In Einstein gravity there is a simple procedure to build \( D \)-dimensional spacetimes starting from \((D - 1)\)-dimensional ones, by stacking any \((D - 1)\)-dimensional Ricci-flat metric into the extra-dimension. We analyze this procedure in the context of Einstein-Gauss-Bonnet gravity, and find that it can only be applied to metrics with a constant Krestschmann scalar. For instance, we show that solutions of the black-string type are not allowed in this framework.
II. STACKING TECHNIQUE

There is a set of solutions to $D$-dimensional Einstein gravity that can be constructed in a simple way starting from vacuum $(D-1)$-dimensional solutions. The procedure is based on the stacking of any vacuum $(D-1)$-dimensional solution into the additional dimension \[18\].

Given any metric $g_{\mu\nu}(x)$ such that the $(D-1)$-dimensional Einstein tensor satisfies $G_{\mu\nu} = 0$, then

$$ds^2 = dy^2 + g_{\mu\nu}(x)dx^\mu dx^\nu$$

is a solution of the $D$-dimensional Einstein equations $(D)G_{AB} = 0$. This procedure can also be used when a $D$-dimensional negative cosmological constant $|\Lambda_D| = (D-2)/(D-1)/2l^2$ is present in the bulk: any metric $(D)$ $g_{AB}$ of the form

$$ds^2 = dy^2 + e^{-2y/l}g_{\mu\nu}(x)dx^\mu dx^\nu, \quad (2)$$

where $g_{\mu\nu}(x)$ is a $(D-1)$-dimensional vacuum solution, is a solution of $(D)G_{AB} = -\Lambda_D(D)g_{AB}$. \[3\]

Starting with solutions of the type \[2\] one can easily construct braneworld geometries by using the standard cut-and-paste procedure. For instance, the metric

$$ds^2 = dy^2 + e^{-2|y|/l}g_{\mu\nu}(x)dx^\mu dx^\nu \quad (4)$$

represents a braneworld geometry with $\mathbb{Z}_2$ symmetry with respect to the location of the brane ($y = 0$). In $D = 5$, if we take $g_{\mu\nu}$ to be the 4-dimensional Schwarzschild metric we reproduce the black string geometry \[19\]. Instead, if we take $g_{\mu\nu}$ to be a pp-wave we have a 5-dimensional pp-wave travelling parallel to the brane \[20\].

This technique can be further generalized to the stacking of any Einstein manifold. Using the ansatz for the $D$-dimensional metric

$$ds^2 = dy^2 + e^{-2A(y)}g_{\mu\nu}(x)dx^\mu dx^\nu \quad (5)$$

the Einstein equations \[4\] can be split into

$$G_{\mu\nu}(x) = -C_2(y)g_{\mu\nu}(x), \quad (6)$$

which correspond to the $\mu\nu$ components, and

$$C_3(y) + C_4(y)R(x) = 0, \quad (7)$$

corresponding to the $yy$ component, where $R = g^{\mu\nu}R_{\mu\nu}$. The coefficients are given by

$$C_2(y) = \left[ \Lambda_D - (D-2)\left( A'' - \frac{(D-1)}{2}A^2 \right) \right] e^{-2A}, \quad (8)$$

$$C_3(y) = \Lambda_D + \frac{(D-1)(D-2)}{2}A^2, \quad (9)$$

$$C_4(y) = -\frac{1}{2}e^{2A}. \quad (10)$$

By differentiating equation \[7\] with respect to any brane coordinate $x^\mu$ we obtain

$$C_4(y)R(x)_{,\mu} = 0, \quad (11)$$

so that the Ricci scalar has to be constant. For convenience we write this constant as

$$R = \pm \frac{(D-1)(D-2)}{L^2}. \quad (12)$$

\footnote{It is interesting to note that by using additional freedom in the way in which a $(D-1)$-dimensional geometry is embedded into a $D$-dimensional Einstein manifold one can locally and isometrically embed whatever metric \[21\].}
From Eq. (7) and for a negative cosmological constant in the bulk, we have an equation for $A(y)$ of the form

$$
\frac{(D-1)(D-2)}{2} \left( -\frac{1}{l^2} + A^2 \mp \frac{1}{L^2} e^{2A} \right) = 0. \tag{13}
$$

Then Eq. (6) becomes

$$
G_{\mu\nu}(x) = \mp \frac{(D-2)(D-3)}{2L^2} g_{\mu\nu}(x), \tag{14}
$$

which shows that $g_{\mu\nu}(x)$ must be a $(D-1)$-dimensional Einstein metric. Equation (13) can be easily solved. When $L \to \infty$ we have $A(y) = y/l + b$, i.e., the solution in Eq. (2) (the constant $b$ is irrelevant for the geometry). For $L \neq 0$ and defining $A = -\ln B$ we have

$$
B'^2 = B^2 \pm \frac{1}{L^2}. \tag{15}
$$

The solution corresponding to the plus sign (positively curved brane) is

$$
ds^2 = dy^2 + \left( \frac{l}{L} \right)^2 \sinh^2 \left( \frac{y - y_0}{l} \right) g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{16}
$$

and that corresponding to the minus sign (negatively curved brane) is

$$
ds^2 = dy^2 + \left( \frac{l}{L} \right)^2 \cosh^2 \left( \frac{y - y_0}{l} \right) g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{17}
$$

Again one can start from these bulk solutions to build braneworld models. The braneworld models with maximally symmetric branes, 4-dimensional de Sitter and anti-de Sitter branes, are the simplest illustration of this procedure. What we have shown here is that by using the same warp factors one can have not only maximally symmetric branes but any brane of the Einstein type.

For the case of a positive cosmological constant in the bulk, $B(y)$ satisfies

$$
B'^2 = -B^2 \pm \frac{1}{L^2}, \tag{18}
$$

which has solutions only for the plus sign (positively curved sections):

$$
B(y) = \frac{l}{L} \sin \left( \frac{y - y_0}{l} \right), \text{ or } B(y) = \frac{l}{L} \cos \left( \frac{y - y_0}{l} \right), \tag{19}
$$

The case with zero bulk cosmological constant can be obtained by taking the limit $l \to \infty$,

$$
B(y) = \frac{y - y_0}{L}. \tag{20}
$$

For completeness, it is interesting to observe that this technique can also be used to generate bulk solutions with positive cosmological constant and arbitrarily curved sections. The result in this case differs from the previous one in the following respect: given any Euclidean metric $g_{\mu\nu}(x)$ such that $R_{\mu\nu} = \pm [(D-2)/L^2] g_{\mu\nu}(x)$, then

$$
ds^2 = -dt^2 + B(t)^2 g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{21}
$$

with

$$
B(t) = e^{2t/l} \text{ for } L \to \infty, \tag{22}
$$

$$
B(t) = \frac{l}{L} \cosh \left( \frac{y - y_0}{l} \right) \text{ for the positive sign}, \tag{23}
$$

$$
B(t) = \frac{l}{L} \sinh \left( \frac{y - y_0}{l} \right) \text{ for the negative sign}, \tag{24}
$$

are Lorentzian $D$-dimensional Einstein manifolds of positive curvature. This might be of interest in standard cosmology. One can deduce this result easily from the above solutions for negative bulk cosmological constant via the substitutions: $l \to -il, t \to it$ and $L \to -iL$. 

Our central aim is to investigate the stacking technique in the simplest modification of Einstein gravity for higher-dimensional spacetimes, the Einstein-Gauss-Bonnet theory. From now on we will set \( D = 5 \). The theory is defined by the action

\[
S = \frac{1}{2 \kappa^2} \int d^5 \sqrt{-g} \left[ (5)^R - 2 \Lambda_5 + \alpha (5)^{L_{GB}} \right],
\]

where

\[
(5)^{L_{GB}} = 2 (5)^{R_{ACDE}R_{B^CDE}} - 4 (5)^{R_{ACBD}R_{C^D}} + 2 (5)^{R_{AB}} - \frac{1}{2} (5)^{g_{AB}L_{GB}}.
\]

This tensor is divergence free, \( \nabla^A (5)^{H_{AB}} = 0 \), and so it can be considered as some sort of source in the Einstein equations. The existence of this term has dramatic consequences for the stacking procedure.

### A. Stacking with a Gauss-Bonnet term

We consider a metric of the form given in Eq. (5), and introduce it into the Einstein-Gauss-Bonnet equations. By doing this, we obtain a set of equations playing the role of the effective 4-dimensional Einstein equations [the \( \mu \nu \)-components of Eq. (27)],

\[
C_1(y)G_{\mu \nu}(x) = -C_2(y)g_{\mu \nu}(x)
\]

and an additional condition [the \( y y \)-component of Eq. (27)],

\[
C_3(y) + C_4(y)R(x) + C_5(y)L_{GB}(x) = 0,
\]

where

\[
\begin{align*}
C_1(y) &= 1 + 4 \alpha \left( A'' - A'^2 \right), \\
C_2(y) &= \left[ \Lambda_5 - 3 \left( A'' - 2 A'^2 \right) + 12 \alpha A'^2 \left( A'' - A'^2 \right) \right] e^{-2A}, \\
C_3(y) &= \Lambda_5 + 6 A'^2 \left( 1 - 2 \alpha A'^2 \right), \\
C_4(y) &= -\left( \frac{1}{2} - 2 \alpha A'^2 \right) e^{2A}, \\
C_5(y) &= -\frac{\alpha}{2} e^{4A}.
\end{align*}
\]

By dividing equation (30) by \( C_5(y) \) and differentiating with respect to \( y \), we obtain

\[
\left( \frac{C_3(y)}{C_5(y)} \right)' - \left( \frac{C_4(y)}{C_5(y)} \right)' = 0.
\]

Now, differentiating it with respect to any brane coordinate \( x \), we arrive at

\[
\left[ \frac{C_4(y)}{C_5(y)} \right]' R(x),_\mu = 0,
\]

so that either \( R(x),_\mu = 0 \) or \( (C_4/C_5)' = 0 \). We call the first case physical because it has a well defined limit when \( \alpha \) tends to zero, the Einsteinian limit. Einstein-Gauss-Bonnet theory is ambiguous from a dynamical point of view. The Lagrangian is quartic in first derivatives of the metric and thus the same initial data can give rise to different evolutions [23]. To resolve this ambiguity, one possibility is to choose as the physical branch that approaching proper general relativity in the limit \( \alpha = 0 \) [24]. The second case, without an Einsteinian limit, we call “purely” Gauss-Bonnet.
We write the constant Ricci scalar as
\[ R = \pm \frac{12}{L^2}. \]  
(38)

By Eq. (30), \( L_{GB} \) is also a constant, \( L_{GB} = S_1 \), and then Eq. (30) gives
\[ 6A^2 (1 - 2\alpha A^2) - \left( \frac{1}{2} - 2\alpha A^2 \right) e^{2A} \left( \pm \frac{12}{L^2} \right) - \frac{\alpha}{2} e^{4A} S_1 + \Lambda_5 = 0, \]  
(39)
or re-arranging,
\[ 2\alpha A^4 - \left( 1 \pm \frac{4\alpha}{L^2} \right) A^2 + \left( \frac{\Lambda_5}{6} \mp \frac{1}{L^2} e^{2A} + \frac{\alpha}{12} e^{4A} S_1 \right) = 0. \]  
(40)

Multiplying Eq. (39) by \( e^{-4A} \) and differentiating we see that equations (29) can be re-expressed as proper Einstein equations for the 4-dimensional geometry,
\[ G_{\mu\nu}(x) = \mp \frac{3}{L^2} g_{\mu\nu}(x). \]  
(41)

In summary, for a metric of the form Eq. (4) to be a solution of Einstein-Gauss-Bonnet theory in five dimensions, the 4-dimensional metric \( g_{\mu\nu} \) must be an Einstein metric. Moreover, and this is the important point, it must have a constant Gauss-Bonnet term, i.e. \( L_{GB} = \) constant. These conditions imply that the Krestschmann scalar, or equivalently in this case the square of the 4-dimensional Weyl curvature, has to be constant. Their values in terms of the constants previously introduced are
\[ R_{\mu\nu\gamma\sigma} R^{\mu\nu\gamma\sigma} = S_1, \quad C_{\mu\nu\gamma\sigma} C^{\mu\nu\gamma\sigma} = S_1 - \frac{24}{L^2}. \]  
(42)

This condition precludes the existence of stacking solutions of the black-string type \(^2\) and most of the solutions of astrophysical interest. The trivial conformal nature of maximally symmetric and pp-wave spacetimes allows them to be stacked to form 5-dimensional bulk solutions and subsequently braneworld models.

For arbitrary values of the constant \( S_1 \) the solution of Eq. (40) cannot be expressed in closed form. However, for the particular value \( S_1 = 24/L^4 \), which is the relevant value for stacking maximally symmetric metrics on the brane, closed-form solutions can be given. For \( 0 > \Lambda_5 = -6/l^2 \),
\[ A^2 = \frac{1}{4\alpha} \left( 1 \pm \frac{4\alpha}{L^2} e^{2A} + \sigma \sqrt{1 - \frac{8\alpha}{l^2}} \right) \]  
(43)
where \( \sigma = \pm 1 \). This expression has a well defined limit for \( \alpha \to 0 \) only when \( \sigma = -1 \). For this case and \( 0 \leq \alpha \leq l^2/8 \), we can define \( \tilde{l} \) and \( \tilde{L} \) as
\[ \frac{1}{\tilde{l}^2} = \frac{1}{4\alpha} \left( 1 - \sqrt{1 - \frac{8\alpha}{l^2}} \right), \]  
(44)
\[ \frac{1}{\tilde{L}^2} = \frac{4\alpha}{L^2}. \]  
(45)

Then, substituting \( A = -\ln B \), Eq. (43) is seen to be equivalent to Eq. (15), i.e.,
\[ B^2 = \frac{B_0^2}{\tilde{l}^2} \pm \frac{1}{\tilde{L}^2}. \]  
(46)

Therefore the warp factors for the Einstein-Gauss-Bonnet theory are formally equal to those discussed in Sec. II but with re-defined values of the curvature of the stacking hypersurfaces and the bulk curvature.

Solutions with \( \alpha < 0 \) are completely equivalent\(^2\). Their only peculiarity is that positively curved and negatively curved stacking hypersurfaces interchange their associated warp factors.

\(^2\) In the Gauss-Bonnet modification to General Relativity arising from string theory, \( \alpha \) must be positive. From this viewpoint, these solutions are not physical.
The solutions for the case $\sigma = 1$ are of purely Einstein-Gauss-Bonnet type in the sense that they do not have a well defined Einsteinian limit. Particularly interesting is the fact that in this case the Gauss-Bonnet term can produce a negatively curved bulk even in the absence of a bulk cosmological constant [7]. Here we take the view that these solutions are unphysical. However, a definitive analysis of their physical nature should consider not their proximity to 5-dimensional Einstein solutions, but their proximity to 4-dimensional Einstein solutions in the reduced 4-dimensional theory that emerges once the compactification (exotic or not) has taken place. This is beyond the scope of the present paper.

C. Purely Gauss-Bonnet case

This case is characterized by $(C_4/C_5)' = 0$, so that condition [7] is satisfied. Therefore,

$$\frac{C_4(y)}{C_5(y)} = \left(\frac{1}{\alpha} - 4A'^2\right)e^{-2A} = S_2,$$

where $S_2$ is a constant. From Eq. (48),

$$\frac{C_3(y)}{C_5(y)} + S_2 R(x) + L_{GB}(x) = 0,$$

and therefore

$$\frac{C_3(y)}{C_5(y)} = -\frac{2}{\alpha}e^{-4A}\left[\Lambda_5 + 6A'^2\left(1 - 2\alpha A'^2\right)\right] = S_3,$$

$$S_2 R(x) + L_{GB}(x) = S_3,$$

with $S_3$ another constant. By Eqs. (47) and (49), we deduce that there are two cases:

1) $S_2^2 = -\frac{2}{\alpha}S_3 \neq 0$, $\Lambda_5 = -\frac{3}{4\alpha}$.

   Again, for $\alpha > 0$ we can define constants $\bar{l}$ and $\bar{L}$ via

$$\frac{1}{\bar{l}^2} = \frac{1}{4\alpha},$$

$$\frac{1}{\bar{L}^2} = \frac{|S_2|}{4}.$$

Setting $A = -\ln B$ we arrive at an equation equivalent to Eq. (45),

$$B'^2 = B^2 + \text{sign}(S_1)\frac{1}{\bar{L}^2},$$

and therefore, to the same formal solutions for the warp factors. For these warp factors, we can check that $C_1(y) = C_2(y) = 0$ and so we reach the intriguing result that there is a single equation for the 4-dimensional metric

$$S_2 R(x) + L_{GB}(x) = -\frac{3}{2}S_2^2.$$

2) $A(y) =$constant.

   Without lost of generality we can set $A(y) = 0$, and then $S_2 = 1/\alpha$ and $S_3 = 2\Lambda_5/\alpha$. In this case the reduced set of equations to solve is, by Eqs. (44) and (49),

$$G_{\mu\nu} = -\Lambda_5 g_{\mu\nu}, \quad L_{GB} = \frac{2\Lambda_5}{\alpha}.$$

If moreover $\Lambda_5 = 0$, these equations do not depend on $\alpha$ and therefore, this case does in fact have a well defined Einsteinian limit.
IV. DISCUSSION AND CONCLUSIONS

We have seen that in the presence of a Gauss-Bonnet term the generation of 5-dimensional bulk solutions via simple stacking of 4-dimensional metrics does not hold as may have been naively expected. Although the Gauss-Bonnet term is a topological invariant in four dimensions, in the process of reducing the system from five dimensions to four dimensions the dynamical degrees of freedom of the Gauss-Bonnet term in five dimensions leave their traces in the reduced theory (cf. [25]).

Let us consider a different point of view. In the simplest case, with zero bulk cosmological constant and no warp factor ($A = 0$), we substitute the ansatz $ds^2 = dy^2 + g_{\mu\nu}(x)dx^\mu dx^\nu$ into the 5-dimensional action,

$$S = \frac{1}{2\kappa_5^2} \int dx^5 \sqrt{-g} \left[ (5)R + \alpha (5)L_{GB} \right],$$

and integrate over $y$ (one can consider a cylindrical condition on $y$ to obtain a finite result.) In this way one finds a reduced action in four dimensions of the Einstein-Hilbert form,

$$S = \frac{1}{2\tilde{\kappa}_5^2} \int dx^4 \sqrt{-g} R.$$

(The trivial additive constant left by the Gauss-Bonnet term has not been written. The Gauss-Bonnet term has become topological in the reduction process.) This is the standard Kaluza-Klein reduction in the absence of the electromagnetic part of the metric (in Kaluza-Klein terminology); therefore we recover vacuum Einstein gravity in four dimensions [26]. In standard Kaluza-Klein reduction, the 5-dimensional Gauss-Bonnet term modifies the way in which the electromagnetic field interacts with gravity [27], but in the absence of this field, there is no other higher-dimensional manifestation of the Gauss-Bonnet term. However, what we have seen is that if one uses first the action Eq. (56) to obtain the 5-dimensional equations of motion, and then one specializes to the above metric ansatz, one does recover the 4-dimensional Einstein equations $R_{\mu\nu} = 0$, but in addition, one finds the condition $L_{GB} = 0$. This condition puts a strong restriction on the allowed geometries. Actually, this condition is equivalent to $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = 0$, so it only permits the existence of conformally trivial 4-geometries of this type. Therefore, in the presence of a Gauss-Bonnet term, the process of dimensional reduction of the action and subsequent variation is not equivalent to the process of first varying the 5-dimensional action and then reducing dimensionally the resulting equations.

What we have seen in this paper is that this phenomenon is general and shows up in braneworld (exotic) compactifications as well. Our simple ansatz does not allow any curvature singularity on the brane, in particular ruling out black string type braneworlds. In this sense our result is in tune with the well known idea that Gauss-Bonnet corrections to Einstein relativity might smooth out the singularities. Trying to find non-trivial brane geometries would involve the consideration of an electromagnetic and/ or a dilaton part for the metric with their corresponding effects.

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