Stefano Bellucci\textsuperscript{1} and Armen Nersessian\textsuperscript{2,3}

\textsuperscript{1} INFN, Laboratori Nazionali di Frascati, P.O. Box 13, I-00044, Italy
\textsuperscript{2} Yerevan State University, Alex Manoogian St., 1, Yerevan, 375025, Armenia
\textsuperscript{3} Yerevan Physics Institute, Alikhanian Brothers St., 2, Yerevan, 375036, Armenia

Abstract

We define the “maximally integrable” isotropic oscillator on $\mathbb{CP}^N$ and discuss its various properties, in particular, the behaviour of the system with respect to constant magnetic field. We show that the properties of the oscillator on $\mathbb{CP}^N$ qualitatively differ in the $N > 1$ and $N = 1$ cases. In the former case we construct the “axially symmetric” system which is locally equivalent to the oscillator. We perform the Kustaanheimo-Stiefel transformation of the oscillator on $\mathbb{CP}^2$ and construct some generalized MIC-Kepler problem. We also define a $N = 2$ superextension of the oscillator on $\mathbb{CP}^N$ and show, that for $N > 1$ the inclusion of a constant magnetic field preserves the supersymmetry of the system.

1 Introduction

The harmonic oscillator plays the distinguished role in theoretical and mathematical physics, due to it overcomplete symmetry group. The wide number of hidden symmetries provide the oscillator by unique properties, e.g. closed classical trajectories and the degeneracy of the quantum-mechanical energy spectrum, the separability of variables in a few coordinate systems. Overcomplete symmetry allows to preserve the exact solvability of the oscillator even after some deformation of the potential breaking the initial symmetry of the system. Particularly, the oscillator remains exactly solvable after coupling to a constant magnetic field, though the latter removes the hidden symmetries of the system. The reduction of the oscillator to low dimensions allows to construct new integrable systems with hidden symmetries (in fact, almost all integrable systems of classical and quantum mechanics are related either with free particle, or with the oscillator) \cite{1}. There is nontrivial relation between oscillator and Coulomb systems: the $(N + 1)$–dimensional Coulomb problem can be obtained from the $2N$–dimensional oscillator by the so-called Levi-Civita (or Bohlin), Kustaanheimo-Stiefel and Hurwitz transformations, when $N = 1, 2, 4$ \cite{2}. These transformations corresponds to the reduction of the oscillator by the actions of $Z_2$, $U(1)$ and $SU(2)$ groups, respectively, and are based on the Hopf maps $S^1/Z_2 = S^1$, $S^3/U(1) = \mathbb{CP}^1 \cong S^2$, $S^7/SU(2) = \text{IHP}^1 \cong S^4$ (relating the the angular parts of the oscillator and Coulomb problems). Indeed, reducing the oscillators, we get some parametric families of Coulomb-like systems specified by the presence of a magnetic flux for $N = 1$; by a Dirac monopole for $N = 2$ (MIC-Kepler system); and by a Yang monopole for $N = 4$ (see, respectively, \cite{3, 4, 5}). Note, that MIC-Kepler system, initially introduced by Zwanziger for the description of a relative motion of two Dirac dyons, also describes the scattering of two well-separated BPS monopoles and dyons. The latter problem was considered in the well-known paper by Gibbons and Manton \cite{6}, where the existence of the hidden Coulomb-like symmetry has also been established. Let us mention also the key role of Hurwitz transformation (and of the second Hopf map) in the recently proposed higher-dimensional quantum Hall effect \cite{7} (see also \cite{8, 9}).

The oscillator is a distinguished system also with respect to supersymmetrization. Supersymmetric oscillator is specified by the splitting of fermionic and bosonic degrees of freedom, thus, inherits the hidden symmetries of initial system. The construction of integrable supersymmetric mechanics is interesting not only in field-theoretical context. Being in deep connection with the factorization problem, supersymmetrization of integrable system could provide us by a new set of integrable systems with isospectral potentials. Since the list of references on supersymmetric mechanics is enormous, we refer to the introductory reviews \cite{10} (mostly devoted to the connection of supersymmetric quantum mechanics with factorization problem) and \cite{11} (containing the most complete list of references on field-theoretical aspects of supersymmetric mechanics).

Recent progress in string theory inspired interest for noncommutative field theories \cite{12}, and, in particular, for noncommutative quantum mechanics \cite{13}. The oscillator was found to be distinguished in noncommutative quantum mechanics too: at the moment it is the only exactly solved (even in the presence of constant magnetic field) noncommutative quantum mechanical system with non-zero potential \cite{14}.
The oscillator on the other spaces of that sort. Kähler manifold. Hence, our model could be easily adopted for the formulation of the oscillator system on the complex projective space \( \mathbb{CP}^N \), from the requirement that it has to possess the hidden symmetries generalizing the hidden symmetries of the planar oscillator, and consider its behaviour with respect to the coupling of constant magnetic field.

The oscillator on \( \mathbb{CP}^1 = S^2 \) it coincides with the Higgs oscillator on the sphere \( S^2 \) (note, that \( \mathbb{CP}^1 = S^2 \)). The oscillator on \( \mathbb{CP}^N, N > 1 \) is defined by the potential

\[
U(z\bar{z}) = \omega^2 r_0^2 z\bar{z},
\]

where the \( z^a, \bar{z}^a \) are inhomogeneous coordinates of \( \mathbb{CP}^N \), corresponding to Fubini-Study metric

\[
g_{ab} dz^a d\bar{z}^b = r_0^2 \frac{dz d\bar{z}}{1 + z\bar{z}} - r_0^2 \frac{z dz \bar{z} d\bar{z}}{(1 + z\bar{z})^2}.
\]

In contrast to the case of the oscillator on \( \mathbb{CP}^1 = S^2 \) which is defined on the disk \(|z| < 1\), the oscillator on \( \mathbb{CP}^N, N > 1 \) is defined on the whole chart. Transition to another chart of \( \mathbb{CP}^N \) transforms the oscillator in the system with potential

\[
U = \omega^2 r_0^2 \left( \frac{1}{z\bar{z}} + \frac{z^2\bar{z}^2 + \ldots + z^N\bar{z}^N}{z^1\bar{z}^1} \right),
\]

which has the oscillator symmetry algebra.

The Kustaanheimo-Stiefel transformation of the oscillator on \( \mathbb{CP}^2 \) yields a generalization of MIC-Kepler system, which could be transformed into the MIC-Kepler system on three-dimensional hyperboloid.

The oscillator on \( \mathbb{CP}^N \) admits, because of its Kähler structure, the simple coupling of the constant magnetic field. It could be done by the following replacement of symplectic structure \( \Omega_0 \rightarrow \Omega_0 + iBg_{ab} dz^a \wedge d\bar{z}^b \). The constant magnetic field preserves the kinematical \( su(N) \) symmetries of the oscillator (for the free particle case, \( \omega = 0 \), it preserves the whole symmetry algebra \( su(N+1) \)), but breaks the hidden symmetries.

We construct the \( \mathcal{N} = 2 \) supersymmetric oscillator on \( \mathbb{CP}^N \) behaviour to the coupling of constant magnetic field (the oscillator on \( \mathbb{CP}^N \), in contrast with the one on \( \mathbb{CP}^N \), does not admit \( \mathcal{N} = 4 \) supersymmetrization).

We show, that in contrast with the \( \mathcal{N} = 2 \) supersoscillator on \( \mathbb{CP}^1 = S^2 \), the \( \mathcal{N} = 2 \) supersoscillator on \( \mathbb{CP}^N, N > 1 \) allows coupling to a constant magnetic field without breaking of supersymmetry.

## 2 Oscillator on \( \mathbb{CP}^N \)

This section is devoted to the construction of the oscillator system on the complex projective space \( \mathbb{CP}^N \). Our consideration essentially exploits the fact that the complex projective space is the constant curvature Kähler manifold. Hence, our model could be easily adopted for the formulation of the oscillator system on the other spaces of that sort.

\[1\] Let us remind, that the Coulomb system on the (pseudo)sphere is defined by the potential \[U_C = -\frac{\gamma}{r_0 |x|} \]

Quantum mechanics of the oscillator and Coulomb system on \( D \)-dimensional sphere and pseudosphere is considered in detail in Ref. [18].
the dynamics of a free particle in Kähler space.

Hence, the inclusion of a constant magnetic field preserves the whole symmetry algebra of a free particle moving in Kähler space.

Let us equip the cotangent bundle $T\mathbb{C}P^N$ with the symplectic structure

$$\Omega_B = dz^a \wedge d\bar{z}^a + d\bar{z}^a \wedge d\bar{z}^a + B g_{ab}dz^a \wedge d\bar{z}^b,$$

which defines, with the Hamiltonian

$$D = g^{ab} \pi_a \bar{\pi}_b$$

the dynamics of a free particle on $\mathbb{C}P^N$ in the presence of a constant magnetic field $B$. The isometries of a Kähler structure define the Noether’s constants of motion of a free particle

$$J_{\mu} = \{ D, J_\nu \} = 0,$$

where

$$J_{ab} = -i z^b \pi_a + i \bar{\pi}_b \bar{z}^a, \quad i J_{a}^+ = \pi_a + \bar{z}^a (\bar{z} \bar{\pi}), \quad -i J_{a}^- = \bar{\pi}_a + z^a (z \pi).$$

Notice that the vector fields generated by $J_{\mu}$ are independent on $B$:

$$\bar{V} = V^a(z) \frac{\partial}{\partial z^a} - V^a_{\bar{b}} \pi_a \frac{\partial}{\partial \pi_a} + \bar{V}^a(\bar{z}) \frac{\partial}{\partial \bar{z}^a} - \bar{V}^a_{\bar{b}} \bar{\pi}_a \frac{\partial}{\partial \bar{\pi}_a}.$$
It is convenient to introduce the generators

\[ \mathcal{H} = g^{ab} \pi_a \pi_b + U(z \bar{z}) , \]  

and require it to have the hidden symmetry (similar to the one of the oscillator) given by the either one of the constants of motion

\[ \begin{align*}
\text{i) } & I_+^a = J_+^a J_0^+ + f_+(z \bar{z}) z^a \bar{z}^b , \\
\text{ii) } & I_{ab} = J_+^a J_0^b + f_0(z \bar{z}) z^a \bar{z}^b .
\end{align*} \]  

Straightforward calculations immediately yield the following constraints:

\[ \begin{align*}
\text{i) } & B = 0 \quad N = 1 \quad U(x) = c_1 x/(1 - x)^2 + c_0 \quad f_+ = c_1/(1 - x)^2 , \\
\text{ii) } & B = 0 \quad N = 1, 2 \ldots \quad U(x) = c_1 x + c_0 \quad f_0 = c_1 .
\end{align*} \]  

Taking into account that \( H = \text{Tr } \hat{I} + \text{Tr } J^2 / 2r_0^2 \), we get the following generalizations of the oscillator on \( \mathbb{CP}^N \).

- \( \mathbb{CP}^1 \). The oscillator is defined by the Hamiltonian system

\[ \mathcal{H} = \frac{(1 + z \bar{z})^2 \pi^2}{r_0^2} + \frac{\omega^2 r_0^2 z \bar{z}}{(1 - z \bar{z})^2} , \quad \Omega_0 = dz \wedge d\pi + d\bar{z} \wedge d\bar{\pi} . \]  

The symmetry algebra is given by the \( U(1) \) generator \( J \) and the complex (or vectorial) constant of motion \( I^{\pm} \)

\[ J = i(\pi z - \bar{\pi} \bar{z}) , \quad I^{\pm} = \frac{J_+^a J_0^b}{r_0^2} \omega^2 r_0^2 z \bar{z}^2 : \{ J, I^{\pm} \} = \pm 2i I^{\pm} , \quad \{ I_-, I_+ \} = 4i \left( \omega^2 J + \frac{\mathcal{H}}{r_0^2} - \frac{J^3}{2r_0^4} \right) . \]  

This is nothing but the well-known Higgs oscillator on the sphere \( S^2 = \mathbb{CP}^1 \) [15].

- \( \mathbb{CP}^N, N > 1 \). The oscillator is defined by the Hamiltonian system

\[ \mathcal{H} = g^{ab} \pi_a \pi_b + \omega^2 r_0^2 z \bar{z} , \quad \Omega_0 = dz \wedge d\pi + d\bar{z} \wedge d\bar{\pi} . \]  

Its symmetries are given by the constants of motion

\[ J_{\bar{a}} = i(z^b \pi_a - \bar{\pi}_b \bar{z}^a) , \quad I_{\bar{a}b} = \frac{J_+^{\bar{a}} J_{-b}}{r_0^2} + \omega^2 r_0^2 z^a \bar{z}^b , \]  

which define the nonlinear (quadratic) algebra

\[ \{ J_{\bar{a}b}, J_{\bar{c}d} \} = i \delta_{\bar{a}d} J_{\bar{c}b} - i \delta_{\bar{c}b} J_{\bar{a}d} , \quad \{ I_{\bar{a}b}, J_{\bar{c}d} \} = i \delta_{\bar{c}} \bar{b} I_{\bar{a}d} - i \delta_{\bar{d}} \bar{a} I_{\bar{c}b} , \]  

\[ \{ I_{\bar{a}b}, I_{\bar{c}d} \} = i \omega^2 \delta_{\bar{b}d} J_{\bar{a}c} - i \omega^2 \delta_{\bar{d}c} J_{\bar{a}b} + i I_{\bar{c}b}(J_{\bar{a}d} - J_0 \delta_{\bar{a}d})/r_0^2 - i I_{\bar{d}a}(J_{\bar{c}b} - J_0 \delta_{\bar{c}b})/r_0^2 . \]  

It is convenient to introduce the generators

\[ J_i = T_i^{ab} J_{\bar{a}b} , \quad J_0 = \text{Tr } \hat{J} , \quad I_i = T_i^{ab} I_{\bar{a}b} , \quad I_0 = \text{Tr } \hat{I} , \]  

where \( T_i \) are traceless \( N \times N \) Hermitian matrices (the generators of the \( su(N) \) algebra). The below generators belong to the center of algebra reads

\[ J_0 = i(\pi z - \bar{\pi} \bar{z}) , \quad \mathcal{H}_{N>1} = I_0 + \frac{\text{Tr } J^2 + J_0^2}{2r_0^2} . \]  

Also the following equality holds

\[ \text{Tr } \hat{I}^2 + \alpha^2 \text{Tr } J^2 = I_0^2 + \alpha^2 J_0^2 . \]  

We have got the “maximally integrable” generalization of the oscillator on complex projective spaces, i.e. the system with maximally possible number of functionally independent constants of motion \(^{3}\). We established the following essential properties of this system.

\(^{3}\)In the theory of integrable systems such systems are called “maximally superintegrable systems”. We prefer to suppress the prefix “super” in this context, in order to avoid any confusion with supersymmetric systems.
0 < |z| < ∞. The oscillator on the CP^1 ∼ S^2 (as well as on higher-dimensional spheres) is defined on the disc |z| < 1 only. The constant magnetic field removes the hidden symmetry of the oscillator on CP^N for any N, while it respects them in the case of free particle, i.e. when ω = 0.

- The above construction could be easily extended for noncompact version of CP^N, provided by the Lobachevski space LN = SU(1.N)/U(1) × SU(N). For this purpose we should replace the Fubini-Study metric by the one generated by the Kähler potential K = −r_0^2 log(1 − z\bar{z}) and subsequently replace the Killing potentials and Nether constants of CP^N by the ones of LN. The Killing potentials of LN are defined by the functions

$$h_{ab} = −r_0^2 \frac{z^a z^b}{1−z\bar{z}}, \quad h_a = −r_0^2 \frac{z^a}{1−z\bar{z}}, \quad h_+ = −r_0^2 \frac{z^a}{1−z\bar{z}}.$$  

Globally, the complex projective space CP^N is covered by N + 1 charts, marked by the indices \( \hat{a} = 0, a \). The transition functions from the \( \hat{b} \)-th chart to the \( \hat{c} \)-th one are of the form

$$z^\hat{a} \rightarrow z^\hat{a}(c) = \frac{\hat{z}^\hat{a}(b)}{\hat{z}^c(\hat{b})}, \quad \text{where} \quad z^\hat{a}(\hat{a}) = 1 .$$  

On CP^1 the transition functions take a simple form \( z \rightarrow 1/z \), corresponding to the transition from one hemisphere to the other. The respective transformation of the momenta is \( π \rightarrow −z^2 π \). The Hamiltonian of oscillator on CP^1 is obviously invariant under the above transformation. In higher-dimensions we get a rather different picture, since the potential term is not covariant under transition (2.26). Let us consider this transformation in more details.

The transition functions (2.26) defines the following canonical transformation, which is singular on the \( z^1 = 0 \) “axes”:

$$z^1 = 1/z^1, \quad π_1 = −z^1(zπ), \quad z^\hat{a} \rightarrow z^\hat{a}/z^1, \quad π_\hat{a} = z^1 π_\hat{a}, \quad \hat{a} = 2, \ldots N .$$  

The kinetic term is covariant with respect to the above transformation, while the potential term is not covariant. As a result, we get the integrable system on CP^N, \( N > 1 \) defined by the Hamiltonian

$$H_{\text{Back}} = g^{ab} π_\hat{a} \bar{π}_b + \omega^2 r_0^2 \left( \frac{1}{z^1\bar{z}^1} + \frac{z^2\bar{z}^2 + \ldots + z^N\bar{z}^N}{z^1\bar{z}^1} \right).$$  

This system inherit the whole symmetry algebra of the oscillator, i.e. it is a “maximally integrable” system. Its constants of motion can be obtained by a straightforward transformation of the oscillators constants of motion (2.20). Note, that in spite of its “maximal integrability”, the system is not invariant under “spatial” u(N) rotations.

On Lobachevski space LN, \( N > 1 \) there is no analog of this system. The “ambient” space for the Lobachevski plane is CP^{1,N}, the transitions (2.26) transform the oscillator on LN into the system on the space with the signature (−, −, +, +, +, ...).

**CP^2: Kustaanheimo-Stiefel transformation**

As we have mentioned in the Introduction, the oscillator on two-, four-, and eight- dimensional planes and spheres could be reduced to the two- , three- and five- dimensional Coulomb systems, and their generalizations specified by the presence of monopoles. Particularly, the oscillator on S^2 = CP^1 and AdS_2 = L could be reduced, by the so-called Levi-Civita transformation, to the Coulomb systems on two-dimensional hyperboloid (Lobachevski plane) L. Similarly, Kustaanheimo-Stiefel transformation of the oscillator on a four-dimensional sphere and a four-dimensional two-sheet hyperboloid leads to the generalization of the MIC-Kepler problem on three-dimensional two-sheet hyperboloid [16].

Let us consider the behavior of the oscillator on CP^2 with respect to the Kustaanheimo-Stiefel transformation. The constants of motion of the oscillator on CP^2 are given by the generators

$$I = \frac{J_+ \sigma J_-}{r_0^2} + \omega^2 r_0^2 z \sigma \bar{z}, \quad J = iz\sigma \pi - i\bar{π} \sigma \bar{z}, \quad J_0 = iz\pi - i\bar{π} \bar{z} .$$  

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Their algebra reads
\[
\{J_0, J_k\} = \{J_0, J_0\} = 0, \quad \{J_k, J_l\} = 2\epsilon_{klm} J_m, \quad \{I_k, I_l\} = 2\epsilon_{klm} I_m, \quad (2.30)
\]

In order to reduce this system by the Hamiltonian action of $J_0$ we have to fix its value
\[
J_0 = 2s, \quad (2.31)
\]

and then factorize the level surface by the $U(1)$ group action. The resulting six-dimensional phase space $T^*M_{\text{red}}$ could be parameterized by the following $U(1)$-invariant functions:
\[
x = z\sigma\tilde{z}, \quad p = \frac{z\sigma\pi + \tilde{z}\sigma\tilde{\pi}}{2z\tilde{z}}: \quad \{x, J_0\} = \{p, J_0\} = 0. \quad (2.32)
\]

In these coordinates the reduced symplectic structure and the generators of the angular momentum are given by the expressions
\[
\Omega_{\text{red}} = dp \wedge dx + s \frac{x \times dx \times dx}{|x|^3}, \quad J_{\text{red}} = J/2 = p \times x + s \frac{x}{|x|}. \quad (2.33)
\]

So, the reduced system is specified by the presence of Dirac monopole.

The reduced Hamiltonian is given by the expression
\[
H_{\text{red}} = \frac{(1 + x)}{r_0^2} \left[ xp^2 + (xp)^2 \right] + s^2 \frac{(1 + x)^2}{r_0^2 x} + \omega^2 r_0^2 x, \quad \text{where} \quad x \equiv |x|. \quad (2.34)
\]

Let us fix the the constant energy surface
\[
H = E_{\text{osc}}, \quad (2.35)
\]

Then, dividing by $2r_0^2 x$, we can represent it in the form
\[
H_{\text{MIC}} = \mathcal{E}, \quad \mathcal{H}_{\text{MIC}} = \frac{(1 + x)}{2r_0^2} \left[ p^2 + \frac{(xp)^2}{x} \right] + s^2 \frac{2}{2r_0^2 x} - \frac{\gamma}{r_0^2 x} \quad (2.36)
\]

where we introduced the notation
\[
\gamma = E_{\text{osc}} / 2 - s^2 / r_0^2, \quad -2\mathcal{E} = \omega^2 + s^2 / r_0^4. \quad (2.37)
\]

The Hamiltonian $H_{\text{MIC}}$ can be interpreted as the Hamiltonian of some generalized MIC-Kepler problem. Notice, that its potential energy term has the same form, as the one of the conventional (flat) MIC-Kepler problem. The hidden symmetries of the system are given by the reduced generators $I_k$.

Let us perform the canonical transformation $(x, p) \rightarrow (\tilde{x}, \tilde{p})$, going to the coordinates where the metric takes a conformally-flat form:
\[
\tilde{x} = f(x)x, \quad p = f\tilde{p} + f \frac{\tilde{p}}{x} - x, \quad (2.38)
\]

where
\[
f(x) = \frac{1}{\sqrt{1 + x} - 1}, \quad (2.39)
\]

In that case the reduced Hamiltonian reads
\[
H_{\text{red}} = \frac{x(1 + x)^2}{4r_0^2} p^2 + s^2 \frac{(x + 1)^4}{4r_0^2 x(1 - x)^2} + \frac{4\omega^2 r_0^2 x}{(1 - x)^2}, \quad x < 1, \quad (2.40)
\]

while the Hamiltonian of the above obtained generalization of MIC-Kepler problem (2.36) takes the form
\[
H_{\text{MIC}} = \frac{(1 - x^2)^2}{32r_0^4} \left( p^2 + \frac{s^2}{x^2} \right) - (\gamma + \frac{4}{2r_0^2}) \frac{1 + x^2}{4r_0^2 x} - \frac{s^2}{4r_0^4}. \quad (2.41)
\]
Performing the Kustaanheimo-Stiefel transformation of the system (2.28) on $\mathbb{CP}^2$, we get the following expression for the reduced Hamiltonian:

$$H_{\text{Back}} = \frac{(1 + x)}{r_0^2} [xp^2 + (xp)^2] + s^2 \frac{1 + x}{r_0^2 x} + 2\omega^2 r_0^2 \frac{1 + x}{x + x_3} - \omega^2 r_0^2, \quad x_3 \neq x. \quad (2.42)$$

In conformal coordinates (2.38) the latter takes the form

$$H_{\text{H}} = \frac{x(1 + x)^2 p^2}{4r_0^2} + s^2 \frac{(x + 1)^4}{4r_0^2 x(x - 1)^2} + \omega^2 r_0^2 \frac{(1 + x)^2}{2(x + x_3)} - \omega^2 r_0^2. \quad (2.43)$$

## $\mathcal{N} = 2$ supersymmetric oscillator on $\mathbb{CP}^N$

In this Section we construct the $\mathcal{N} = 2$ superextension of the oscillator on $\mathbb{CP}^N$ coupled to constant magnetic field.

It is well-known, that any Hamiltonian system of the form

$$H_0 = g^{ij}(p_i p_j + W_i W_j), \quad \Omega^{\text{can}} = dp_i \wedge dx^i \quad (3.1)$$
could be easily extended to the system with exact $\mathcal{N} = 2$ supersymmetry

$$\{Q^+, Q^-\} = \mathcal{H}, \quad \{Q^\pm, Q^\pm\} = 0. \quad (3.2)$$

The function $W(x)$ is called superpotential.

The oscillator on sphere $S^D$ belongs to the above class of the systems. Its superpotential is given by the expression

$$W = \omega \log(2 + x^2)/(2 - x^2) \quad (3.3)$$

where $x$ denotes the conformal coordinates of sphere $S^D$.

For the supersymmetrization of the system (3.1) we have to define the supersymplectic structure

$$\Omega = dp_i \wedge dx^i + \frac{1}{2} R_{ijkl} \theta^k_+ \theta^l_+ dx^i \wedge dx^k + g_{ij} D\theta^i_+ \wedge D\theta^j_-, \quad D\theta^i_+ = d\theta^i_+ + \Gamma^i_{kl} \theta^k_- dx^l, \quad \alpha = 1, 2$$

and the supercharges $Q_\pm = (p_i \pm iW_i)\theta^i_\pm$ which obey the condition $\{Q_\pm, Q_\pm\} = 0$. Then we immediately get the $\mathcal{N} = 2$ supersymmetric Hamiltonian

$$\mathcal{H} = \{Q_+, Q_+\} = H_0 + W_i \theta^i_+ \theta^i_- + R_{ijkl} \theta^i_+ \theta^j_- \theta^k_- \theta^l_+. \quad (3.4)$$

The inclusion of a magnetic field $\Omega \to \Omega + F_{ij} \theta^i_+ \theta^j_- \theta^i_- \theta^j_+$ breaks the $\mathcal{N} = 2$ supersymmetry of the system

$$\{Q_\pm, Q_\pm\} = F_{ij} \theta^i_+ \theta^j_+, \quad \{Q_+, Q_-\} = \mathcal{H} + iF_{ij} \theta^i_+ \theta^j_-.$$  

For a construction of supersymmetric oscillator on $\mathbb{CP}^N$, let us represent the initial (bosonic) Hamiltonian in the form

$$\mathcal{H} = g^{ab}(\pi_a \bar{\pi}_b + \partial_a W \bar{\partial}_b W). \quad (3.4)$$

If the superpotential may be represented in the form $W(z, \bar{z}) = W_+(z) + W_-(\bar{z})$, then one can construct the $\mathcal{N} = 4$ supergeneralization of the system on Kähler space [19]. Otherwise, the system can be endowed with $\mathcal{N} = 2$ supersymmetry.

Hence, we can construct the $\mathcal{N} = 4$ supersymmetric oscillator on $\mathbb{CP}^N$ choosing the superpotential $2W = \omega z^2 + \omega z \bar{z}$. However, we cannot construct the (anti)holomorphic superpotential for the oscillator on $\mathbb{CP}^N$, and, consequently, obtain its $\mathcal{N} = 4$ superextension. On the other hand, for the oscillators on $\mathbb{CP}^N$ and $\mathbb{CP}^N$, one can find the superpotentials with explicit $su(N)$ symmetry,

$$W = \omega K = \omega z \bar{z} \quad \text{for} \quad \mathbb{CP}^N$$

$$W = \omega r_0 \log(1 - z \bar{z})/(1 + z \bar{z}) \quad \text{for} \quad \mathbb{CP}^1$$

$$2W = \omega r_0 \log(1 - z \bar{z})/(1 + z \bar{z}) \quad \text{for} \quad \mathbb{CP}^1$$

$$W = \omega K = \omega r_0 \log(1 + z \bar{z}) \quad \text{for} \quad \mathbb{CP}^N, \quad N > 1. \quad (3.5)$$
that the linear dependence of the superpotential $W$ on Kähler potential $K$ leads to an interesting behaviour of the supersymmetric system with respect to constant magnetic field. Thus, the superoscillator on $\mathbb{CP}^N$, $N > 1$ has more similarities with the planar one, than the oscillator on $\mathbb{CP}^1$.

Let us consider a $(2N,2N)\mathbb{CP}$-dimensional phase space equipped with the symplectic structure
\[
\Omega = d\pi_a \wedge dz^a + d\pi_a \wedge dz^a + i(Bg_{ab} + iR_{abcd}\eta^c_{\alpha}\eta^d_{\alpha})dz^a \wedge dz^b + g_{ab}D\eta^a_{\alpha} \wedge D\eta^b_{\alpha}
\] (3.6)
where $D\eta^a_{\alpha} = d\eta^a_{\alpha} + \Gamma^a_{bc}\eta^b_{\alpha}dz^c$, $\alpha = 1, 2$ and $\Gamma^a_{bc}$, $R_{abcd}$ are respectively the connection and curvature of the Kähler structure.

The corresponding Poisson brackets are defined by the following non-zero relations (and their complex-conjugates):
\[
\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \bar{\eta}^b_{\alpha}\} = -\Gamma^b_{ac}\eta^c_{\alpha}, \\
\{\pi_a, \bar{\pi}_b\} = i(Bg_{ab} + iR_{abcd}\bar{\eta}^c_{\alpha}\bar{\eta}^d_{\alpha}), \quad \{\eta^a_{\alpha}, \bar{\eta}^b_{\beta}\} = g^{ab}\delta_{\alpha\beta}.
\]

The symplectic structure (3.6) becomes canonical in the coordinates ($p_a, \chi^\alpha$)
\[
p_a = \pi_a - \frac{i}{2}\partial_\alpha g, \quad \chi^m = e^m_a\eta^b_{\alpha}, \quad \Omega_{Scan} = dp_a \wedge dz^a + d\bar{p}_a \wedge d\bar{z}^a + d\chi^m_a \wedge d\bar{\chi}^m_a,
\]
where $e^m_a$ are the einbeins of the Kähler structure: $e^m_a \delta_{mn} e^m_b = g_{ab}$.

So, in order to quantize the system, one chooses
\[
\hat{p}_a = -i\frac{\partial}{\partial z^a}, \quad \hat{\pi}_a = -i\frac{\partial}{\partial \bar{z}^a}, \quad [\chi^m_a, \bar{\chi}^m_{\beta}]^+ = \delta^m_n \delta_{\alpha\beta}.
\]
In order to construct the system with the exact $N = 2$ supersymmetry (3.2) we have to find the appropriate candidate for $Q^\pm$, which obey the equations $\{Q^+, Q^+\} = 0$. Let us search the realization of supercharges among the functions
\[
Q^\pm = \cos \lambda \Theta^\pm_1 + \sin \lambda \Theta^\pm_2,
\]
where
\[
\Theta^+_1 = \pi_a\eta^a_{\bar{1}} + i\partial_a W\bar{\eta}^a_{\bar{2}}, \quad \Theta^+_2 = \bar{\pi}_a\bar{\eta}^a_{\bar{1}} + i\partial_a W\eta^a_{\bar{2}}, \quad \Theta^- = \Theta^+_1 = \Theta^+_2,
\]
and $\lambda$ is some parameter.

Calculating the Poisson brackets of these functions, we get
\[
\{Q^+, Q^+\} = \sin 2\lambda Bg_{ab} + \cos 2\lambda W_{ab}\eta^a_{\bar{1}}\eta^b_{\bar{1}}, \quad \{Q^+, Q^-\} = \mathcal{H}_{3USY} + \cos 2\lambda Bg_{ab} - \sin 2\lambda Z_3,
\]
(3.10)
(3.11)
Here and further we use the notation
\[
\mathcal{H}_{3USY} = \mathcal{H} - R_{abcd}\eta^a_{\bar{1}}\eta^b_{\bar{1}}\eta^c_{\bar{2}}\eta^d_{\bar{2}} - iW_{ab}\eta^a_{\bar{1}}\eta^b_{\bar{2}} + iW_{ab}\bar{\eta}^a_{\bar{1}}\bar{\eta}^b_{\bar{2}} + B\mathcal{F}_3^2,
\]
(3.12)
where $\mathcal{H}$ denotes the oscillator Hamiltonian on $\mathbb{CP}^N$ (see expressions in (2.17), (2.19)), and
\[
\mathcal{F}_3 = ig_{ab}(\eta^a_{\bar{1}}\eta^b_{\bar{2}} - \eta^b_{\bar{1}}\eta^a_{\bar{2}}), \quad Z_3 = iW_{ab}(\eta^a_{\bar{1}}\eta^b_{\bar{2}} - \eta^b_{\bar{1}}\eta^a_{\bar{2}}), \quad g = ig_{ab}\eta^a_{\bar{1}}\eta^b_{\bar{2}}.
\]
(3.13)
In what follows we will also need the generators
\[
\mathcal{F}_+ = ig_{ab}\eta^a_{\bar{1}}\eta^b_{\bar{2}}, \quad \mathcal{F}_- = \mathcal{F}_+^*,
\]
(3.14)
which obey the commutation relations
\[
\{\mathcal{F}_\pm, \mathcal{F}_3\} = \mp 2i\mathcal{F}_\pm, \quad \{\mathcal{F}_+, \mathcal{F}_-\} = i\mathcal{F}_3
\]
(3.15)
\[
\{\Theta^\pm_\alpha, \mathcal{F}_\mp\} = 0, \quad \{\mathcal{F}_\mp, \mathcal{G}\} = \pm i\alpha_{\beta}\mathcal{F}_\mp, \quad \{\Theta^\pm_\alpha, \mathcal{F}_3\} = \pm i\Theta^\pm_\alpha,
\]
(3.16)
\[
\{\mathcal{F}_\pm, \mathcal{G}\} = \{\mathcal{F}_3, \mathcal{G}\} = 0, \quad \{\Theta^\pm_\alpha, \mathcal{G}\} = -i(\pi_a - i\partial_a W\bar{\eta}^a_{\bar{2}}), \quad \text{and so on}.
\]
(3.17)
Supersymmetric Oscillators Coupled to the Constant Magnetic Field.

In the absence of a magnetic field, i.e. for \( B = 0 \), we have

\[
\cos 2\lambda = 0, \quad \sin 2\lambda = \pm 1.
\]

Hence, we could choose two copies of the superscharges and Hamiltonians

\[
Q^\pm_\alpha = \frac{\Theta^\pm_\alpha - (-1)^{\alpha} \Theta^\pm_2}{\sqrt{2}}, \quad \{Q^\pm_\alpha, Q^\pm_\beta\} = \mathcal{H}_\alpha = \mathcal{H}^0_{\text{SUSY}} + (-1)^\alpha Z_3, \quad \alpha = 1, 2.
\]

We constructed two copies of the \( N = 2 \) supersymmetric oscillator on \( \mathbb{CP}^1 \). The inclusion of a constant magnetic field \( B \) breaks their \( N = 2 \) supersymmetry down to \( N = 1 \).

Note that

\[
\{Q^\pm_\alpha, Q^\pm_\beta\} = 2\epsilon_{\alpha\beta} Z_\pm \quad Z_\pm = A(z\bar{z}) F_\pm, \quad Z_3 = A(z\bar{z}) F_3,
\]

where \( A(z\bar{z}) = \omega \frac{1+(z\bar{z})^2}{1-(z\bar{z})^2} \).

Hence, in the planar limit one has \( A \to \omega \), so that the generators \( Q^\pm_\alpha, Z_\pm, Z_3, \mathcal{H} \) form a closed Lie superalgebra.

Supersymmetric Oscillators on \( \mathbb{CP}^N, N > 1 \). On the higher-dimensional complex projective spaces one has

\[
W_{ab} = \omega g_{ab}, \quad \Rightarrow \quad \{Q^\pm_\alpha, Q^\pm_\beta\} = 0 \iff B \sin 2\lambda + 2\omega \cos 2\lambda = 0.
\]

Let us introduce the parameter \( \lambda_0 \):

\[
\cos 2\lambda_0 = \frac{B/2}{\sqrt{\omega^2 + (B/2)^2}}, \quad \sin 2\lambda_0 = -\frac{\omega}{\sqrt{\omega^2 + (B/2)^2}},
\]

so that

\[
\lambda = \lambda_0 + (1-\alpha)\pi/2, \quad \alpha = 1, 2.
\]

Hence, we get the following superscharges:

\[
Q^\pm_\alpha = \cos \lambda_0 \Theta^\pm_1 + (1-\alpha) \sin \lambda_0 \Theta^\pm_2,
\]

and the pair of corresponding \( N = 2 \) supersymmetric Hamiltonians

\[
\mathcal{H}^0_{\text{SUSY}} = \{Q^+_\alpha, Q^-_\beta\} = \mathcal{H}^0_{\text{SUSY}} - (1-\alpha) \left( \cos 2\lambda_0 \frac{B}{2}\overline{g} - \sin 2\lambda_0 \omega F_3 \right).
\]

We constructed, on the higher-dimensional complex projective spaces, the two copies of exact \( N = 2 \) supersymmetric oscillators coupled to the constant magnetic field.

Calculating the commutators of \( Q^\pm_1 \) and \( Q^\pm_2 \) we get

\[
\{Q^+_1, Q^+_2\} = 2\omega F_\pm, \quad \{Q^+_1, Q^-_2\} = \cos 2\lambda_0 \mathcal{H}^0_{\text{SUSY}} + \frac{B}{2}\overline{g},
\]

where the Poisson brackets between \( F_\pm \) and \( Q^\pm_\alpha \) look as follows:

\[
\{Q^\pm_\alpha, F_\pm\} = 0, \quad \{Q^\pm_\alpha, F^\pm_\pm\} = \pm \epsilon_{\alpha\beta} Q^\pm_\beta, \quad \{Q^\pm_\alpha, F_3\} = \pm i Q^\pm_\alpha.
\]

In the absence of a magnetic field, i.e. for \( B = 0, \cos 2\lambda_0 = 0, \sin \lambda_0 = -1 \), the two systems form the superalgebra

\[
\{Q^\pm_1, Q^\pm_2\} = 2\omega F_\pm, \quad \{Q^+_\alpha, Q^-_\beta\} = \delta_{\alpha\beta} \mathcal{H}^0_{\text{SUSY}} - \sigma^3_{\alpha\beta} \omega F_3, \quad \{Q^\pm_\alpha, F_\pm\} = 0, \quad \{Q^\pm_\alpha, F^\pm_\pm\} = \pm \epsilon_{\alpha\beta} Q^\pm_\beta, \quad \{Q^\pm_\alpha, F_3\} = \pm i Q^\pm_\alpha, \quad \{F_\pm, F_\pm\} = i F_3,
\]

The symmetry superalgebra of the oscillator on \( \mathbb{CP}^N \) coincides with the above one in any dimension, i.e., once again we find a quite different behaviour for the oscillators on \( \mathbb{CP}^1 \) and \( \mathbb{CP}^N, N > 1 \) spaces, respectively.

Finally, let us give the explicit expression of the Noether constants of motion corresponding to the susy\((N)\) symmetry:

\[
\mathcal{J}^\text{SUSY}_{ab} = \mathcal{J}_{ab} + \frac{\partial^2 p_{\bar{a} b}}{\partial \bar{z}^2 \partial \bar{z}^d} \sigma_3 \eta^d.
\]
We suggested an integrable system on CP^N, with 4N – 1 functionally independent constants of motion, which could be viewed as the generalization of a 2N-dimensional oscillator. On the complex projective plane CP^1 = S^2 this systems coincide with Higgs oscillator; the Kustaanheimo-Stiefel transformation of the system on CP^2 leads to the three-dimensional Coulomb-like system, which is equivalent to the MIC-Kepler problem on the three-dimensional hyperboloid. obtained by the Kustaanheimo-Stiefel transformation of the oscillator on S^3. On the other hand, while the spherical oscillator remains unchanged upon transition from one hemisphere to another, the oscillator on CP^1, N > 1, after transition to another chart, yields a system, which, in spite of the absence of a rotational symmetry, remains “maximally integrable".

The oscillators on CP^3 and CP^4, in our opinion, deserve a separate study due to their relevance to higher-dimensional quantum Hall effect [7]. This theory, based on the quantum mechanics of the particle on S^4 interacting with a SU(2) monopole field, later has been extended on CP^N spaces in the presence of a constant U(1)(magnetic) field [8]. Since CP^3 can be viewed as fiber bundle of S^4 with S^2 in the bundle, the higher-dimensional quantum Hall system can be formulated in as a system on CP^3 [8, 9]. Thus, the oscillator on CP^3 could be used for a formulation of the higher-dimensional quantum Hall effect in the presence of a potential field (in its present version the potential field is used in this theory for the reduction to three dimensions). Performing the Hurwitz transformation of the oscillator on CP^4, we will get the five-dimensional Coulomb-like system specified by the presence of SU(2) Yang monopole. This system will have the degenerate ground state, and hence, could be used in higher-dimensional quantum Hall effect for the same purposes as the oscillator on CP^3.

The Kähler structure makes the study of the coupling of a constant magnetic field to the oscillator on CP^N much simpler than on 2N-dimensional sphere. In particular, we have shown that the oscillators on CP^N, N > 1 coupled with constant magnetic field behave similarly with respect to N = 2 supersymmetrization. While the constant magnetic field breaks the N = 2 supersymmetry of the oscillator on sphere (and on the CP^1 = S^2), it preserves the N = 2 supersymmetry of the oscillators on CP^N, N > 1 and on CP^N. On the other hand, in the absence of a magnetic field, the oscillator on CP^N allows us to introduce the N = 4 supersymmetry, while the oscillators on spheres and CP^N admit only N = 2 superextensions. It is easy to see, that the similarity of the oscillators on CP^N and CP^N, N > 1 in their behaviour with respect to supersymmetrization is due to the special form of the Hamiltonian

\[ H = g^a \bar{b} (\pi_a \bar{\pi}_b + \omega^2 \partial_a K \bar{\partial}_b K), \]

where K is a Kähler potential of the metric.

Therefore, from the viewpoint of N = 2 supersymmetry, the above Hamiltonian could be viewed as the generalization of oscillator on arbitrary Kähler manifold. In that case the existence of hidden symmetries of the oscillator on CP^N could be viewed as an “accidental" one. Simultaneously, it is clear, that the oscillators on other symmetrical Kähler spaces, say, on the Lobachewski spaces L, or Grassmanians Gr_{N,M}, will have hidden symmetries, due to translational invariance of the above spaces.

Acknowledgments.

We thank Erni Kalnins, for stimulating questions that prompted us to this study, and Anton Galajinsky for his the interest in this work. The work of S.B. was supported in part by the European Community’s Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime, the INTAS-00-0254 grant and the Iniziativa Specifica MI12 of the Commissione IV of INFN. The work of A.N was supported by grants INTAS 00-00262 and ANSEF PS124-01. A.N. thanks LNF, Frascati for hospitality during the completion of this work.

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