Estimating the post-measurement state

We study generalized measurements (GMMs), which are a generalization of the traditional quantum measurement. GMMs allow for the measurement of an unknown state of a quantum system, where the outcome of the measurement is not determined by a single measurement, but instead by a probability distribution over a set of measurements. This is in contrast to traditional quantum measurements, where the outcome is determined by a single measurement.

There are two important properties in which measures on a quantum system differ from measurements on classical systems. These properties are related to the concept of quantum uncertainty and the possibility of superposition. The first property is that the measurement outcome of a quantum system is not determined by a single measurement, but instead by a probability distribution over a set of measurements. This is in contrast to classical systems, where the outcome of a measurement is determined by a single measurement.

The second property is that the measurement of a quantum system can affect the system itself. This is known as the measurement problem and is a fundamental aspect of quantum mechanics. The measurement problem is related to the concept of quantum superposition, which states that a quantum system can exist in multiple states simultaneously. When a measurement is performed, the superposition is collapsed, and the system is in a single state.

In conclusion, GMMs are a useful tool for understanding the behavior of quantum systems. They allow for the measurement of an unknown state of a quantum system, where the outcome of the measurement is not determined by a single measurement, but instead by a probability distribution over a set of measurements. This is in contrast to traditional quantum measurements, where the outcome is determined by a single measurement.

1. Jochen Fröhlich, Universität Konstanz, E-51674, Konstanz, Germany
2. Fachhochschule Kempten, Universitätsstraße 10, 87800 Kempten, Germany
3. Research Institute for Particle and Nuclear Physics, P.O. Box 133, Budapest 114, POB 45.
They are positive operators satisfying a completeness relation \( \sum_{s=1}^{n} E_s = I \) which guarantees \( \sum_{s=1}^{n} p_s = 1 \) for the probabilities. The set \( \{ E_s \} \) is called a positive-operator-valued measure (POVM) and the individual operators \( E_s \) are also known as POVM elements or effects.

To prepare later calculations we introduce the spectral decomposition of \( E_s \)

\[
E_s = \sum_{i=1}^{d} a_i^{(s)} |r_i^{(s)}\rangle \langle r_i^{(s)}|.
\]

(4)

\( a_i^{(s)} \) are the positive eigenvalues. The eigenvectors \( \{ |r_i^{(s)}\rangle \} \) form an orthonormal basis. Due to the polar decomposition theorem (cf. e.g. [8]), we may split the measurement operator \( M_s \) into a product of a unitary operator \( U_s \) and the square root of \( E_s \)

\[
M_s = U_s \sqrt{E_s}.
\]

(5)

This implies

\[
M_s M_s^\dagger = U_s E_s U_s^\dagger.
\]

(6)

Thus the positive operators \( M_s M_s^\dagger \) and \( E_s \) have the same eigenvalues \( a_i^{(s)} \) and the diagonal representation of \( M_s M_s^\dagger \) becomes

\[
M_s M_s^\dagger = \sum_{i=1}^{d} a_i^{(s)} |\psi_i^{(s)}\rangle \langle \psi_i^{(s)}|.
\]

(7)

The eigenvectors \( |\psi_i^{(s)}\rangle = U_s |r_i^{(s)}\rangle \) form again an orthonormal basis. Herewith and with the help of eqs.(4) and (5) we obtain as result the useful bi-orthogonal expansions of the unitary operators \( U_s \) and of the measurement operators \( M_s \):

\[
U_s = \sum_{i=1}^{d} |\psi_i^{(s)}\rangle \langle r_i^{(s)}|.
\]

(8)

\[
M_s = \sum_{i=1}^{d} \sqrt{a_i^{(s)}} |\psi_i^{(s)}\rangle \langle r_i^{(s)}|.
\]

(9)

\( |\psi_i^{(s)}\rangle \) and \( |r_i^{(s)}\rangle \) are the l.h.s. and r.h.s. eigenvectors of \( M_s \), respectively. The number of non-zero eigenvalues \( \sum a_i^{(s)} \) equals the rank of \( M_s \).

Based on this we can now move to the problems of quantum state estimation. We assume a single \( d \)-level quantum system prepared in a completely unknown pure pre-measurement state \( |\psi\rangle \). A particular generalized measurement specified by the known set \( \{ M_s \} \) of operators is performed with measurement result \( s \) which is read off. What is the optimal strategy for the estimation of the post-measurement state \( |\psi^{(s)}\rangle \) prepared by the measurement? It is worthwhile to emphasize that the only data available for the estimation are the set \( \{ M_s \} \) specifying the measurement and the value \( s \) of the actual readout.

If the state \( |\chi^{(s)}\rangle \) is proposed as an estimate of the unknown post-measurement state \( |\psi^{(s)}\rangle \), the fidelity

\[
f_s = |\langle \chi^{(s)} | \psi^{(s)} \rangle|^2 = \frac{1}{p_s} |\langle \chi^{(s)} | M_s | \psi \rangle|^2
\]

(10)

is a measure of the quality of the estimation. The fidelity \( \int \) averaged over all measurement outcomes reads:

\[
\mathcal{F} = \sum_{s=1}^{n} f_s p_s.
\]

The mean estimation fidelity \( G_{\text{post}}(\chi) \) in case the in-going (pre-measurement) state is completely unknown, is the result of an integration over all possible states \( |\psi\rangle \):

\[
G_{\text{post}}(\chi) := \int d\psi = \frac{1}{\mathcal{F}} \int d\psi \sum_{s=1}^{n} \langle \chi^{(s)} | M_s | \psi \rangle \langle \psi | M_s^\dagger | \chi^{(s)} \rangle,
\]

(11)

with respect to the normalized unitary invariant measure on the state space, yielding:

\[
G_{\text{post}}(\chi) = \frac{1}{\mathcal{F}} \sum_{s=1}^{n} \langle \chi^{(s)} | M_s M_s^\dagger | \chi^{(s)} \rangle.
\]

(12)

By virtue of eq.(7), each component in the sum over \( s \) in (12) is maximized if \( |\chi^{(s)}\rangle \) is chosen to be the eigenvector \( |\psi_{\text{max}}^{(s)}\rangle \) of \( M_s M_s^\dagger \) of the maximum eigenvalue \( a_{\text{max}}^{(s)} \).

For the measurement result \( s \) the best estimate of the post-measurement state is therefore given by

\[
|\chi_{\text{post}}^{(s)}\rangle = |\psi_{\text{max}}^{(s)}\rangle.
\]

(13)

In case of degeneracy of the greatest eigenvalue \( a_{\text{max}}^{(s)} \), any state vector from the corresponding eigenspace represents an optimal estimation of the post-measurement state. The maximum value of \( G_{\text{post}}(\chi) \) reads

\[
G_{\text{post}} = \frac{1}{\mathcal{F}} \sum_{s=1}^{n} a_{\text{max}}^{(s)}.
\]

(14)

\( G_{\text{post}} \) is the mean post-measurement estimation fidelity, \( |\chi_{\text{post}}^{(s)}\rangle \) and \( G_{\text{post}} \) are determined solely by the operators \( M_s \) which specify the generalized measurement.

We now address the question, how \( G_{\text{post}} \) is related to the mean operation fidelity \( F \) which describes how much the state after the measurement resembles the original one. The larger the value \( F \) of a measurement is, the weaker is its disturbing influence. Arguing as above, \( F \) is obtained from eq.(11) if we replace \( |\chi^{(s)}\rangle \) by \( |\psi\rangle \):

\[
F = \int d\psi \sum_{s=1}^{n} |\langle \psi | M_s | \psi \rangle|^2.
\]

(15)

It may be rewritten as [5]:

\[
F = \frac{1}{d(d+1)} \left( d + \sum_{s=1}^{n} \text{tr} M_s^2 \right).
\]

(16)
To derive a relation between $G_{\text{post}}$ and $F$, it is useful to first relate $G_{\text{post}}$ to the estimation fidelity of the pre-measurement state. Denoting this estimate by $|\chi^{(s)}_{\text{pre}}\rangle$, the corresponding mean estimation fidelity, in analogy to $G_{\text{post}}(\chi)$ of eq.(11), reads:

$$G_{\text{pre}}(\chi) = \int d\psi \sum_{s=1}^{n} p_s |\langle \chi^{(s)}|\psi \rangle|^2$$

which may be rewritten according to Banaszek [5] as

$$G_{\text{pre}}(\chi) = \frac{1}{d(d+1)} \left( d + \sum_{s=1}^{n} \langle \chi^{(s)} | E_s | \chi^{(s)} \rangle \right)^{\frac{1}{2}}.$$  

The optimum pre- and post-measurement fidelities are closely related. For a given measurement result $s$, the best estimate $|\chi_{\text{pre}}^{(s)}\rangle$ of the pre-measurement state is the one which maximizes the corresponding component in the sum in eq.(18). Because of eq.(4), it is given by the eigenvector $|r_{\text{max}}^{(s)}\rangle$ of $E_s$ belonging to the maximum eigenvalue $[5,9]$. But this eigenvalue is again $a_{\text{max}}^{(s)}$. The best estimate of the pre-measurement state related to the outcome $s$ is therefore

$$|\chi_{\text{pre}}^{(s)}\rangle = |r_{\text{max}}^{(s)}\rangle.$$

We denote the corresponding maximum value of $G_{\text{pre}}(\chi)$ by $G_{\text{pre}}$ and call it the mean pre-measurement estimation fidelity. Comparing it to the form (14) of $G_{\text{post}}$, we obtain the simple new relationship:

$$G_{\text{pre}} = \frac{1}{d+1} (1 + G_{\text{post}}).$$

This result allows us to transcribe Banaszek’s constraint [5] between $F$ and $G_{\text{pre}}$ into a constraint relating $F$ and $G_{\text{post}}$:

$$\sqrt{(d+1)F - 1} \leq \sqrt{G_{\text{post}}} + \sqrt{(d-1)(1-G_{\text{post}})}.$$  

To illustrate how state disturbance and information gain are related for the post-measurement situation, we display the domain of possible combination of $F$ and $G_{\text{post}}$ in the $G_{\text{post}}$-$F$ plane. If the system is not influenced at all, the measurement has the operation fidelity $F = 1$. In this case the guess of the pre- and post-measurement state is totally random which amounts to $G_{\text{pre}} = G_{\text{post}} = 1/d$. On the other hand there are measurements which allow to predict the post-measurement state exactly (e.g. projection measurements), i.e., with maximum fidelity $G_{\text{post}} = 1$. This leads via eq.(20) to $G_{\text{pre}} = 2/(d+1)$. This result for $G_{\text{pre}}$ has also been obtained in [1,4-6]. It is known [2] that it corresponds to $F = 1/(d+1)$. To summarize, the domain of possible combinations $(G_{\text{post}},F)$ is limited by $1/d \leq G_{\text{post}} \leq 1$ and $2/(d+1) \leq F \leq 1$ as well as by the inequality (21). The boundaries of the domain are indicated in Fig. 1 for $d = 2$, including the dashed lines. In this domain, every particular generalized measurement $\{M_s\}$ corresponds to a point. Its position illustrates to what extent the information about the outgoing (post-measurement) state is gained at the cost of disturbing the in-going (pre-measurement) one. Large values of $F$ combined with large values of $G_{\text{post}}$ characterize the most optimal type of generalized measurement. For increasing dimension $d$ of the state space all types of measurements become less advantageous (cf. Fig. 1).

To complete this discussion we return to the question: What type of generalized measurements apart from projection measurements make it possible to know the post-measurement state $|\psi^{(s)}\rangle$ exactly? As we mentioned earlier, the necessary condition is the rank of Kraus-operator $M_s$ be 1:

$$M_s = \sqrt{a^{(s)}} |r^{(s)}\rangle \langle r^{(s)}|.$$  

From eq.(1) it follows that the post-measurement state is always $|r^{(s)}\rangle$ independently of the otherwise unknown pre-measurement state. If we apply our general rule (13) to this trivial case we find that, indeed, the best estimate is the one: $|\chi_{\text{post}}^{(s)}\rangle = |r^{(s)}\rangle$. The rule (19) yields $|\chi_{\text{pre}}^{(s)}\rangle = |r^{(s)}\rangle$ for the best estimate of the pre-measurement state. Hence the ultimate form of the rank-one Kraus-operators is this:

$$M_s = \sqrt{a^{(s)}} |\chi_{\text{post}}^{(s)}\rangle \langle \chi_{\text{post}}^{(s)}|.$$  

The corresponding effects $E_s$ are then given by

$$E_s = a^{(s)} |\chi_{\text{pre}}^{(s)}\rangle \langle \chi_{\text{pre}}^{(s)}|.$$  

The completeness relation $\sum E_s = 1$ constrains the pre-measurement state estimates to form an orthonormal basis in general. The set of post-measurement states is not constrained at all. Note that the multiplicity of different measurement results $s$ may exceed the number $d$ of levels in our system. Since neither $|\chi_{\text{pre}}^{(s)}\rangle$ nor $|\chi_{\text{post}}^{(s)}\rangle$ have to form orthogonal systems they are in general not the

![FIG. 1. Maximal operation fidelity $F$ for given estimation fidelity $G_{\text{post}}$ of the post-measurement state in dimensions $d = 2,4,8,16,\ldots$. The dashed lines mark for dimension $d = 2$ the domain for possible combinations of $F$ and $G_{\text{post}}$.](image-url)
eigenstates of any Hermitian observable. So we are still having a generalized measurement and not a projective measurement. The post-measurement state is nevertheless exactly known ($G_{\text{post}} = 1$) and the optimal estimate of the pre-measurement state $\chi^{(s)}_{\text{pre}}$ is of maximal estimation fidelity $G_{\text{pre}} = 2/(d+1)$.

We turn to a further aspect of information gain and state disturbance. In eq.(5) we have uniquely decomposed the measurement operation $M_s$ which corresponds to the measurement result $s$, into the positive operator $\sqrt{E_s}$ and a unitary operator $U_s$. The unitary part does not change the von Neumann entropy. By virtue of eq.(2), all information, which is contained in a measurement result, goes back to $\sqrt{E_s}$. In particular, the estimation fidelities $G_{\text{post}}$ and $G_{\text{pre}}$ (14,18) depend only on the eigenvalues of $E_s$. The part $\sqrt{E_s}$ of $M_s$ represents, at a given information gain, the unavoidable minimal disturbance of the state vector. We call $\sqrt{E_s}$ the pure measurement part of a generalized measurement and a measurement with $U_s = 1$ a pure measurement. The operation fidelity $F$ depends on the unitary parts $U_s$, too. The inequality (21) shows that the maximal operation fidelity $F$ is limited by $G_{\text{post}}$ and therefore by the pure measurement part.

Having connected the three mean fidelities $F, G_{\text{pre}}$ and $G_{\text{post}}$, we connect now the best guesses $\chi^{(s)}_{\text{pre}}$ and $\chi^{(s)}_{\text{post}}$ for the pre- and post-measurement states, respectively. The two best guesses are the distinguished pair of l.h.s. and r.h.s. eigenvectors to the same eigenvalue, cf. eqs.(13) and (19). Invoking the expansions (8) and (9), this leads directly to the results

$$U_s \chi^{(s)}_{\text{pre}} = \chi^{(s)}_{\text{post}}$$

and

$$\frac{M_s \chi^{(s)}_{\text{pre}}}{\sqrt{G_{\text{max}}}} = \chi^{(s)}_{\text{post}}.$$  

Equation (25) shows that the best estimate for the post-measurement state can be obtained from the best estimate of the pre-measurement state by applying merely the unitary part $U_s$ of the measurement operator. This has the surprising consequence that for all pure measurements the best estimations for the pre- and post-measurement state always agree if the ingoing state $\chi^{(s)}$ is completely unknown. This is the case regardless of the values of the operation fidelity $F$ and the estimation fidelities $G_{\text{pre}}$ and $G_{\text{post}}$.

Finally we give a physical interpretation of relation (26). As a matter of fact, both the pre- and post-measurement states become only partially revealed by the estimation procedure. Nonetheless, even non-optimal estimates $\chi^{(s)}_{\text{pre}}, \chi^{(s)}_{\text{post}}$ must obey the constraint (26) expressing the certain fact that the post-measurement state results from the generalized measurement (1) of the pre-measurement one. Recall that we estimated the optimum pre- and post-measurement states by maximizing independently the pre- and post-measurement fidelities. We did not guarantee explicitly that the two optimum states $\chi^{(s)}_{\text{pre}}, \chi^{(s)}_{\text{post}}$ satisfy the exact constraint. The derived result (26) proves that they do.

In conclusion, we have studied generalized measurement $\{M_s\}$ on a single $d$-level quantum system. For the case when the initial state is pure and otherwise completely unknown, we pointed out that the best estimates of the pre- and post-measurement states for a given measurement readout $s$ are the respective right and left eigenvectors of $M_s$, belonging to the (common) largest eigenvalue. The mean post-measurement estimation fidelity of the measurement device is also calculated and shown to satisfy a simple relationship with the mean pre-measurement estimation fidelity. A constraint between the post-measurement estimation fidelity and the operation fidelity of the measurement illustrates how state disturbance and information gain about the post-measurement state are competing with each other. We have shown that for pure generalized measurements the independent best estimates of the pre- and post-measurement states agree. We have proved that, in general, they are related via the corresponding measurement operator as we expect of them.

This work was supported by the Optik Zentrum Konstanz. L.D. acknowledges the support from the Hungarian Science Research Fund under Grant 32640.