D M APPLEBY  
Department of Physics, Queen Mary, University of London, Mile End Rd, London  
E1 4NS, UK  
(E-mail: D.M.Appleby@qmul.ac.uk)  

Abstract  
Hess and Philipp have constructed what, they claim, is a local hidden variables model reproducing the empirical predictions of quantum mechanics. In this paper explicit expressions for the conditional probabilities for the outcomes of the measurements at the two detectors are calculated. These expressions provide a conclusive demonstration of the falsity of the authors' claim. The authors give two different accounts of their model. The published version omits a crucial detail. As a result it disagrees with quantum mechanics. It also violates signal locality. The unpublished version agrees with quantum mechanics. However, it violates the condition of parameter independence, as Myrvold has previously shown.
1. Introduction

Hess and Philipp [1, 2] (also see Hess and Philipp [3, 4]) have constructed what, they claim, is a local hidden-variables model reproducing the quantum mechanical predictions for the singlet state—contrary to the result proved by Bell [5]. The falsity of their claim has been shown by, among others, Myrvold [6] (also see Gill et al [7, 8] and Mermin [9]). However, Hess and Philipp [10] dispute Myrvold’s conclusion. The purpose of this note is to present some additional considerations which, we hope, may help to settle the question.

Hess and Philipp describe their model in two different places. The two versions are not identical. By some oversight the published version [1] (version 1) omits a crucial detail. As a result it disagrees with quantum mechanics. It also violates signal locality. The unpublished version [2] of their model (version 2) does not suffer from these deficiencies. However, the same equations which show that version 1 violates signal locality also conclusively demonstrate that version 2 violates parameter independence, as Myrvold has argued.

The concept of parameter independence was analyzed in detail by Jarrett [11] and Shimony [12, 13]. Consider a pair of spin-1/2 particles prepared in the singlet state. Particle 1 is sent to station $S_1$, which measures its spin in the direction $a$, obtaining the outcome $A = \pm 1$. Particle 2 is sent to station $S_2$, which measures its spin in the direction $b$, obtaining the outcome $B = \pm 1$. Let $\lambda$ denote the complete hidden state of the pair $1 + 2$. Let $p_1(A|a, b, \lambda)$ be the probability of obtaining outcome $A$ at $S_1$ for detector settings $a, b$ and hidden state $\lambda$. Similarly, let $p_2(B|a, b, \lambda)$ be the probability of obtaining outcome $B$ at $S_2$ for given $a, b, \lambda$.

A hidden variables model satisfies the condition of parameter independence (“locality”, in the terminology of Jarrett [11]) if and only if $p_1$ is independent of $b$, and $p_2$ is independent of $a$, so that

$$p_1(A|a, b, \lambda) = p_1(A|a, \lambda)$$  

and

$$p(B|a, b, \lambda) = p(B|b, \lambda)$$

for suitable $p_1(A|a, \lambda)$ and $p_2(B|b, \lambda)$.

Version 2 of the Hess-Philipp model [2] violates parameter independence. This clearly implies that the model is non-local because it means that if, per impossible, one could prepare an ensemble of pairs all in the same hidden state $\lambda$, then experimenters at $S_1$ and $S_2$ could use these pairs to communicate superluminally.

This point has already been made by Myrvold [6]. However, Myrvold’s paper is largely concerned with a simplified version of the Hess-Philipp model. The extension to the full model is only made briefly, at the end of his paper. In particular, Myrvold does not actually calculate the conditional probabilities appearing in Eqs. (1) and (2). This gives Hess and Philipp [10] some scope to challenge his conclusion.

In the following we do calculate the conditional probabilities, and explicitly show that they do not satisfy Eqs. (1) and (2)—thereby providing a clear-cut mathematical demonstration that Myrvold is correct. Hess and Philipp are only entitled to disagree with this statement if they can identify a flaw in our calculations, and if they can present some alternative, fully explicit calculations leading to expressions which do satisfy Eqs. (1) and (2).

In the following we begin by showing that version 1 of the Hess-Philipp model violates signal locality. We then go on to show that, for what is essentially the same reason, version 2 violates the condition of parameter independence.
2. Version 1 of the Model Violates Signal Locality

Hess and Philipp construct their model in two stages. The first stage is the theorem proved on pp. 14231–2 of ref. [1]. In Eqs. (22), (23) and (27) they define functions \( A_a(u), B_b(v) \) and a density \( \rho_{ab}(u,v) \) with the property

\[
\int A_a(u)B_b(v)\rho_{ab}(u,v)dudv = -a \cdot b \tag{3}
\]

(Eq. (18) in the statement of their theorem). The reader may confirm that one also has

\[
\int A_a(u)\rho_{ab}(u,v)dudv = a \cdot |b| + \frac{1}{2} \sum_{k=1}^{3/2} N_{2r}(|a_k|)\psi_{2r}(|b_k|) \tag{4}
\]

\[
\int B_b(u)\rho_{ab}(u,v)dudv = -|a| \cdot b \tag{5}
\]

Here \(|a|, |b|\) are the vectors with components \(|a_1|, |a_2|, |a_3|\) and \(|b_1|, |b_2|, |b_3|\) respectively. \(N_{2r}, \psi_{2r}\) and the even integer \(n\) are the quantities defined in the lemma on p. 14231 of ref. [1].

The second stage in the construction of the Hess-Philipp model is the complicated combinatoric argument leading to Eq. (35) of ref. [1]. It should be noted that there are two problems with this part of the construction, one major and one minor.

The major problem is the point made by Gill et al [8], that the index \(m\) featuring in the construction apparently introduces an element of non-locality into the model. In our view Hess and Philipp [10] do not satisfactorily answer this objection. However, we will not insist on the point here because it is tangential to our argument.

The minor problem is a technical point, concerning the details of the combinatorics. We discuss it in the appendix.

The full probability space for the Hess-Philipp model consists of all pairs \((\lambda, \omega)\), where \(\lambda\) describes the state of the source particles and \(\omega\) refers to the detectors (second paragraph on p. 14230 of ref. [1]). Let \(\nu_\omega\) be the probability measure on the space of \(\omega\), and let \(\nu_\lambda\) be the probability measure on the space of \(\lambda\). For any random variable \(X\), let \(E_\lambda(X) = \int X d\nu_\omega\) be the conditional expectation value obtained by integrating out \(\omega\) for a fixed value of \(\lambda\).

The combinatorics leading to Eq. (35) of ref. [1] are such that

\[
E_\lambda(A_aB_b) = \int A_a(u)B_b(v)\rho_{ab}(u,v)dudv = -a \cdot b \tag{6}
\]

holds trivially. It is also easily seen that

\[
E_\lambda(B_b) = \int B_b(u)\rho_{ab}(u,v)dudv = -|a| \cdot b \tag{7}
\]

However, the evaluation of \(E_\lambda(A_a)\) is slightly less straightforward. It is shown in the appendix that

\[
E_\lambda(A_a) = a \cdot |b| + \frac{1}{2} (1 - |a| \cdot |b|) + \frac{\theta}{16n^2} \tag{8}
\]

with \(0 \leq \theta \leq 1\).

Now let \(E(X) = \int E_\lambda(X) d\nu_\lambda\) be the unconditioned expectation value of \(X\). A striking feature\(^2\) of Eqs. (6–8) is that the right-hand sides are independent of \(\lambda\).

\(^1\)The reader who does wish to check this statement should bear in mind that there is a missing summation sign in Eq. (26) of ref. [1], as noted in Hess and Philipp [10].

\(^2\)It is a curious feature because it means that all the work is being done by the detectors. The source particles might as well not be there.
Consequently, the unconditioned expectation values are the same as the conditional ones:

\[ E(A_a B_b) = -a \cdot b \]  \hspace{1cm} (9)
\[ E(B_b) = -|a| \cdot b \]  \hspace{1cm} (10)
\[ E(A_a) = a \cdot |b| + \frac{1}{2}(1 - |a| \cdot |b|) + \frac{\theta}{16n_2} \]  \hspace{1cm} (11)

whatever the probability measure \( \nu_\lambda \).

At this stage we notice that the model disagrees with quantum mechanics. For the singlet state quantum mechanics predicts \( E(A_a) = E(B_b) = 0 \) for all \( a, b \). It can be seen that Eqs. (10) and (11) disagree with this prediction.

Not only does the model disagree with quantum mechanics. It also violates signal locality\(^3\). To see this, let \( p_2(B = \pm 1|a, b) \) be the probability of obtaining the measurement outcome \( B = \pm 1 \) at station \( S_2 \), for detector settings \( a, b \). Then

\[ p_2(B = \pm 1|a, b) = \frac{1}{2}(1 \pm E(B_b)) = \frac{1}{2}(1 \mp |a| \cdot b) \]  \hspace{1cm} (12)

Suppose that Alice at station \( S_1 \) and Bob at station \( S_2 \) have previously agreed that Bob will always measure in the direction \((-1, 0, 0)\), while Alice will measure in one of the two alternative directions \((1, 0, 0)\) or \((0, 1, 0)\). Then, if Alice measures in the direction \((1, 0, 0)\) there will be probability 1 of Bob obtaining the result +1, while if Alice measures in the direction \((0, 1, 0)\) there will only be probability \(1/2\) of Bob obtaining the result +1. In this way they can use the arrangement to communicate superluminally, with a probability of error that, with sufficient redundancy, can be made arbitrarily small.

3. **Version 2 of the Model Violates Parameter Independence**

We now turn to version 2 of the Hess-Philipp model, described in the e-print [2]. Unlike version 1, version 2 of the model does reproduce the empirical predictions of quantum mechanics (for the singlet state), and it does not violate signal locality. Nevertheless, it is still non-local. Furthermore, it is non-local for a reason that is closely related to the reason that version 1 violates signal locality—as we now show.

The crucial detail, which is omitted from version 1 of the model, is described at the end of ref. [2], in Section 5.3. Let \( A_a(\lambda, \omega), B_b(\lambda, \omega) \) be the functions describing the measurement outcomes in version 1 of the model. Version 2 is obtained by making the replacements

\[ A_a(\lambda, \omega) \rightarrow r(\lambda) A_a(\lambda, \omega) \]  \hspace{1cm} (13)
\[ B_b(\lambda, \omega) \rightarrow r(\lambda) B_b(\lambda, \omega) \]  \hspace{1cm} (14)

where \( r(\lambda) \) is a function taking the values \( \pm 1 \), and having the property \( \int r(\lambda) d\nu_\lambda = 0 \) (\( \nu_\lambda \) being the probability measure on the space of the source variables \( \lambda \), as before)\(^4\).

With these replacements the model does reproduce the correct quantum mechanical predictions for the unconditioned expectation values \( E(A_a B_b), E(A_a) \), \( E(B_b) \) (in the singlet state). Consequently, it does not violate signal locality.

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3 We are indebted to M. Żukowski for this observation.

4 Hess and Philipp stipulate that \( r \) depends on any “parameter specific to the source (e.g. \( \lambda_1, \lambda_2 \) or time \( t \))”. By “time \( t \)” they presumably mean the time at which the particles are emitted by the source. In the definition of parameter independence, as given in Eqs. (1), (2) above, \( \lambda \) denotes a complete specification of the state of the source. On this definition \( \lambda \) must be taken to include a specification of the time of emission.
Suppose, however, that one considers the conditional expectation values $E_\lambda(A_a), E_\lambda(B_b)$. Then one has, in place of Eqs. (7), (8) above,

$$E_\lambda(A_a) = r(\lambda) \left( a \cdot |b| + \frac{1}{2} (1 - |a| \cdot |b|) + \frac{\theta}{16n^2} \right)$$  \hspace{2cm} (15) \\
$$E_\lambda(B_b) = -r(\lambda)|a| \cdot b$$  \hspace{2cm} (16)

Let $p_1$, $p_2$ be the probabilities appearing on the left-hand sides of Eqs. (1), (2) above. Then

$$p_1(A = \pm 1|a, b, \lambda) = \frac{1}{2} (1 \pm E_\lambda(A_a))$$  \hspace{2cm} (17) \\
$$p_2(B = \pm 1|a, b, \lambda) = \frac{1}{2} (1 \pm E_\lambda(B_b))$$  \hspace{2cm} (18)

implying

$$p_1(A = \pm 1|a, b, \lambda) = \frac{1}{2} \left( 1 \pm r(\lambda) \left( a \cdot |b| + \frac{1}{2} (1 - |a| \cdot |b|) + \frac{\theta}{16n^2} \right) \right)$$  \hspace{2cm} (19) \\
$$p_2(B = \pm 1|a, b, \lambda) = \frac{1}{2} (1 \mp r(\lambda)|a| \cdot b)$$  \hspace{2cm} (20)

These expressions clearly fail to satisfy the condition of parameter independence, stated in Eqs. (1), (2) above. It follows that version 2 of the model is non-local.

The violation of parameter independence in version 2 of the model is closely related to the violation of signal locality in version 1. This can be seen by comparing the expression for $p_2(B = \pm 1|a, b, \lambda)$ in version 2 (see Eq. (20) above) with the expression for $p_2(B = \pm 1|a, b)$ in version 1 (see Eq. (12) above).

A model which violates parameter independence is one which would violate signal locality in an imaginary world, where it was possible to obtain complete information regarding the state of the source, including the values of all the quantities which are in fact “hidden”. For version 2 of the Hess-Philipp model complete information regarding the state of the source would have to include the value of the function $r(\lambda)$.

Suppose that we are in such an imaginary world. As in the discussion following Eq. (12) above, consider two experimenters, Alice at $S_1$ and Bob at $S_2$. Suppose that there is also a third experimenter Xenophon located in the intersection of Alice and Bob’s backward light cones. Xenophon prepares a succession of pairs in the singlet state. For each pair he determines the value of $r(\lambda)$ and then sends the particles to Alice and Bob. He also sends the value of $r(\lambda)$ to Bob (by telephone, say). Bob always measures in the direction $(-1, 0, 0)$. If Alice measures in the direction $(1, 0, 0)$ then Bob obtains the value $r(\lambda)$ with probability 1. If, on the other hand, Alice measures in the direction $(0, 1, 0)$ there is only probability 1/2 of Bob obtaining the value $r(\lambda)$. The value $r(\lambda)$ is known to Bob. It is therefore possible for Alice to send superluminal signals to Bob, just as in the case discussed in the passage following Eq. (12) above.

Finally, let us note that it does not help if one modifies the model again, by making $r$ a function of the state of the detectors instead of $\lambda$. It is easily verified that the model, thus modified, would still violate Shimony’s [13, 14] condition of outcome dependence (“completeness” in the terminology of Jarrett [11])—meaning that the model would still be non-local. Moreover, the model would still allow Alice and Bob to communicate superluminally in an imaginary world were it was possible to obtain complete information regarding the state of any system (although the reading of Bob’s detector is given by $rB$, in such a world Bob could still find out the value of $B$ by inspecting the hidden state of his detector). This means that, on the level of the hidden variables, there must be superluminal exchanges of information between the two detectors.
4. Conclusion

We conclude that the expressions for the probabilities \( p_1 \) and \( p_2 \) calculated in this paper have the inescapable implication that the Hess-Philipp model is non-local. Hess and Philipp are only entitled to challenge this conclusion if they can find an error in our calculations, and if they can provide alternative calculations leading to expressions for \( p_1 \) and \( p_2 \) which do not have the implication of non-locality.

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Appendix A. Remarks on the Combinatoric Part of the Construction

In this appendix we first discuss a minor technical difficulty with the construction leading to Eq. (35) of reference [1]. We then go on to derive the expression for \( E_3(A_n) \) given in Eq. (8) above.

Hess and Philipp choose in turn each of the \( (n+1)^2 \) squares \( Q_{jk} \). For each choice of \( Q_{jk} \) they define \( L \) measures, where \( L \) is the binomial coefficient \( \binom{n}{2} \). Each of these \( L \) measures assigns the weights \(|a_1b_1|, |a_2b_2|, |a_3b_3| \) to 3 of the unit squares \( \in Q_{jk} \), and the weights \((1/2)N_1(|a_1|)|\psi_1(|b_1|)|, \ldots, (1/2)N_n(|a_3|)|\psi_n(|b_3|)| \) to 3\( n \) of the \( 9n^2 \) unit squares not contained in the vertical and horizontal strips defined by \( Q_{jk} \). It assigns every other unit square weight 0. The assignment is such that each unit square \( \in Q_{jk} \) is assigned the weight \(|ab_1| \) by \( L/9 \) measures, and each unit square not in the vertical and horizontal strips defined by \( Q_{jk} \) is assigned the weight \((1/2)N_i(|a_i|)|\psi_i(|b_i|)| \) by \( L/(9n^2) \) measures \( (t = 1, 2, 3 \text{ and } i = 1, \ldots, n) \).

The problem with this construction is that it tacitly assumes that the binomial coefficient \( \binom{n}{2} \) is divisible by \( 9n^2 \). However, this is typically not the case (the only even integers \( \leq 100 \) for which it is true are \( n = 10, 40, 44 \) and 84). Moreover, it is not clear to us that the other requirements can be satisfied even when \( n \) does satisfy this condition. The difficulty is, however, easily resolved if, instead of taking \( L = \binom{n}{2} \), we take it to be the permutation \( \binom{n}{2} = (9n^2)/(9n^2 - 3n)! \).

Now let us turn to the derivation of Eq. (8). Let \( I_{jk} \) be the set of indices \( m \) such that \( \mu_m \) is one of the \( L \) measures associated with \( Q_{jk} \). It is easily seen that

\[
\sum_{m \in I_{jk}} \int_{Q_{jk}} A_m(u) \, du \, \mu_m = L \mathbf{a} \cdot \mathbf{b}
\]  
(21)

Let \( S_{jk} \) be the subset of \([-3, 3n)^2 \) which is obtained by deleting the horizontal and vertical strips defined by \( Q_{jk} \). For each \( m \in I_{jk} \), \( \int_{U} A_m(u) \, du \, dv = 1 \) for half the unit squares \( U \subset S_{jk} \), and it = 0 for the other half. Consequently

\[
\sum_{m \in I_{jk}} \int_{S_{jk}} A_m(u) \, du \, \mu_m = L \left( \frac{3}{4} \sum_{t=3}^{n} \sum_{i=1}^{n} N(|a_t|)|\psi(|b_t|)| = \frac{L}{2} \left( 1 - |\mathbf{a}| \cdot |\mathbf{b}| + \frac{\theta}{8n^2} \right) \right)
\]  
(22)

with \( 0 \leq \theta \leq 1 \) (where we have used Eq. (21) of ref. [1]). Hence

\[
\sum_{m \in I_{jk}} \int_{[-3, 3n)^2} A_m(u) \, du \, \mu_m = L \left( \mathbf{a} \cdot \mathbf{b} + \frac{1}{2} (1 - |\mathbf{a}| \cdot |\mathbf{b}|) + \frac{\theta}{16n^2} \right).
\]  
(23)

The right-hand side of this equation is independent of \( j, k \). Consequently, the effect of summing over all \( j, k \) is simply to multiply the expression by \( (n+1)^2 \). After
dividing by $N = (n + 1)^2 L$ this gives

$$E_\lambda (A_a) = \frac{1}{N} \sum_{m=1}^{N} \int_{[-3.3n]^2} A_m(u) d\mu_m = a \cdot |b| + \frac{1}{2}(1 - |a| \cdot |b|) + \frac{\theta}{16n^2} \tag{24}$$

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