Partially-Quenched Nucleon-Nucleon Scattering

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Abstract

Nucleon-nucleon scattering is studied to next-to-leading order in a partially-quenched extension of an effective field theory used to describe multi-nucleon systems in QCD. The partially-quenched nucleon-nucleon amplitudes will play an important role in relating lattice simulations of the two-nucleon sector to nature.
I. INTRODUCTION

The next decade promises to be a very exciting time for strong interaction physics. With ever increasing computer power and impressive progress in developing new techniques to simulate quantum field theories, one hopes that lattice simulations of simple hadronic systems will provide rigorous and reliable predictions of QCD for strong interaction observables. While one looks forward to fully-unquenched QCD simulations performed with the physical values of the light-quark masses, \( m_q \), such simulations are presently prohibitively time-consuming. At present, and for the foreseeable future, unphysical theories will be simulated because, in contrast with QCD, the simulations can be performed in a reasonable time frame [1]. A second motivation for simulating unphysical theories is particular to nuclear physics, and is related to the unnaturally large values of the S-wave scattering lengths. We will discuss this point in the final section of the paper. A commonly simulated unphysical theory is quenched QCD (QQCD) where disconnected quark-loop diagrams (the quark determinant) are omitted. While QQCD simulations of strong interaction observables can be performed with small \( m_q \), they have the distinct disadvantage of not being related to QCD except in the large-\( N_c \) limit [2]. A more interesting unphysical theory is partially-quenched QCD (PQQCD) [3–7] in which the quark masses, \( m_S \) (the “S” stands for “sea”), used in evaluating the disconnected quark-loops are larger than the masses of the quarks connected to external sources, \( m_V \) (the “V” stands for “valence”). By computing strong interaction observables in PQQCD and performing the extrapolation \( m_S, m_V \to m_q \) one recovers the QCD observables that one is interested in. It is for this last step—the extrapolation to the physical values of \( m_q \)—that effective field theory (EFT) is required. The EFT’s describing QCD in the low-momentum regime in the pseudo-Goldstone boson sector (chiral perturbation theory, \( \chi_P T \)), and in the single baryon sector (heavy baryon chiral perturbation theory, HB\( \chi_P T \)), are well established, and their extension to PQQCD in the form of PQ\( \chi_P T \) [3–7] and PQHB\( \chi_P T \) [8–10] have been accomplished relatively recently.

The construction of an EFT to describe the low-momentum dynamics of multi-nucleon systems has proven to be extremely challenging. In the very low-momentum regime, where the typical momentum of the external particles involved in a given process is much less than the mass of the pion, \( p \ll m_\pi \), and hadronic production is therefore kinematically forbidden, an EFT, EFT(\( \pi/\pi \)) [11–14], can be constructed from nucleons and photons (and any other low-momentum transfer probes) quite simply. The fact that there is a bound state near threshold in the \( ^3S_1 - ^3D_1 \) coupled-channels, and a pole on the second-sheet near threshold in the \( ^1S_0 \) channel means that at least one operator in the EFT(\( \pi/\pi \)) Lagrange density must be treated non-perturbatively in these channels. The choice of operators to be resummed and details of the perturbative expansion are defined by the power-counting in EFT(\( \pi/\pi \)). Despite chiral symmetry not being a good symmetry for \( p \ll m_\pi \), isospin remains a good symmetry. The only input into the construction of EFT(\( \pi/\pi \)) is Lorentz invariance, electromagnetic gauge invariance, baryon number conservation and the approximate isospin symmetry, the breaking of which can be included perturbatively. In the kinematic regime where the momenta involved in a given process are larger than \( m_\pi \), the pion must be included as a dynamical field. It was Weinberg’s pioneering efforts [15] in the early 1990’s in this kinematic regime that initiated interest in developing EFT for nuclear physics. Weinberg attempted to construct an EFT for nuclear processes and nuclei involving momenta all the way up to the chiral
symmetry breaking scale $\Lambda$, and necessarily included the pion as a dynamical degree of freedom. The power-counting that he developed, known as Weinberg power-counting (W), involves a chiral expansion of the nucleon-nucleon potential using the same power-counting rules that are used in the meson and single nucleon sectors. The chirally expanded potential is inserted into the Schrödinger equation to determine observables, such as phase shifts. Unfortunately, there is a formal problem with this power counting [16] in some of the scattering channels, particularly the $^1S_0$ channel. However, extensive phenomenological studies with W power-counting appear to be in good agreement with data [17–20], where such comparisons are possible, and the formal problems appear to have little impact when a massive regulator is used with a mass scale that is not radically different from a few hundred MeV. The formal problems with W power-counting led Kaplan, Savage and Wise (KSW) to develop a power-counting [21] in which the momentum-independent four-nucleon operator is promoted to one lower order in the chiral expansion, and consequently pion exchanges are subleading and treated in perturbation theory. This power-counting is formally consistent and gives renormalization group invariant amplitudes order-by-order in the EFT expansion. However, Fleming, Mehen and Stewart [22] (FMS) showed that the scattering amplitude in the $^3S_1 - ^3D_1$ coupled channels diverges at next-to-next-to-leading order (NNLO) at relatively small momenta and KSW power-counting fails. FMS found that a contribution that remains large in the chiral limit destroys the convergence: it is the chiral limit of the tensor force that “does the damage”. Recently, it was suggested that one should expand observables about the chiral limit [23] (BBSvK power-counting). BBSvK power-counting has all the nice features of W and KSW counting: the chiral limit of the tensor force is resummed at leading order (LO) along with the momentum- and $m_q$-independent four-nucleon operator in the $^3S_1 - ^3D_1$ coupled channels, while pions are perturbative in the $^1S_0$ channel, and in higher partial waves, where analytic calculations are possible.

In recent work [24], we showed that hairpin diagrams in PQQCD give rise to a component of the nucleon-nucleon (NN) potential that falls exponentially at long-distance and therefore does not have the Yukawa behavior found in QCD. Thus, measuring the long-distance behavior of the NN potential [25–28] does not provide information about QCD unless the hairpin contribution can be removed in a rigorous way. While the presence of this behavior is quite discouraging, one might focus on the behavior of S-matrix elements rather than on the NN potential itself (for a recent survey of the status of lattice calculations of the NN potential see Ref. [29]). In this work we develop the partially-quenched EFT that describes the two-nucleon sector using BBSvK power-counting. In channels with higher partial waves, we give analytic expressions for scattering amplitudes and a few characteristic scattering volumes to next-to-leading order (NLO) in the partially-quenched EFT. In the $^1S_0$-channel we give analytic expressions for the scattering amplitude, scattering length and effective range to NLO in terms of the valence and sea quark masses. As in QCD, the $^3S_1 - ^3D_1$ coupled channels in the partially-quenched EFT are somewhat more complicated; we provide the NN potential at NLO that is required to generate the NN phase-shifts, $\delta_{0,2}$ and mixing-parameter, $\epsilon_1$, by solving the Schrödinger equation.
II. THE PARTIALLY-QUENCHED EFT CALCULATION

In BBSvK power-counting S-matrix elements are an expansion about the chiral limit, where the expansion parameter, $Q$, is $Q \sim 1/3 \sim m_\pi/\Lambda_{NN}$, where the constant $\Lambda_{NN} = 8\pi f^2/(g_A^2 M_N)$ is determined by the relative size of pion exchange, and $m_\pi$ is the physical value of the pion mass. In the two S-wave channels, the momentum and $m_q$-independent four-nucleon operators, along with the chiral limit of one-pseudo-Goldstone-Boson-exchange (OPGBE), are resummed to all orders (each iteration is the same order in $Q$) and this sum constitutes the LO scattering amplitude. In the $^1S_0$-channel this corresponds to KSW power-counting and the scattering amplitude can be computed analytically, while in the $^3S_1 - ^3D_1$ coupled channels, the scattering amplitude must be determined numerically. In the higher partial waves, the chiral limit of OPGBE provides the LO contribution to the scattering amplitude as the four-nucleon operators are suppressed by additional powers of momentum.

It is the ultra-violet (UV) behavior of the theory that requires an expansion about the chiral limit, and in particular allows control of the very singular diagrams as $r \to 0$ in coordinate-space. The deviations from the chiral limit in OPGBE that formally occur at higher orders in BBSvK power-counting are UV safe and can therefore be included at NLO without compromising the renormalizability of the theory. The price for not including them is that the long-distance behavior of the theory must be recovered order-by-order in perturbation theory and convergence is somewhat slow [23]. By contrast, the two-pseudo-Goldstone-Boson-exchange (TPGBE) diagrams are singular away from the chiral limit, and therefore only the chiral limit can be retained at NLO; keeping the full TPGBE introduces divergences that cannot be renormalized at NLO. In this work we compute to NLO in the EFT with BBSvK power-counting. In the S-wave channels at NLO there are contributions from OPGBE (the full meson mass dependence is retained), from momentum- and $m_q$-independent four-nucleon operators, from the leading momentum-dependent four-nucleon operators ($p^2$) and from the four-nucleon operators with a single insertion of $m_q$. In the higher partial waves, the four-nucleon operators contribute beyond NLO and thus only OPGBE contributes at NLO.

We will work in the isospin limit of the SU(4)\(_L\otimes\)SU(4)\(_R\) PQ\(_\chi\)PT. This means that the $u$, $d$, $\bar{u}$ and $\bar{d}$ quarks in the valence and ghost sectors are degenerate and the $j$ and $l$ quarks in the sea sector are degenerate. The formalism for this theory can be found in Refs. [8–10] and we will not describe it here.

A. Partial Waves with $L > 0$

At NLO ($O(Q^0)$), the partial waves with $L > 0$ (higher partial waves) receive contributions only from OPGBE, as shown in Fig. 1, and no resummation of diagrams is required. In PQ\(_\chi\)PT the potential between two-nucleons due to OPGBE is [24]

$$V^{(PQ)}(r) = \frac{1}{8\pi f^2} \sigma_1 \cdot \nabla \sigma_2 \cdot \nabla \left( g_A^2 \frac{\tau_1 \cdot \tau_2}{r} - g_0^2 \frac{(m_{SS}^2 - m_\pi^2)}{2m_\pi} \right) e^{-m_\pi r},$$

arising from the interaction Lagrange density in PQ\(_Q\)CD (in the isospin limit)
\[ L = N \left[ \frac{g_A}{\sqrt{2}f} \tau^\alpha \sigma \cdot \nabla \pi^\alpha + \frac{g_0}{\sqrt{2}f} \sigma \cdot \nabla \eta \right] N. \] (2)

The \( \pi \) and \( \eta \) propagators are of the form

\[ G_\pi = \frac{i}{q^2 - m_{\pi}^2 + i\epsilon}, \quad G_\eta = \frac{i(m_{SS}^2 - m_{\pi}^2)}{(q^2 - m_{\pi}^2 + i\epsilon)^2}, \] (3)

where \( f \sim 132 \text{ MeV} \), \( g_A \) is the isovector axial coupling constant and \( g_0 \) is the isoscalar axial coupling. The mass \( m_{SS} \) is that of a meson composed of two sea quarks while \( m_\pi \) is the pion mass which is, of course, composed of two valence quarks. The propagators in eq. (3) clearly exhibit the correct behavior in the QCD limit, where the coefficient of the double-pole contribution in the \( \eta \) propagator vanishes, and the single pole contribution is absent. By treating \( m_\pi, m_{SS} \) and \( |q| \) all of \( \mathcal{O}(Q) \), both OPGBE contributions are the same order in the power-counting.

\[ \begin{array}{c}
\text{(a)} \\
\text{Diagram (a) corresponds to the exchange of } \pi, \text{ while diagram (b) corresponds to the exchange of } \eta \text{ with a double-pole propagator, as given in eq. (3), denoted by “X”.}
\end{array} \]

FIG. 1. The LO contribution, \( \mathcal{O}(Q^0) \), to scattering in the higher partial waves from OPGBE. Diagram (a) corresponds to the exchange of \( \pi \), while diagram (b) corresponds to the exchange of \( \eta \) with a double-pole propagator, as given in eq. (3), denoted by “X”.

In QCD, the scattering amplitudes in the higher partial waves, \( \mathcal{A}^{(QCD)}_{J,J,J',L,L'} \), for total angular momentum \( J \), total spin \( S \), and initial and final state orbital angular momentum \( L \) and \( L' \) respectively, are well known to \( \mathcal{O}(Q) \) [30]. In the spin-singlet channel \((S = 0)\) the scattering amplitude in partial waves with \( J = L > 0 \), \( \mathcal{A}^{(QCD)}_{J,J,J,0} \), is

\[ \mathcal{A}^{(QCD)}_{J,J,J,0} = -\frac{(-)^I}{2I+1} \frac{3g_A^2}{2f^2} (z-1) \quad Q_J(z), \] (4)

where \( Q_n(z) \) is an irregular Legendre Polynomial of order \( n \), using the conventions of Ref. [30],

\[ Q_0(z) = \frac{1}{2} \log \left( \frac{z+1}{z-1} \right), \quad Q_1(z) = \frac{1}{2} z \log \left( \frac{z+1}{z-1} \right) - 1, \quad ..., \] (5)

and where the variable \( z \) is given in terms of the pion mass and nucleon center-of-mass momentum as \( z = 1 + m_{\pi}^2/(2p^2) \). The quantity \( I \) is the isospin of the channel under consideration. In the spin-triplet channel \((S = 1)\) the expression is somewhat more complicated due to the fact that the spin and orbital angular momentum can couple to produce three different total angular momentum states, \( J = L - 1, L, L + 1 \). It is straightforward to show that the amplitudes for scattering between states of the same orbital angular momentum are [30]
\[ A_{J,LL,1}^{(QCD)} = \frac{(-)^{I}}{2I+1} \frac{g_{A}^{2}}{2f^{2}} \left[ (z-1)Q_{L}(z) + S_{12}^{JLL} \left( (z-1)Q_{L}(z) + \frac{3}{2(2L+1)} (Q_{L-1}(z) - Q_{L+1}(z)) \right) \right] , \]  

where the constant \( S_{12}^{JLL} \) is given by

\[ S_{12}^{JLL} = \left( - \frac{2(L+1)}{2L-1}, +2, - \frac{2L}{2L+3} \right) \quad \text{for} \quad J = (L-1, L, L+1) . \]  

The amplitudes for scattering between states with orbital angular momenta that differ by two units, \( \Delta L = 2 \), induced by the tensor component of the interaction are

\[ A_{J,LL+2,1}^{(QCD)} = - \frac{(-)^{I}}{2I+1} \frac{g_{A}^{2}}{4f^{2}} S_{12}^{JLL+2} \left[ Q_{L+2}(z) + Q_{L}(z) - 2Q_{L+1}(z) \right] , \]

where \( S_{12}^{JLL+2} \) is given by

\[ S_{12}^{JLL+2} = \frac{6\sqrt{J(J+1)}}{2J+1} . \]

In order to arrive at the partially-quenched amplitudes it is convenient to note that the contribution from single \( \eta \) exchange can be obtained from OPGBE by taking a derivative of \( A_{J,LL,S}^{(QCD)} \) with respect to \( m_{\eta}^{2} \), and multiplying by appropriate constant factors. Generically,

\[ A_{OPGBE}^{(PQ)} = \left( 1 - (-)^{I}(2I+1) \frac{g_{A}^{2}}{3g_{A}^{2}} \left( m_{\pi}^{2} - m_{\eta}^{2} \right) \frac{\partial}{\partial z} \right) A_{OPGBE}^{(QCD)} . \]

It is straightforward to show that the partially-quenched amplitudes in the spin-singlet higher partial waves are

\[ A_{J,LL,0}^{(PQ)} = A_{J,LL,0}^{(QCD)} + \frac{g_{A}^{2}}{4f^{2}} \frac{m_{\pi}^{2} - m_{\eta}^{2}}{p^{2}} \left[ Q_{L}(z) + \frac{L+1}{z+1} \left( Q_{L+1}(z) - zQ_{L}(z) \right) \right] , \]

and in the spin-triplet higher partial waves are

\[ A_{J,LL,1}^{(PQ)} = A_{J,LL,1}^{(QCD)} - \frac{g_{A}^{2}}{24f^{2}} \frac{m_{\pi}^{2} - m_{\eta}^{2}}{p^{2}} \left[ 2 \left( 1 + S_{12}^{JLL} \right) \left( Q_{L}(z) + \frac{L+1}{z+1} \left( Q_{L+1}(z) - zQ_{L}(z) \right) \right) \right] + \frac{3}{(z-1)(2L+1)} \left( L \left[ Q_{L}(z) - zQ_{L-1}(z) \right] + (L+2) \left[ zQ_{L+1}(z) - Q_{L+2}(z) \right] \right) , \]

\[ A_{J,LL,2,1}^{(PQ)} = A_{J,LL,2,1}^{(QCD)} + \frac{g_{A}^{2}}{24f^{2}} \frac{m_{\pi}^{2} - m_{\eta}^{2}}{p^{2}} \frac{S_{12}^{JLL+2}}{z^{2} - 1} \left[ L \left( Q_{L+1}(z) + Q_{L+3}(z) - 2Q_{L+2}(z) \right) - z \left( Q_{L}(z) + 3Q_{L+2}(z) - 4Q_{L+1}(z) \right) - zL \left( Q_{L}(z) + Q_{L+2}(z) - 2Q_{L+1}(z) \right) + \left( Q_{L+1}(z) + 3Q_{L+3}(z) - 4Q_{L+2}(z) \right) \right] . \]

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The phase-shifts in the spin-singlet channels can be easily extracted from the scattering amplitudes given in eq. (11) by using the relation (for non-relativistic systems)

$$\delta_{JJ,0} = \frac{1}{2i} \log \left( 1 + \frac{i M_N p}{2\pi} A_{JJ,0} \right), \quad (13)$$

from which parameters in the effective range expansion can be determined for \( p < m_\pi, m_{SS} \). It is somewhat more complicated to determine the phase-shifts in the spin-triplet channels as one has to disentangle them from the mixing parameters. However, at NLO (\( O(Q^0) \)), the mixing effects are higher order in the EFT and one can straightforwardly determine parameters in the effective range expansion. In the \( P \)-waves, the scattering volumes, defined to be

$$a^{(2S+1)P_J} = -\lim_{p \to 0} \frac{\tan \delta_{J11.S}}{p^2}, \quad (14)$$

are found to be, at NLO,

$$a^{(1)P_1} = \frac{g_A^2 M_N}{4\pi f^2 m_\pi^2} + \frac{g_0^2 M_N}{12\pi f^2 m_\pi^2} \frac{m_{SS}^2 - m_\pi^2}{m_\pi^2}$$

$$a^{(3)P_0} = -\frac{g_A^2 M_N}{4\pi f^2 m_\pi^2} + \frac{g_0^2 M_N}{4\pi f^2 m_\pi^2} \frac{m_{SS}^2 - m_\pi^2}{m_\pi^2}$$

$$a^{(3)P_1} = \frac{g_A^2 M_N}{6\pi f^2 m_\pi^2} - \frac{g_0^2 M_N}{6\pi f^2 m_\pi^2} \frac{m_{SS}^2 - m_\pi^2}{m_\pi^2}$$

$$a^{(3)P_2} = 0, \quad (15)$$

for which the QCD limit agrees with the well-known results [30].

**B. The \( ^1S_0 \) Channel**

The scattering amplitude in the \( ^1S_0 \) channel can be determined analytically order-by-order in perturbation theory as BBSvK power-counting coincides with KSW power-counting in this channel. The momentum and \( m_q \)-independent four nucleon operator with coefficient \( C_0^{(1S_0)} \) enters at LO, and the bubble chains that it generates, as shown in Fig. 2, are resummed to all orders to produce the LO scattering amplitude [21]. At NLO there are several different contributions. There is a contribution from OPGBE that can be dressed in a variety of ways by the LO amplitude as shown in Fig. 3, and each dressing remains \( O(Q^0) \). There is a contribution from a momentum-dependent \( (p^2) \) operator with coefficient \( C_2^{(1S_0)} \) that is

![FIG. 2. The LO contribution, \( O(Q^{-1}) \), to the scattering amplitude in the \( ^1S_0 \) channel.](image)
FIG. 3. The NLO contributions, $O(Q^0)$, to the scattering amplitude in the $^1S_0$ channel. Diagram (a) corresponds to an insertion of the momentum-dependent operator with coefficient $C_2^{(s_0)}$, diagrams (b)-(d) correspond to dressed OPGBE, while diagram (e) denotes an insertion of the $m_q$-dependent operator with coefficient $D_2^{(s_0)}$.

dressed by the LO amplitude. Also, there are two contributions from a single insertion of $m_q$, with coefficients $D_2^{(s_0)}$ and $D_2^{(s_0)}$, which are also dressed by the LO amplitude.

The scattering amplitude at NLO, $A_{s_0}^{(QCD)}$, is the sum of the contributions shown in Figs. 2 and 3,

$$A_{s_0}^{(QCD)} = A_{s_0,0}^{(QCD)} + \sum_i A_{s_0,0}^{(QCD)(i)} .$$

(16)

It is straightforward to show that the individual contributions are

$$A_{s_0,0}^{(QCD)} = -\frac{C_0^{(s_0)}}{1 + C_0^{(s_0)} M_N \frac{\mu + \i p}{4\pi}} ,$$

$$A_{s_0,0}^{(QCD)(I)} = -C_2^{(s_0)} p^2 \left[ \frac{A_{s_0,0}^{(QCD)}}{C_0^{(s_0)}} \right]^2 , A_{s_0,0}^{(QCD)(II)} = \left( \frac{g_A^2}{2f^2} \right) \left( -1 + \frac{m^2}{4p^2} \ln \left( 1 + \frac{4m^2}{m^2} \right) \right) ,$$

$$A_{s_0,0}^{(QCD)(III)} = \frac{g_A^2}{f^2} \left( \frac{m\pi M_N A_{s_0,0}^{(QCD)}}{4\pi} \right) \left( -\frac{\mu + \i p}{m} + \frac{m}{2p} X(p, m) \right) ,$$

$$A_{s_0,0}^{(QCD)(IV)} = \frac{g_A^2}{2f^2} \left( \frac{m\pi M_N A_{s_0,0}^{(QCD)}}{4\pi} \right) \left( 1 - \left( \frac{\mu + \i p}{m} \right)^2 + \i X(p, m) - \ln \left( \frac{m}{\mu} \right) \right) ,$$

$$A_{s_0,0}^{(QCD)(V)} = -D_2^{(s_0)} m^2 \left[ \frac{A_{s_0,0}^{(QCD)}}{C_0^{(s_0)}} \right]^2 , X(p, m) = \tan^{-1} \left( \frac{2p}{m\pi} \right) + \i \ln \left( 1 + \frac{4p^2}{m^2} \right) ,$$

(17)

where $D_2^{(s_0)} = D_{2A}^{(s_0)} + D_{2B}^{(s_0)}$, and $\mu$ is the renormalization scale. The PDS subtraction procedure [21] has been used in defining the power-law divergent loop diagrams.
The partially-quenched amplitude in the $^1S_0$ channel can be found straightforwardly from the QCD amplitude by taking derivatives with respect to $m^2_\pi$ of the OPGBE contributions (see eq. (10)) and by constructing the local operators that can contribute in the $SU(4|2)_L \otimes SU(4|2)_R$ EFT. The additional diagrams that contribute are shown in Fig. 4, and the scattering amplitude in the $^1S_0$ channel at NLO is

\[ A_{^1S_0}^{(PQ)} = A_{^1S_0}^{(QCD)} - \left( m^2_{SS} - m^2_\pi \right) \left( \frac{A_{^1S_0,-1}^{(QCD)}}{C_0^{(^1S_0)}} \right)^2 D_{^1S_0}^{(2B)}(\mu) \]

\[ + \frac{g^2_0}{2f^2} \frac{m^2_{SS} - m^2_\pi}{2p^2} \left[ \frac{1}{2} \log \left( 1 + \frac{4p^2}{m^2_\pi} \right) - \frac{2p^2}{m^2_\pi + 4p^2} \right] \]

\[ + \frac{iM_{NP}}{2\pi} A_{^1S_0,-1}^{(QCD)} \left( 1 + \frac{iM_{NP}}{4\pi} A_{^1S_0,-1}^{(QCD)} \right) \left( \log \left( 1 - i \frac{2p}{m_\pi} \right) + i \frac{p}{m_\pi - i2p} \right) \]

\[ - \frac{M_{NP}^2}{8\pi^2} \left( A_{^1S_0,-1}^{(QCD)} \right)^2 \left( \log \left( \frac{m_\pi}{\mu} \right) - \frac{1}{2} \right) \right], \tag{18} \]

where we have worked to LO in the relation between the quark masses and the meson masses. The explicit renormalization-scale dependence of the amplitude due to the $\log \left( \frac{m_\pi}{\mu} \right)$ contribution is exactly compensated by the renormalization-scale dependence of the coefficient $D_{^1S_0}^{(2B)}(\mu)$ to yield a $\mu$-independent amplitude.

As there is no mixing between different partial waves in the $^1S_0$ channel it is straightforward to determine the phase-shift, $\delta_{^1S_0} = \delta_{0,0,0}$, from the scattering amplitude without approximation using eq. (13), and, in turn, to construct the effective range expansion

\[ p \cot \delta_{^1S_0} - ip = \frac{4\pi}{M_N A_{^1S_0}} - \frac{1}{a^{(^1S_0)}} + \frac{1}{2} r^{(^1S_0)} p^2 + \ldots. \tag{19} \]

Here $a^{(^1S_0)}$ and $r^{(^1S_0)}$ are the scattering length and effective range in the $^1S_0$ channel, respectively. The scattering length in the partially-quenched EFT is found to be
\[
\frac{1}{a^{1S_0}} = \gamma - \frac{M_N}{4\pi} (\mu - \gamma)^2 D_2^{(1S_0)}(\mu) m_\pi^2 - \frac{M_N}{4\pi} (\mu - \gamma)^2 D_2^{(1S_0)}(\mu) \left( m_{SS}^2 - m_\pi^2 \right) \\
+ \frac{g_\Lambda^2 M_N}{8\pi f^2} \left[ m_\pi^2 \log \left( \frac{\mu}{m_\pi} \right) + (m_\pi - \gamma)^2 - (\mu - \gamma)^2 \right] \\
+ \frac{g_3^2 M_N}{8\pi f^2} \left( m_{SS}^2 - m_\pi^2 \right) \left[ \log \left( \frac{\mu}{m_\pi} \right) + \frac{1}{2} - \frac{\gamma}{m_\pi} \right]
\]

where \( \gamma \) is a \( \mu \)-independent linear combination of \( C_0^{(1S_0)} \) and \( \mu \) that enters at LO in the expansion and must be determined from data. Furthermore, the effective range is found to be

\[
r^{(1S_0)} = \frac{M_N}{2\pi} (\mu - \gamma)^2 C_2(\mu) + \frac{g_\Lambda^2 M_N}{12\pi f^2} \left( 3 - \frac{8\gamma}{m_\pi} + \frac{6\gamma^2}{m_\pi^2} \right) \\
+ \frac{g_3^2 M_N}{6\pi f^2} \frac{m_{SS}^2 - m_\pi^2}{m_\pi^2} \left( \frac{2\gamma}{m_\pi} - \frac{3\gamma^2}{m_\pi^2} \right)
\]

In QCD with KSW power-counting, the scattering amplitude in the \( 1S_0 \) channel has been determined up to NNLO [22], and it has been found that the expansion is convergent. However, the chiral expansion of the effective range parameters in this channel suggests that the convergence of the expansion is quite slow [31]. Consequently, in order to have confidence in the chiral extrapolation of the partially-quenched amplitude and effective range parameters we have presented here, the NNLO amplitude (and even higher orders) should be computed in order to understand the convergence properties of the chiral expansion.

**C. The \( 3S_1 - 3D_1 \) Coupled Channels**

Due to the non-perturbative nature of OPGBE—particularly the chiral limit of the tensor force—in the \( 3S_1 - 3D_1 \) channel, the method for computation in this channel is fundamentally different from that in the \( 1S_0 \) channel and the higher partial waves where OPGBE can be included in perturbation theory. The details of the calculation of scattering lengths, phase shifts and bound state energies in the \( 3S_1 - 3D_1 \) coupled channels in QCD can be found in Refs. [23,32], and we do not repeat them here. In the \( 3S_1 - 3D_1 \) coupled-channels, OPGBE generates both central and tensor potentials,

\[
V_C^{(QCD)(\pi)}(r; m_\pi) = -\alpha_\pi m_\pi^2 \frac{e^{-m_\pi r}}{r} \\
V_T^{(QCD)(\pi)}(r; m_\pi) = -\alpha_\pi \frac{e^{-m_\pi r}}{r} \left( \frac{3}{r^2} + \frac{3m_\pi}{r} + m_\pi^2 \right)
\]

where \( \alpha_\pi = g_3^2 (1 - 2m_\pi^2 d_{18}/g_A^2)/(8\pi f^2) \). The constant \( d_{18} \) is somewhat uncertain [33], with different extractions yielding \(-0.78 \pm 0.27, -0.83 \pm 0.06, -1.4 \pm 0.24 \) [34] and \(-10.14 \pm 0.45 \) GeV\(^{-2} \) [35]. Since the chiral limit of the potentials in eq. (22) contribute at LO, as shown in Fig. 5, the \( m_\pi \)-dependence of \( g_A, f \) and \( M_N \) are required at NLO [23]. Each of these observables has been studied extensively, the results of which can be found in Refs. [33,36–38], and up to NNLO it is found that
FIG. 5. Lippmann-Schwinger equation for the LO contribution to the scattering amplitude (large solid rectangle) in the $^3S_1 - ^3D_1$ coupled-channels. The small solid circles denote an insertion of $C_{0}^{(3S_1)}$ or $g_A$. The “o” appearing below the OPGBE diagram implies the chiral limit.

\[
f = f^{(0)} \left[ 1 - \frac{1}{4\pi^2(f^{(0)})^2} m^2_\pi \log \left( \frac{m_\pi}{m_{\pi^{(PHYS)}}} \right) + \frac{m^2_\pi}{8\pi^2(f^{(0)})^2} \tilde{t}_4 \right]
\]

\[
M_N = M_N^{(0)} - 4m^2_\pi c_1
\]

\[
g_A = g_A^{(0)} \left[ 1 - \frac{2(g_A^{(0)})^2 + 1}{4\pi^2(f^{(0)})^2} m^2_\pi \log \left( \frac{m_\pi}{m_{\pi^{(PHYS)}}} \right) - \frac{(g_A^{(0)})^2 m^2_\pi}{8\pi^2(f^{(0)})^2} + \frac{4m^2_\pi}{g_A^{(0)} d_{16}} \right] , \quad (23)
\]

where $m_\pi^{(PHYS)} = 139$ MeV, $\tilde{t}_4 = 4.4 \pm 0.2$ [36,37], $c_1 \sim -1$ GeV$^{-1}$ [33] are $m_\pi$-independent constants (we have explicitly separated the logarithmic contribution from $\tilde{t}_4$). A complete analysis by Fettes [39] of the $\pi N$ sector provides three determinations of $d_{16}$, $d_{16} = -0.91 \pm 0.74$, $-1.01 \pm 0.72$ and $-1.76 \pm 0.85$ GeV$^{-2}$. At NLO there is a contribution from the chiral limit of TPGBE and from an insertion of a momentum dependent ($p^2$) operator with coefficient $C_{2}^{(3S_1)}$, as shown in Fig. 6. At this order there are two contributions arising from a single insertion of $m_q$, with coefficients $D_{2A}^{(3S_1)}$ and $D_{2B}^{(3S_1)}$, which in QCD are combined together into $D_{2}^{(3S_1)}$. The TPGBE potential in coordinate space has been computed in Ref. [17,40], and in the chiral limit is given by

\[
V_C^{(QCD)(\pi\pi)}(r; 0) = \frac{3(22g_A^4 - 10g_A^2 - 1)}{64\pi^3 f_\pi^4} \frac{1}{r^5} , \quad V_T^{(QCD)(\pi\pi)}(r; 0) = -\frac{15g_A^4}{64\pi^3 f_\pi^4} \frac{1}{r^5} . \quad (24)
\]

FIG. 6. Chiral limit of the crossed TPGBE diagram, deviations from the chiral limit of OPGBE, and the $C_2$ (large solid circle) and $D_2$ (large solid square) operators, all of which contribute at NLO in the $^3S_1 - ^3D_1$ coupled-channels. The “o” appearing below a diagram implies the chiral limit.

The singular nature of the tensor force has so far precluded regularization of this channel using dimensional regularization, or any other mass-independent regulator. We therefore
use a spatial square-well of radius \( R \) [23,32,41], where the potential outside the square well, \( V_{L}^{(QCD)}(r; m_{\pi}) = V_{L}^{(QCD)} \), is

\[
V_{L}^{(QCD)} = M_{N} \left( \begin{array}{l}
- V_{C}^{(QCD)}(r; m_{\pi}) \\
- 2\sqrt{2} V_{T}^{(QCD)}(r; m_{\pi}) \end{array} \right) - \frac{6}{M_{N} r^{2}} ,
\]

where

\[
\begin{align*}
V_{C}^{(QCD)}(r; m_{\pi}) &= V_{C}^{(QCD)(\pi)}(r; m_{\pi}) + V_{C}^{(QCD)(\pi\pi)}(r; 0) \\
V_{T}^{(QCD)}(r; m_{\pi}) &= V_{T}^{(QCD)(\pi)}(r; m_{\pi}) + V_{T}^{(QCD)(\pi\pi)}(r; 0) .
\end{align*}
\]

The energy and \( m_{q} \)-dependent potential, \( V_{S}^{(QCD)}(r; m_{\pi}, k^{2}) = V_{S}^{(QCD)} \), inside the square well is

\[
V_{S}^{(QCD)} = -M_{N} \left( \begin{array}{c}
V_{eq}^{(QCD)} \\
0
\end{array} \right)
\]

where \( V_{eq}^{(QCD)} = V_{0}^{(SS)} + m_{\pi}^{2} V_{D_{0}^{(SS)}} + p^{2} V_{C_{1}^{(SS)}} \), and where we have again used the LO relation between \( m_{q} \) and the pion mass. \( V_{0}^{(SS)} \), \( V_{D_{0}^{(SS)}} \) and \( V_{C_{1}^{(SS)}} \) are constant potentials corresponding to the renormalized local operators with coefficients \( C_{0}^{(SS)} \), \( D_{0}^{(SS)} \) and \( C_{1}^{(SS)} \) in the \( ^{3}S_{1} - ^{3}D_{1} \) coupled-channels, respectively. An identification can be made between the coefficients of the local operators, \( C_{i}^{(SS)} \) and \( D_{i}^{(SS)} \), and the constant potentials of the square-wells that enter into eq. (27), \( V_{C_{i}^{(SS)}} \) and \( V_{D_{i}^{(SS)}} \). For example

\[
C_{i}^{(SS)} \delta^{(3)}(r) \to \frac{3C_{i}^{(SS)} \theta(R - r)}{4\pi R^{3}} \equiv V_{C_{i}^{(SS)}} \theta(R - r) .
\]

It is important to recall that there is implicit \( m_{q} \)-dependence in this potential arising from the chiral expansions of \( g_{A}, M_{N} \) and \( f \), in addition to the explicit dependence from the \( D_{2}^{(SS)} m_{\pi}^{2} \) contribution. Defining the two-component wavefunction \( \Psi \) to be

\[
\Psi(r) = \begin{pmatrix} u(r) \\ w(r) \end{pmatrix} ,
\]

where \( u(r) \) is the S-wave wavefunction and \( w(r) \) is the D-wave wavefunction, the regulated Schrödinger equation is

\[
\Psi''(r) + \left[ p^{2} + V_{L}^{(QCD)}(r; m_{\pi}) \theta(r - R) + V_{S}^{(QCD)}(r; m_{\pi}, k^{2}) \theta(R - r) \right] \Psi(r) = 0 .
\]

At LO in the partially-quenched theory one has contributions to the NN potential from the momentum and \( m_{q} \)-independent four-nucleon operator with coefficient \( C_{0}^{(SS)} \) and from the chiral limit of OPGBE. In fact, as discussed earlier, it is consistent to retain the full \( m_{q} \) dependence of OPGBE potential at NLO, which in PQQCD leads to

\[
\begin{align*}
V_{C}^{(PQ)(\pi)}(r) &= -\alpha_{\pi} m_{\pi} \frac{e^{-m_{\pi} r}}{r} - \frac{\alpha_{0}}{6} \left( m_{SS}^{2} - m_{\pi}^{2} \right) \left[ m_{\pi} - \frac{2}{r} \right] e^{-m_{\pi} r} \\
V_{T}^{(PQ)(\pi)}(r) &= -\alpha_{\pi} \frac{e^{-m_{\pi} r}}{r} \left( \frac{3}{r^{2}} + \frac{3 m_{\pi}^{2}}{r} + m_{\pi}^{2} \right) - \frac{\alpha_{0}}{6} \left( m_{SS}^{2} - m_{\pi}^{2} \right) \left[ m_{\pi} + \frac{1}{r} \right] e^{-m_{\pi} r} ,
\end{align*}
\]
where it is implicit that both $\alpha_\pi$ and $\alpha_0$ are defined in the partially-quenched theory, whose chiral expansions differ from those of QCD. Given that only the tensor component of OPGBE contributes at LO, and $\alpha_0$ contributes only at NLO, only the partially-quenched expansion of $\alpha_\pi$ is required for this NLO calculation. While one can straightforwardly construct all the operators that contribute to the process in PQ$\chi$PT as one does in $\chi$PT, this is a tedious procedure. Ultimately one arrives at the following relations appropriate for an NLO calculation

$$
\alpha_\pi = \frac{g_A^2}{8\pi f^2} \left( 1 - \frac{2m_\pi^2 T_{1SB}}{g_A} \right)^2, \quad \alpha_0 = \frac{g_0^2}{8\pi f^2},
$$

where $T_{1SB}$ is an additional coefficient that must be determined from lattice calculations. In the chiral expansion of the OPGBE potentials in eq. (31) it is important to note that there is no contribution of the form $1/r^2$. The appearance of such a term would destroy the renormalization program we have constructed for QCD and it is encouraging that such a term does not arise from OPGBE in PQQCD either.

The chiral expansions of $f$, $M_N$ and $g_A$ are also required, and these are known. The chiral expansion of $f$ at NLO is [42]

$$
f = f^{(0)} \left[ 1 - \frac{m_{SV}^2}{4\pi^2 f^{(0)}(0)^2} \log \left( \frac{m_{SV}}{\mu} \right) + l_1 m_\pi^2 + l_2 m_{SS}^2 \right],
$$

where $m_{SV}$ is the mass of a meson composed of one valence and one sea quark, at LO in the chiral expansion, and $l_{1,2}$ are coefficients that need to be determined from lattice calculations and are directly related to the constant $T_4$ in QCD. At NLO, the nucleon mass receives contributions from counterterms only

$$
M_N = M_0 + c_1 m_\pi^2 + c_2 m_{SS}^2 + \ldots,
$$

where the $c_{1,2}$ are to be determined from lattice calculations. The matrix element of the axial current is somewhat more complicated, as it depends upon how one extends the axial currents from QCD to PQQCD, as discussed in Refs. [43,9,10]. For vanishing ghost and sea-quark axial charge $y^{(s)}$, the axial matrix element is found to be [10]

$$
g_A = g_A^{(0)} + \frac{1}{16\pi^2 f^2} \left( \eta^{(0)} - g_A^{(0)} w_N \right) + c^{(0)},
$$

where

$$
\eta^{(0)} = \frac{(g_A^{(0)})^3}{2} L_\pi - 2g_A^{(0)} L_{SV} - \frac{g_0^{(0)} - g_A^{(0)}}{6} \left( 3(g_A^{(0)})^2 + (g_0^{(0)})^2 \right) (L_\pi - L_{SV}),
$$

$$
w = (g_A^{(0)})^2 \left( \frac{1}{2} L_\pi + 4L_{SV} \right) - \frac{g_0^{(0)} - g_A^{(0)}}{2} \left( 5g_0^{(0)} - g_A^{(0)} \right) (L_\pi - L_{SV}),
$$

$$
c^{(0)} = r_1 m_\pi^2 + r_2 m_{SS}^2,
$$

and $L_\pi = m_\pi^2 \log (m_\pi^2/\mu^2)$, and $L_{SV} = m_{SV}^2 \log (m_{SV}^2/\mu^2)$. Generalization to arbitrary ghost and sea-quark axial charges is straightforward [43,10]. The $r_i$ are constants that must be determined from the lattice. Unlike in the single nucleon sector [10], we have not included
the $\Delta$ resonance as an explicit degree of freedom, as the $\Delta$-nucleon mass splitting is approximately the same as $\Lambda_{NN}$, the scale at which the EFT is expected to break-down. If the range of the EFT is somehow extended to higher momenta (or if this one is shown to be valid at higher momenta that we presently expect), the $\Delta$ should be included in the theory explicitly, as the $\Delta N\pi$ intermediate states—in contrast with the $\Delta\Delta$ states—are likely to make a sizable contribution that cannot be described by local counterterms at momenta beyond $\Lambda_{NN}$.

![Diagram](image)

**Fig. 7.** Additional contributions to the scattering amplitude in the $^3S_1$ channel at NLO, $O(Q^0)$, in the partially-quenched EFT. The large solid square denotes an insertion of an $m_q$-dependent operator with coefficient $D_{2B}^{(3S_0)}$.

At NLO in the partially-quenched EFT there are potentially additional contributions from TPGBE. However, it is only the chiral limit of TPGBE that contributes at NLO and the chiral limit of PQQCD is the same as the chiral limit of QCD, by construction, and thus the TPGBE potentials are those given in eq. (24) \(^1\). Thus, the additional contributions at NLO have the same form as in the $^1S_0$ channel: a single insertion of $m_q$ and the exchange of a single $\eta$, as shown in Fig. 7, which generates the potentials given in eq (31). The additional contribution from a single insertion of $m_q$ leads to a modification of the short-distance potential in eq. (27). The short distance potential in the partially-quenched theory is

$$V_{sq}^{(PQ)} = V_{sq}^{(QCD)} + \left( m_{SS}^2 - m_{\pi}^2 \right) V_{D_{2B}^{(3S_1)}}.$$  

(37)

This completes the construction of a partially-quenched EFT describing the $^3S_1 - ^3D_1$ coupled channels. The potential defined above is inserted into the Schrödinger equation in eq. (30) to generate observables. Due to the lack of partially-quenched lattice data in this channel we have not generated phase-shifts or scattering lengths. However, this is a straightforward procedure and requires only limited numerical work [23,32].

**III. DISCUSSION**

In this work we have formulated the effective field theory required to describe the low-energy behavior of partially-quenched QCD in the two-nucleon sector. In fact, it is quite simple to construct the partially-quenched effective field theory from the known QCD results

\(^1\)In $W$ counting one keeps the full TPGBE contribution at NLO [17], for which partial-quenching is somewhat more complicated [24].
FIG. 8. The scattering length in the $^3S_1$-channel in QCD as a function of the pion mass for characteristic strong-interaction parameters. The two shaded regions correspond to different allowed ranges of $D_2(^3S_1)$ that are both consistent with naive dimensional analysis. For a detailed discussion, see Ref. [32]. At the physical value of the pion mass the scattering length is $a(^3S_1) \sim +5.425$ fm.

and it is gratifying to see that one can obtain analytic results for many observables in the $^1S_0$ channel and in the higher partial waves. While a numerical solution is required in the $^3S_1 - ^3D_1$ coupled channels, it amounts to a simple problem in non-relativistic quantum mechanics.

One should be concerned about the range of sea and valence quark masses for which this theory converges. In QCD it is found that the NN EFT converges for $m_\pi$ and momenta less than of order $\Lambda_{NN} \sim 300$ MeV, and one suspects that the same radius of convergence will exist in the partially-quenched theory. If this is indeed the case, lattice calculations will be required with meson masses of less than $\sim 300$ MeV in order to match to the EFT and use it to make predictions about nature. This is somewhat more restrictive than in the meson and single nucleon sectors and therefore one would like to see convergent results in those sectors before being confident in results obtained in the multi-nucleon sectors.

It is very encouraging to see that partially-quenched calculations of quantities in PQ$\chi$PT describing the dynamics of the PGB’s are presently being performed [44], and linear combinations of the Gasser-Leutwyler coefficients appearing at $O(p^4)$ in $\chi$PT are being determined as a result. The situation is far less advanced in the two-nucleon sector. At present, NN scattering lengths must be extracted at finite volume using Lüscher’s formula, which expresses the energy of a two-particle state as a perturbative expansion in the scattering length divided by the size of the box [46]. There are (at least) two potential problems with this approach. First, one may worry that lack of unitarity in PQQCD may invalidate Lüscher’s formula for the NN scattering lengths. However, Lüscher’s formula is easily obtained in EFT($\bar{\chi}$) [47], and one can convince oneself using the arguments of Ref. [24] that, while the pionful NN EFT described above is not unitary in PQQCD, EFT($\bar{\chi}$) is unitary in PQQCD; all the effects of partial-quenching in EFT($\bar{\chi}$) are in the coefficients of the contact operators. It follows by continuity that the NN scattering lengths can be extracted in a lattice simulation of PQQCD by using Lüscher’s formula and extrapolating to the physical quark masses using the for-
malism presented in this paper. A second worry is truly cause for concern: the S-wave NN scattering lengths are extremely large (which is understood as proximity to an infrared fixed point [21,48]) as compared to the sizes of state-of-the-art lattices: $a^{(1S_0)} \sim -23.714$ fm and $a^{(3S_1)} \sim +5.425$ fm. A recent study [32] of the pion-mass dependence of the NN scattering lengths in QCD suggest that the $3S_1$ scattering length relaxes to natural values of $\sim 1$ fm as the pion mass is increased beyond $\sim 200$ MeV (see Fig. 8). One anticipates similar behavior in the partially-quenched theory. Given current uncertainties in strong interaction parameters, particularly $D_2^{(3S_0)}$, it is at present unclear whether the same is true in the $1S_0$ channel (see Fig. 9).

To our knowledge, a single lattice determination of the $1S_0$ and $3S_1$ NN scattering lengths in QQCD exists [45] at a pion mass of $\sim 500$ MeV. This pion mass is beyond the range of applicability of the EFT described in this paper, or of an analogous EFT that one can easily construct to describe QQCD. Given the unknown $D_2$ coefficients that encode the short-distance quark-mass dependence in the S-wave channels one may question the motivation for an improved, partially-quenched simulation of the NN scattering lengths with pion masses within the NN EFT, since such simulation will, at best, simply determine the $D_2$ operators. It has recently been shown [23,32,49] that to leading order in the NN EFT, and assuming perfect knowledge of the single-nucleon sector, the $D_2$ operators determine the quark-mass dependence of the deuteron and, more generally, of dinucleon binding. Since small changes in $m_q$ can in principle lead to drastic changes in the positions of nuclear energy levels, much attention has been given to light-element abundances predicted by big-bang nucleosynthesis and to the abundance of isotopes produced by the Oklo “natural reactor” in the hope that these abundances can be used to constrain high-energy physics [50]. This is powerful motivation indeed for improved lattice simulations of NN scattering lengths. We look forward to a significant lattice effort in the multi-nucleon sector.
ACKNOWLEDGMENTS

This work is supported in part by the U.S. Department of Energy under Grant No. DE-FG03-00-ER-41132 (SRB) and Grant No. DE-FG03-97-ER-4014 (MJS).
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