Universal features of the holographic duality: conformal anomaly and brane gravity trapping from 5d AdS Black Hole

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ABSTRACT

We calculate the holographic conformal anomaly and brane Newton potential when bulk is 5d AdS BH. It is shown that such anomaly is the same as in the case of pure AdS or (asymptotically) dS bulk spaces, i.e. it is (bulk) metric independent one. While Newton potential on the static brane in AdS BH is different from the one in pure AdS space, the gravity trapping still occurs for two branes system. This indicates to metric independence of gravity localization.

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1 Introduction

The spectacular realization of the holographic principle in string theory in the form of (A)dS/CFT correspondence and braneworld scenario shows that there are some close relations between so far distant areas of the theory of fundamental interactions. Moreover, some really new questions should be addressed in the connection with holography. Indeed, the first impression of the AdS/CFT correspondence is the fundamental role of bulk AdS space. On the same time, it became clear that different bulk spaces (say, pure AdS or AdS Black Hole or even dS) simply correspond to different dual CFTs when duality may be established. In this connection, the fundamental question may be: what are the universal features of holographic duality? In other words, what properties do not depend on the choice of bulk space?

In the present contribution we discuss two issues: the calculation of the 4d holographic conformal anomaly and of the brane Newton potential when bulk space is 5d AdS BH. It is shown that the holographic conformal anomaly has the same form as in case of pure AdS or pure dS bulk space. This is demonstration of its bulk metric (as well as horizon) independence. The Newton potential on the (adiabatically) static brane (when bulk is AdS BH) is different from the one in case of pure AdS bulk. Nevertheless, for large BH limit the gravity trapping occurs for two branes model. This indicates that gravity trapping is also universal phenomenon which does not depend from the choice of bulk space.

2 Metric independence of holographic conformal anomaly: bulk black hole spacetime

In this section our purpose is to calculate holographic conformal anomaly from bulk AdS black hole, using AdS/CFT correspondence[1]. We prove that such holographic anomaly turns out to be the same as from pure bulk AdS space. The same phenomenon occurs for 5d cosmological deSitter bulk space. This explicitly shows (bulk) metric independence as well as horizon independence of holographic conformal anomaly.
One starts from 5-dimensional AdS-Schwarzschild black hole background:

\[ ds^2_{\text{AdS-S}} = - \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right) dt^2 + \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right)^{-1} dr^2 + r^2 \sum_{i=1}^{3} (dx^i)^2 . \]  

(1)

The last term in (1) is replaced as

\[ ds^2 = - \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right) dt^2 + \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right)^{-1} dr^2 + r^2 \sum_{i=1}^{3} g_{ij} dx^i dx^j , \]  

(2)

and we introduce a new coordinate \( \rho \) by

\[ \rho \equiv r^{-2} , \]  

(3)

and rewrite the metric (2) in the following form:

\[ ds^2 = - \left( \frac{1}{l^2 \rho} - \frac{\mu}{\rho^4} \right) dt^2 + \frac{1}{4} \left( \frac{\rho^2}{l^2} - \frac{\mu}{\rho^4} \right)^{-1} d\rho^2 + \rho^{-1} \sum_{i,j=1}^{3} g_{ij} dx^i dx^j , \]  

(4)

In the limit \( \rho \to 0 \) (\( r \to \infty \)), the metric behaves as

\[ ds^2 \to \frac{l^2}{4\rho^2} d\rho^2 + \rho^{-1} \left( - \frac{1}{l^2} + \sum_{i=1}^{3} g_{ij} dx^i dx^j \right) = \frac{l^2}{4\rho^2} d\rho^2 + \rho^{-1} \sum_{m,n=0}^{3} \tilde{g}_{mn} dx^m dx^n . \]  

(5)

Then one can regard that for the general metric \( \tilde{g}_{mn} \), we have chosen gauge conditions:

\[ \tilde{g}_{tt} = - \frac{1}{l^2} , \quad \tilde{g}_{ti} = \tilde{g}_{it} = 0 , (i = 1, 2, 3) . \]  

(6)

For the metric \( \tilde{g}_{mn} \), the non-vanishing components of the connection \( \tilde{\Gamma}^l_{ij} \) and the Ricci curvature \( \tilde{R}_{mn} \) are

\[ \tilde{\Gamma}_{ij} = \frac{l^2}{2} g_{ij,t} , \quad \tilde{\Gamma}_{tj} = \tilde{\Gamma}_{jt} = \frac{1}{2} g_{ik,t} g_{kj,t} , \quad \tilde{\Gamma}_{jk} = \gamma_{jk} , \]  

\[ \tilde{R}_{tt} = - \frac{1}{2} g_{ij,t} g_{ij,t} + \frac{1}{4} g_{ik,t} g_{kj,t} g_{kl,t} , \]  

\[ \tilde{R}_{ij} = r_{ij} + \frac{l^2}{2} g_{ij,t} - \frac{l^2}{2} g_{kl,t} g_{ik,t} g_{kj,t} + \frac{l^2}{4} g_{ij,t} g_{kl,t} g_{kl,t} , \]  

\[ \tilde{R} = r + 2l^2 g_{ij} g_{ij,t} - \frac{3l^2}{4} g_{ij,t} g_{ik,t} g_{kj,t} + \frac{l^2}{4} g_{kl,t} g_{kl,t} . \]  

(7)
Here $r^i_{jk}$, $r_{ij}$ and $r$ are the connection, the Ricci tensor and the scalar curvature given by $g_{ij}$.

For the metric (4), the non-trivial components of the connection are

\[
\Gamma^{\rho}_{\rho\rho} = -\frac{1}{\rho} - \frac{2\mu^2 \rho^2}{1 - \mu^2 \rho^2} \sim -\frac{1}{\rho} \left( 1 - \mu^2 \rho^2 + \mathcal{O} \left( \rho^4 \right) \right),
\]

\[
\Gamma^{\rho}_{tt} = -\frac{2}{l^4} \left( 1 - \mu^2 \rho^2 \right) \left( 1 + \mu^2 \rho^2 \right) \sim -\frac{2}{l^4} \left( 1 + \mathcal{O} \left( \rho^4 \right) \right),
\]

\[
\Gamma^{t}_{\rho\rho} = -\frac{1}{2\rho} - \frac{\mu^2 \rho^2}{1 - \mu^2 \rho^2} \sim -\frac{1}{2\rho} \left( 1 + 2\mu^2 \rho^2 + \mathcal{O} \left( \rho^4 \right) \right),
\]

\[
\Gamma^{\rho}_{ij} = \frac{2}{l^2} \left( 1 - \mu^2 \rho^2 \right) \left( g_{ij} - \rho g_{ij,\rho} \right), \quad \Gamma^{i}_{\rho j} = \Gamma^{i}_{\rho j} = -\frac{1}{2} \left( 1 - g^{ik} g_{kj,\rho} \right),
\]

\[
\Gamma^{t}_{ij} = \frac{l^2}{2} \left( 1 - \mu^2 \rho^2 \right) g_{ij,t}, \quad \Gamma^{i}_{jt} = \Gamma^{i}_{jt} = \frac{1}{2} g^{ik} g_{kj,t}, \quad \Gamma^{i}_{jk} = \gamma^{i}_{jk}. \quad (8)
\]

As we are interested in the holographic conformal anomaly, we calculate the curvatures up to relevant order in the power of $\rho$. Then the non-trivial components are given by

\[
R_{\rho\rho} = -\frac{1}{\rho^2} - \frac{\mu^2}{1 - \mu^2 \rho^2} - \frac{1}{2} g^{ik} g_{ik,\rho\rho} + \frac{1}{4} g^{ik} g^{jl} g_{ij,\rho} g_{kl,\rho} + \mathcal{O} \left( \rho \right),
\]

\[
R_{tt} = \frac{4}{l^4} - \frac{4\mu \rho}{l^2} - \frac{1}{l^2} g^{ik} g_{ki,\rho} + \tilde{R}_{tt} + \mathcal{O} \left( \rho^2 \right),
\]

\[
R_{ij} = -\frac{4}{l^2} g_{ij} + \frac{2}{l^2} g_{ij,\rho} + \frac{1}{l^2} g_{ij} g^{kl} g_{kl,\rho} - \frac{2\rho}{l^2} g_{ij,\rho\rho} + \frac{2\rho}{l^2} g_{ki,\rho} g_{kj,\rho} - \frac{\rho}{l^2} g_{ij,\rho} g^{kl} g_{kl,\rho} + \tilde{R}_{ij},
\]

\[
R = -\frac{20}{l^2} + \rho \tilde{R} + \rho^2 \left( -\frac{4}{l^2} g^{ij} g_{ij,\rho\rho} + \frac{3}{l^2} g^{ik} g^{jl} g_{ij,\rho} g_{kl,\rho} - \frac{1}{l^2} \left( g^{ij} g_{ij,\rho} \right)^2 \right) + \mathcal{O} \left( \rho^3 \right). \quad (9)
\]

When one calculates the holographic conformal anomaly, $\tilde{g}_{mn}$ or $g_{ij}$ are expanded as a power serie of $\rho$,

\[
g_{ij} = g^{(0)}_{ij} + \rho g^{(1)}_{ij} + \rho^2 g^{(2)}_{ij} + \cdots. \quad (10)
\]

By using the Einstein equation, $g^{(1)}_{ij}$, $g^{(2)}_{ij}$, $\cdots$ can be solved with respect to $g^{(0)}$. After that, substituting these expressions into the Einstein-Hilbert action, one can find the holographic anomaly from the coefficient of $\rho^{-1}$ term[2].
From Eq. (9), the μ-dependent term does not contribute to the Einstein equation in the relevant order. Since \( R \) does not contain μ-dependent term (they are cancelled with each other), the μ-dependent term does not appear in the expression for the conformal anomaly, what is consistent with the usual field theory calculation.

If we put \( \mu = 0 \), the metric (5) is invariant if we change \( \rho \) and \( \tilde{g}_{ij} \) by

\[
\delta \rho = \delta \sigma \rho, \quad \delta \tilde{g}_{ij} = \delta \sigma g_{ij} .
\] (11)

Here \( \delta \sigma \) is a constant parameter of the transformation. The transformation (11) can be regarded as the scale transformation. When one substitutes the expressions in (10) (after solving \( g_{ij}^{(1)} \) etc. with respect to \( g_{ij}^{(0)} \)) into the action, the action diverges in general since the action contains the infinite volume integration on the asymptotically AdS space. The action is regularized by introducing the infrared cutoff \( \epsilon \), which generates a boundary at finite \( \rho (= \epsilon) \)

\[
\int d^5x \to \int d^4x \int_\epsilon d\rho, \quad \int \text{Boundary} d^4x (\cdots) \to \int d^4x (\cdots) \bigg|_{\rho = \epsilon} .
\] (12)

The terms proportional to the (inverse) power of \( \epsilon \) in the regularized action are invariant under the scale transformation

\[
\delta g_{(0)\mu\nu} = 2 \delta \sigma g_{(0)\mu\nu}, \quad \delta \epsilon = 2 \delta \sigma \epsilon ,
\] (13)

which corresponds to (11). Then the subtraction of these terms proportional to the inverse power of \( \epsilon \) does not break the invariance under the scale transformation. When \( d \) is even, however, the term proportional to \( \ln \epsilon \) appears. This term is not invariant under the scale transformation (13) and the subtraction of the \( \ln \epsilon \) term breaks the invariance. The variation of the \( \ln \epsilon \) term under the scale transformation (13) is finite when \( \epsilon \to 0 \) and should be canceled by the variation of the finite term (which does not depend on \( \epsilon \)) in the action since the original action is invariant under the scale transformation. Therefore the \( \ln \epsilon \) term \( S_{\ln} \) gives the Weyl anomaly \( T \) of the action renormalized by the subtraction of the terms which diverge when \( \epsilon \to 0 \) by

\[
S_{\ln} = -\frac{1}{2} \int d^4x \sqrt{-g_{(0)}} T .
\] (14)

The explicit form of \( T \) is found to be (for explicit calculations, see [2])

\[
T = \frac{l^3}{8 \pi G} \left[ \frac{1}{8} R_{(0)ij} R_{(0)}^{ij} - \frac{1}{24} R_{(0)}^2 \right] .
\] (15)
Comparing with the field theory calculation, the conformal anomaly coming from the multiplets of $\mathcal{N} = 4$ supersymmetric $U(N)$ or $SU(N)$ Yang-Mills, we obtain
\[
\frac{l^3}{16\pi G} = \frac{2N^2}{(4\pi)^2}.
\]  
Hence, the calculation of holographic anomaly represents explicit check of AdS/CFT correspondence. Moreover, the previous calculations were limited to pure (or asymptotically) AdS spaces [2]. From above AdS BH calculation one arrives at the conclusion of bulk metric independence of holographic anomaly. This is supported also by bulk deSitter space case where dual calculation for anomaly gives precisely above result[3].

Moreover, one can also consider asymptotically deSitter space instead of asymptotically anti-deSitter space. If one replaces the length parameter $l^2$, the time coordinate $t$ and the radial coordinate $r$ by
\[
l^2 \to -l^2, \quad t \to r, \quad r \to t,
\]  
in Eq.(1), we obtain the asymptotically deSitter space as
\[
ds_{\text{dS-S}}^2 = -\left(\frac{t^2 + \mu}{l^2}\right)^{-1} dt^2 + \left(\frac{t^2}{l^2} + \frac{\mu}{r^2}\right) dr^2 + l^2 \sum_{i=1}^{3} (dx^i)^2.
\]  
This spacetime can be regarded as a cosmological one, which has a singularity at $t = 0$, which may be identified with the big-bang singularity. When $t \to \infty$, the spacetime approaches to the deSitter spacetime. If we put a brane at $t \to \infty$, there will exist a dual conformal field theory (dS/CFT correspondence [4, 5]). By repeating a similar calculation as the asymptotically anti-deSitter case after replacing the $l^2$ by $-l^2$, we can evaluate the conformal anomaly at $t \to \infty$ and obtain the expression identical with (15). This finishes our proof of independence of holographic conformal anomaly from the bulk space choice. However, it could be that in different dualities (say AdS or dS) the corresponding dual CFTs having same central charges could be essentially different.
3 Newton potential of the gravity induced on the brane in the bulk AdS black hole spacetime

In the present section our problem is to describe the calculation of Newton potential of the gravity localized on the 3-brane in the 5 dimensional Schwarzschild-AdS background. Metric is chosen in the warped form:

$$ds^2 = dy^2 + e^{2A(y)} \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu,$$

(19)

the action of the brane at \( y = y_0 \) is given by

$$S_b = \frac{2\kappa_0}{\kappa^2} \delta (y - y_0) \int d^4x \sqrt{-g}.$$

(20)

The metric (19) can be rewritten in the Schwarzschild-like form:

$$ds^2 = \sum_{m,n=0}^{4} G_{mn} dx^m dx^n = e^{-2\rho(r)} dr^2 - e^{2\rho} dt^2 + r^2 \sum_{i,j=1}^{3} \tilde{g}_{ij} dx^i dx^j.$$

(21)

Here

$$dy = e^{-\rho} dr.$$

(22)

Then since \( dy \delta (y - y_0) = dr \delta (r - r_0) \), the brane action (20) is rewritten as

$$S_b = \frac{2\kappa_0}{\kappa^2} \delta (r - r_0) e^{-\rho(r_0)} \int d^4x \sqrt{-g}.$$

(23)

As a solution of the vacuum Einstein equation, we consider the Schwarzschild AdS metric (21)

$$e^{2\rho} = \frac{\mu^{2/3}}{r_0} - \frac{\mu}{r^2}, \quad \sum_{i,j=1}^{3} \tilde{g}_{ij} dx^i dx^j = \sum_{i,j=1}^{3} (dx^i)^2.$$

(24)

Here we assume that the shape of the event horizon is flat. If we orbifoldize the spacetime by identifying \( r - r_0 = - (r - r_0) \), the dynamics of the brane is described by the FRW like equation:

$$H^2 = \frac{1}{l^2} + \frac{\kappa_0^2}{4} + \frac{\mu}{a^4}.$$

(25)
Here $l^2$ is the length parameter, which is related with the bulk cosmological constant $\Lambda$ by $\Lambda = -\frac{l^2}{4}$. We also assume that there is a brane at $r = a(\tau)$ ($\tau$ is the proper time on the brane) and the Hubble parameter $H$ is defined by $H \equiv \frac{1}{a} \frac{da}{d\tau}$. Then if the brane is static ($H = 0$), we obtain

$$a^4 = r_0^4 = \frac{\mu}{l^2 - \frac{\kappa_0}{4}}. \quad (26)$$

On the other hand, with the metric assumption (21), the $(i,j)$-component of the bulk Einstein equation gives

$$-\frac{2}{r^2} (1 + \rho r') e^{2\rho} - \frac{1}{2} e^{2\rho} \left\{ -2\rho'' - 4 (\rho')^2 - \frac{12}{r} \rho' - \frac{6}{r^2} \right\} + \frac{\Lambda}{2} = \kappa_0 e^{-\rho(r_0)} \delta (r - r_0) \quad (27)$$

and

$$\left( e^{2\rho} \right)'' \bigg|_{r=r_0} = 2\kappa_0 e^{-\rho(r_0)} \delta (r - r_0). \quad (28)$$

The above equation (28) is rewritten as

$$\kappa_0 = -\frac{2}{l} \left( 1 + \frac{\mu l^2}{r_0^2} \right) \left( 1 - \frac{\mu l^2}{r_0^2} \right)^{\frac{1}{2}} \quad (29)$$

By combining (26) and (29), one obtains

$$\alpha^3 + \alpha^2 - 3\alpha = 0, \quad \alpha \equiv \frac{\mu l^2}{r_0^4}. \quad (30)$$

Then the brane can exist at

$$\alpha = \frac{\mu l^2}{r_0^4} = -1 + \sqrt{13} \quad (31)$$

We should note, however, the brane is not stable. In fact, if we rewrite the FRW equation (25) as

$$\left( \frac{da}{d\tau} \right)^2 = -V(a), \quad V(a) = -\left( \frac{\kappa_0^2}{4} - \frac{1}{l^2} \right) a^2 - \frac{\mu}{a^2}, \quad (32)$$
we find the effective potential has no minimum since assumption $\frac{\kappa^2}{l^2} - \frac{1}{l^2} < 0$ in order that static brane exists, the effective potential $V(a)$ is monotonically increasing function. Then the brane is attracted by the black hole and falls into the black hole.

In the following we assume, for simplicity, that the brane is adiabatically static by considering the behavior around the case of (26), where the brane can be static for a short time interval, or by introducing the another force like electromagnetic interaction, which might stabilize the brane.

The Einstein equation is given by

$$ R_{mn} - \frac{1}{2} G_{mn} R + \frac{\kappa^2 \Lambda}{2} G_{mn} = -2 \left( \frac{\mu}{r_0^3} + \frac{r_0}{l^2} \right) \delta(r - r_0) G_{mn}, \quad (33) $$

Then if we consider the perturbation of the metric $G_{mn} \rightarrow G_{mn} + \delta G_{mn}$, the Einstein equation (33) gives

$$ 0 = \frac{1}{2} \left\{ \nabla^l \nabla_m \delta G_{nl} + \nabla^l \nabla_n \delta G_{ml} - \nabla^2 \delta G_{mn} ight. 
\left. - \frac{1}{2} \left( \nabla_m \nabla_n + \nabla_n \nabla_m \right) \left( G^{kl} \delta G_{kl} \right) \right\} 
\left. - \frac{1}{2} \delta G_{mn} \left( R - \kappa^2 \Lambda \right) + 4 \kappa^2 \left( \frac{\mu}{r_0^3} + \frac{r_0}{l^2} \right) \delta G_{mn} \delta(r - r_0) \right. 
\left. - \frac{1}{2} G_{mn} \left( - \delta G_{kl} R^{kl} - \nabla^k \nabla^l \delta G_{kl} - \nabla^2 \left( G^{kl} \delta G_{kl} \right) \right) \right\}. \quad (34) $$

Here the curvature $R$ and $R_{mn}$, the covariant derivative $\nabla_m$ are defined by the unperturbative part $G_{mn}$ of the metric. We now choose the following gauge conditions

$$ \nabla^m \delta G_{mn} = \delta G_{rm} = \delta G_{mr} = \delta G_{tm} = \delta G_{mt} = 0, \quad (35) $$

and we write $\delta G_{ij} = h_{ij}$. Then by using the solution (24) of the bulk Einstein equation, the $(m, n) = (i, j)$ components of Eq.(34) are given by

$$ 0 = -\frac{1}{2} \left[ \frac{1}{r^2} \triangle h_{ij} - \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right) \frac{1}{r^2} \nabla^2 h_{ij} + \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right) \left\{ \partial_r^2 h_{ij} - \frac{1}{r} \partial_r h_{ij} 
\right. 
\left. + 2 \left( \frac{r}{l^2} + \frac{\mu}{r^3} \right) \left( \partial_r h_{ij} - \frac{2}{r} h_{ij} \right) \right\} \right] 
\left. - 2 \kappa^2 \left( \frac{\mu}{r_0^3} + \frac{r_0}{l^2} \right) \delta(r - r_0) h_{ij} \right), $$

$$ \triangle \equiv \sum_{k=1}^{3} (\partial_k)^2. \quad (36) $$
Taking the plane wave on the brane as
\[ h_{ij} = h_{ij}^{(0)} e^{i \sum_{l=1}^{3} k_{l} x^{l} - i \omega t} \left( \frac{r}{l} - \frac{\mu}{r^3} \right)^{-\frac{3}{2}} \phi(r) , \] (37)

one gets
\[
\begin{align*}
0 &= -\frac{1}{2} \left[ \partial_{r}^2 + \frac{1}{r^2} \left\{ -\frac{15}{4} + \frac{4\mu^2}{r^4} \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right)^{-2} \right\} \\
&\quad - \left( \frac{r_0^2}{l^2} - \frac{\mu}{r_0^2} \right)^{-1} \left( \frac{3r_0}{l^2} + \frac{\mu}{r_0^2} \right) \delta (r - r_0) \\
&\quad - \left\{ \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right)^{-1} \frac{k^2}{r^2} - \left( \frac{r^2}{l^2} - \frac{\mu}{r^2} \right)^{-2} \omega^2 \right\} \right] \phi .
\end{align*}
\] (38)

When \( \mu = 0 \), the above equation reduces to that in the Randall-Sundrum model[6]:
\[
\begin{align*}
0 &= -\frac{1}{2} \left[ \partial_{y}^2 - \frac{15}{4r^2} - \frac{3}{r_0} \delta (r - r_0) - \frac{l^4}{r_0^4} \left( \frac{k^2}{l^2} - \omega^2 \right) \right] \tilde{\phi} .
\end{align*}
\] (39)

In fact, by identifying \( r = r_0 e^{-\frac{|y|}{l}} \) and redefining \( \phi \) as \( \phi = e^{-\frac{|y|}{l}} \tilde{\phi} \), Eq.(39) can be rewritten as
\[
\begin{align*}
0 &= -\frac{1}{2} \left[ \partial_{y}^2 - \frac{4}{l^2} - \frac{2}{l} \delta (y) + m^2 e^{\frac{2|y|}{l}} \right] \tilde{\phi} , \quad m^2 \equiv -\frac{l^2}{r_0^2} \left( \frac{k^2}{l^2} - \omega^2 \right) ,
\end{align*}
\] (40)

which is identical with the corresponding equation (8) in [6].

For small \( \mu \), the graviton will be localized on the brane. We should note, however, the condition that the graviton is massless on the brane is given by
\[
-m^2 \equiv \frac{k^2}{r_0^2} - \left( \frac{r_0^2}{l^2} - \frac{\mu}{r_0^2} \right)^{-1} \omega^2 = 0 .
\] (41)

Then not as in the case of the Randall-Sundrum model, the equation (38) does not reduce to the eigenvalue equation.

The radius \( r_H \) of the event horizon is given by
\[
r_H = \mu^\frac{3}{2} l^\frac{3}{2} .
\] (42)
We now consider the large black hole $\mu \to \infty$ or $r_H \to \infty$ by fixing $r_0 - r_H$.

Then by defining a new coordinate $\xi$ as

$$r = r_H + \xi ,$$

Eq.(38) reduces into

$$0 = \partial^2_\xi \phi - \frac{15}{4r^2_H} - \frac{l^2 k^2}{4r^3_H \xi} + \left( \frac{1}{4} + \frac{l^4 \omega^2}{16r^2_H} \right) \frac{1}{\xi^2} \phi - \frac{1}{\xi_0} \delta (\xi - \xi_0) \phi . $$

Here we assume that there is a brane at $\xi = \xi_0 \equiv r_0 - r_H$. When $k^2 = 0$, we can solve Eq.(44) by

$$\phi = \left( \frac{\xi}{l} \right)^{\frac{1}{2}} \left( \alpha I_{\nu} \left( \frac{\sqrt{15}}{2r_H} \xi \right) + \alpha^* I_{-\nu} \left( \frac{\sqrt{15}}{2r_H} \xi \right) \right) .$$

Here $\alpha$ is a complex constant and $I_\nu(z)$ is a deformed Bessel function. Since we are now considering the large black hole, we may approximate the modified Bessel functions by

$$I_\nu(z) \sim \frac{1}{\Gamma(1 + \nu)} \left( \frac{z}{2} \right) ^\nu + \cdots .$$

Then the phase of $\alpha$ can be determined by requiring that the solution satisfies the delta function in (44), that is, by the equation;

$$-2 \frac{\partial_\xi \phi}{\phi} \bigg|_{\xi = \xi_0} = \frac{1}{\xi_0} ,$$

which leads

$$\frac{-\alpha^*}{\alpha} = \left( \frac{\sqrt{15}}{2r_H} \xi_0 \right)^{\frac{1}{2}} \Gamma \left( 1 - \nu \frac{l^2 \omega}{4r_H} \right) \frac{1 + i \frac{l^2 \omega}{4r_H}}{\Gamma \left( 1 + \nu \frac{l^2 \omega}{4r_H} \right) 1 - i \frac{l^2 \omega}{4r_H}} .$$

There are some ambiguities how one should treat the boundary condition of $\phi$ at the horizon. In order to avoid this problem, we put one more brane with tension $-\frac{1}{8}$ at $\xi = \xi_1 < \xi_0$ as in the first Randall-Sundrum model. Then in addition to Eq.(48), we obtain another condition;

$$\frac{-\alpha^*}{\alpha} = \left( \frac{\sqrt{15}}{2r_H} \xi_1 \right)^{\frac{1}{2}} \Gamma \left( 1 - \nu \frac{l^2 \omega}{4r_H} \right) \frac{1 + i \frac{l^2 \omega}{4r_H}}{\Gamma \left( 1 + \nu \frac{l^2 \omega}{4r_H} \right) 1 - i \frac{l^2 \omega}{4r_H}} .$$
Combining (48) and (49) leads to

\[ 1 = \left( \frac{\xi_0}{\xi_1} \right)^{2i\frac{\omega}{4r_H}} e^{2i\frac{\omega}{4r_H} \ln \left( \frac{\xi_0}{\xi_1} \right)}. \]  

That is

\[ \frac{\omega}{4r_H} \ln \left( \frac{\xi_0}{\xi_1} \right) = \pi n, \quad n = 0, 1, 2, 3, \cdots. \]  

Then by using (41), we find

\[ m^2 = \frac{4r_H}{\xi_0 l^2} \left( \frac{\pi n}{\ln \left( \frac{\xi_0}{\xi_1} \right)} \right)^2. \]  

Then for finite \( \xi_1 \), the Kaluza-Klein modes, which correspond to \( n = 1, 2, 3, \cdots \), become very heavy since we are now considering the big black hole, where \( r_H \) is large. Then the Kaluza-Klein modes decouple on the brane and only massless mode, corresponding to \( m^2 = 0 \) or \( n = 0 \), will contribute to the Newton potential. Therefore if \( r \) is the distance between two particles on the brane, the Newton potential behaves as \( \frac{1}{r} \) and the gravity would localize on the brane for the two brane model. We should note that such a decoupling of the Kaluza-Klein modes makes the Newton potential on both branes corresponding to \( \xi_0 \) and \( \xi_1 \) to behave as \( \frac{1}{r} \). If the inner brane, which exists at \( \xi = \xi_1 \), approaches to the horizon \( \xi_1 \rightarrow 0 \), however, the logarithmic term in (50) becomes dominant and the Kaluza-Klein modes become light. Then the Newton potential behaves as \( \frac{1}{r^2} \), as in the five dimensional one. This may indicate that, for one brane model, the gravity would not localize on the brane.

Thus, we demonstrated that at sufficiently reasonable conditions the RS2 (two branes) model realized in bulk AdS BH shows the properties similar to the properties of RS2 model in the pure AdS bulk [6]. Specifically, gravity trapping on the brane occurs. Note that gravity trapping on the brane occurs also when bulk is 5d de Sitter space [7]. In this sense one can say again about (bulk metric) independence of the brane gravity trapping from the choice of bulk spacetime. Moreover, the fact that bulk space is AdS BH is not essential too. In other words, again the localization is universal feature of holographic duality (no horizon dependence).
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References


