cos p_j a \]
\[- \frac{K}{2 a N_c} \bar{v} \sum_{p, f_1, f_2} \left( \tilde{\Pi}_{f_1 f_2}(p) \tilde{\Pi}_{f_2 f_1}(-p) + \tilde{\Pi}_{f_1 f_2}(-p) \tilde{\Pi}_{f_2 f_1}(p) \right) \sum_j \cos p_j a. \] (22)

The Bogoliubov transformation\^[18]\]
\[ \tilde{\Pi}(p) \rightarrow \tilde{\Pi}(p) \cosh u_p + \tilde{\Pi}^\dagger(-p) \sinh u_p, \]
\[ \tilde{\Pi}^\dagger(p) \rightarrow \tilde{\Pi}^\dagger(p) \cosh u_p + \tilde{\Pi}(p) \sinh u_p, \] (23)
diagonalizes \( H_{\Pi} \) if
\[ \tanh 2 u_p = - \frac{2 G_2}{G_1} \sum_j \cos p_j a. \]
\[ G_1 = 2 M - \frac{K d}{a N_c} \bar{v}(1 - r^2) + \frac{K r^2}{a N_c} \bar{v} \sum_j \cos p_j a, \]
\[ G_2 = - \frac{K}{2 a N_c} \bar{v}. \] (24)

The resulting \( H_{\Pi} \) is
\[ H_{\Pi} = G_1 \sum_{p, f_1, f_2} (1 - \tanh^2 2 u_p)^{\frac{1}{2}} \tilde{\Pi}_{f_1 f_2}^\dagger(p) \tilde{\Pi}_{f_2 f_1}(p) \]
\[ - \frac{G_1}{2} N_f^2 \sum_p \left[ 1 - (1 - \tanh^2 2 u_p)^{\frac{1}{2}} \right] + \frac{2 G_2 r^2}{G_1} \sum_j \cos p_j a \right) \]. (25)

According to Eq. (25), the difference between the pseudo-scalar meson energy and vacuum energy is
\[ E_{\Pi} = G_1 \left[ 1 - \tanh^2 2 u_p \right]^{\frac{1}{2}} \]
\[ = \left[ 2 M - \frac{K d}{a N_c} \bar{v}(1 - r^2) - \frac{K}{a N_c} (1 - r^2) \bar{v} \sum_j \cos p_j a \right]^{\frac{1}{2}} \]
\[ \times \left[ 2 M - \frac{K d}{a N_c} \bar{v}(1 - r^2) + \frac{K}{a N_c} (1 + r^2) \bar{v} \sum_j \cos p_j a \right]^{\frac{1}{2}}, \] (26)
which gives the pseudo-scalar mass when \( p_j = 0 \). The pseudo-scalar mass square is
\[ M_{\Pi}^2 = E_{\Pi}^2 |_{p_j=0} = 4 \left( M + \frac{K d r^2}{a N_c} \bar{v} \right) \left( M + \frac{K d r^2}{a N_c} \bar{v} - \frac{K d \bar{v}}{a N_c} \right). \] (27)
In order to define the chiral limit, one has to fine tune $M \to M_{\text{chiral}}$ so that the pion becomes massless. From Eq. (27), we get

$$M_{\text{chiral}} = -\frac{K d_r^2}{a N_c} \bar{v}. \quad (28)$$

In this limit, the pseudo-scalar mass square behaves as $M^2_\Pi \propto M - M_{\text{chiral}}$, which is the PCAC relation.

The vector meson sector can be considered in a similar way. After a Fourier transformation

$$V_l(x) = \sum_p e^{ipx} \tilde{V}_l(p) \quad (29)$$

and a Bogoliubov transformation

$$\tilde{V}_l(p) \to \tilde{V}_l(p) \cosh w_p l + \tilde{V}^\dagger_l (-p) \sinh w_p l, \quad \tilde{V}_l^\dagger(p) \to \tilde{V}_l^\dagger(p) \cosh w_p l + \tilde{V}_l(p) \sinh w_p l, \quad (30)$$

$H_V$ becomes

$$H_V = G_1 \sum_{p,l,f_1,f_2} \left[ 1 - \tanh^2 2w_p^{(l)} \right] \tilde{V}^\dagger_{f_1 f_2} (p) V_{f_2 f_1} (p)$$

$$- G_1^2 N_f^2 \sum_{p,l} \left[ 1 - \left( 1 - \tanh^2 2w_p^{(l)} \right) \right] + \frac{2G_2}{G_1^2} r^2 \sum_j \cos p_j a \right), \quad (31)$$

if

$$\tanh 2w_p^{(l)} = -\frac{2G_2}{G_1} (\sum_{j=1}^d \cos p_j a - 2 \cos p_l a). \quad (32)$$

The vector mass is

$$M_V = G_1 \left( 1 - \tanh^2 2w_0^{(l)} \right) \frac{1}{2} \sqrt{M - M_c} - \frac{2K \sqrt{d - 1}}{a N_c} \bar{v}. \quad (33)$$

According to Eqs. (19), (25) and (31), the vacuum energy reads

$$E_\Omega = \langle \Omega | H_{\mu \nu}^\text{eff} | \Omega \rangle$$

$$= E^{(0)}_\Omega - \mu N_f N_s v^\dagger - \frac{G_1}{2} N_f^2 \sum_p \left[ 1 - \left( 1 - \tanh^2 2w_p^{(l)} \right) \right] + \frac{2G_2}{G_1} r^2 \sum_j \cos p_j a \right)$$

$$- \frac{G_1^2}{2} N_f^2 \sum_{p,l} \left[ 1 - \left( 1 - \tanh^2 2w_p^{(l)} \right) \right] + \frac{2G_2}{G_1^2} r^2 \sum_j \cos p_j a \right). \quad (34)$$

This also gives the thermodynamic potential (grand potential) at $T = 0$.

For an generic nucleon operator $O_B$ consisting of three quarks, the thermo mass is given by

$$M_N = \langle \Omega | H_{\mu \nu}^\text{eff} - \mu N | O_B^\dagger \Omega | H_{\mu \nu}^\text{eff} - E_\Omega \rangle.$$
4 Phase structure at $T = 0$ and $\mu \neq 0$

For simplification, we will use the mean field approximation to study the critical behavior of the system. It is easy to prove that this is sufficient for determining the critical chemical potential. From Eqs.(7), (8), and (16), we obtain the normalized vacuum energy at $T = 0$ in the chiral limit

\[
\epsilon_{\Omega} = \frac{E_{\Omega}}{2NcN_fN_s} = M_{\text{chiral}}(n + \bar{n} - 1) - \frac{Kd\sigma^2}{a}(n - \bar{n} + 1) + \frac{Kd\sigma^2}{a}(n^2 + \bar{n}^2 + 1 - 2\bar{n}) + \frac{Kd\sigma}{a}(n + \bar{n} - 2n\bar{n} - 1) - \mu(n - \bar{n} + 1).
\]

(37)

The ground state of the system corresponds to the lowest value of $\epsilon_{\Omega}$. At some given inputs of Wilson parameter $r$ and chemical potential $\mu$, we can find the value of $n$ and $\bar{n}$ when $\epsilon_{\Omega}$ is minimized. The result is

\[
n = \Theta(\mu - \mu_C),
\]

\[
\bar{n} = 0.
\]

(38)

Here $\Theta$ is the step function and $\mu_C$ is the critical chemical potential

\[
\mu_C = \frac{Kd}{a}(1 + 2r^2).
\]

(39)

According to Eq. (16), the results for the chiral condensate and quark number density under mean-field approximation are

\[
\frac{1}{N_fN_s}\langle \bar{\psi}\psi \rangle = 2N_c[\Theta(\mu - \mu_C) - 1],
\]

\[
n_q = \Theta(\mu - \mu_C).
\]

(40)

The inclusion of the meson fields gives qualitatively the same results for these quantities. There is clearly is a first order chiral phase transition. For $\mu < \mu_C$, the system is in the chiral-symmetry breaking phase; in this phase the results for the chiral condensate, the pseudo-scalar and the vector masses are the same as those in Refs. [17, 18] at $\mu = 0$, where the system is in the confinement phase. For $\mu > \mu_C$, chiral symmetry is restored.

According to Eqs. (8), (35) and (38), the thermo mass of the nucleon is

\[
M_N = M_N^{(0)} - 3\mu,
\]

(41)

where

\[
M_N^{(0)} = 3m_{d_{\text{dyn}}}.
\]

(42)
is the nucleon mass at $\mu = 0$, and

$$m_{\text{dyn}}^{(0)} = \frac{K d(1 + r^2)}{a}$$

(43)

is the dynamical mass of quark at $\mu = 0$. At $\mu = M_N^{(0)}/3$, the nucleon mass vanishes before the chiral phase transition takes place. This is not surprising because Wilson fermions break explicitly the chiral symmetry. The value of $\mu_C$ should coincide with $M_N^{(0)}/3$ when $r$ is very small, i.e., the case of Ref. [9].

5 Discussions

In the preceding sections, we have investigated (d+1)-dimensional lattice QCD at finite density with Wilson fermions in the strong coupling regime. We compute the vacuum energy, meson and nucleon masses, chiral condensate and quark number density. At finite chemical potential, there is an interplay between the bare fermion mass in the chiral limit and the chiral condensate, which have to be determined self-consistently. There is a first order chiral phase transition. The thermo mass of the nucleon vanishes before the chiral transition takes place. This is due to the explicit breakdown of chiral symmetry by Wilson fermions.

We have not yet specified the nature of the chiral-symmetric phase for $\mu > \mu_C$. Is it a QGP phase or a color-superconducting phase[20, 21]? Up to now, there has been no first principle investigation of such a phase in SU(3) gauge theory. The answer to this question might be very important to our understanding of the formation of the neutron star or quark star.

We also know that the strong coupling regime is far from the continuum limit. One has to develop a numerical method to study continuum physics. The Monte Carlo Hamiltonian method developed recently[22, 23] might be useful for such a purpose. We hope to discuss these interesting issues in the future.

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