Abstract
We show that the Born-Infeld action with the Wess-Zumino terms for the Ramond-Ramond fields, which is the D3-brane effective action, is a solution to the Hamilton-Jacobi (H-J) equation of type IIB supergravity. Adopting the radial coordinate as time, we develop the ADM formalism for type IIB supergravity reduced on $S^5$ and derive the H-J equation, which is the classical limit of the Wheeler-De Witt equation and whose solutions are classical on-shell actions. The solution to the H-J equation reproduces the on-shell actions for the supergravity solution of a stack of D3-branes in a $B_2$ field and the near-horizon limit of this supergravity solution, which is conjectured to be dual to noncommutative Yang Mills and reduces to $AdS_5 \times S^5$ in the commutative limit. Our D3-brane effective action is that of a probe D3-brane, and the radial time corresponds to the vacuum expectation value of the Higgs field in the dual Yang Mills. Our findings can be applied to the study of the holographic renormalization group.
1 Introduction

Recent studies of D-branes have revealed many aspects of the connections between gauge theories and gravities (string theories). In particular, the AdS/CFT correspondence [1, 2] was originally based on an observation that the dynamics of D3-branes can be described under some circumstances both by super Yang Mills and by type IIB supergravity. First, the low velocity dynamics of $N$ D3-branes that are located almost on top each other can be described by super Yang Mills, since in this case the higher excited modes of open strings can be ignored [3]. Second, the region near the horizon of the $N$ D3-branes, whose geometry is $AdS_5 \times S^5$, is described well by supergravity when the curvature radius $R (= (4\pi g_s N)^{1/2} = (4\pi g_{YM}^2 N)^{1/2})$ of $AdS_5$ is sufficiently large. Thus the open-closed string duality leads to a conjecture that $N = 4$ super Yang Mills with large ’t Hooft coupling is dual to type IIB supergravity on $AdS_5 \times S^5$.

Although the correspondence between $N = 4$ super Yang Mills and type IIB supergravity on $AdS_5 \times S^5$ has been tested mainly at the conformally invariant point [4], the above consideration motivates us to conjecture that it is also valid in the Coulomb branch [5, 6]. Indeed, on the one hand, the effective action of the D3-brane probing the $N$ D3-branes, which should take the form of the Born-Infeld action [7] on the $AdS_5$ background, is determined only by the broken conformal invariance [1]. On the other hand, the effective action of $N = 4$ super Yang Mills in which the $SU(N + 1)$ gauge symmetry is broken to $U(1) \times SU(N)$ due to the vacuum expectation value of the Higgs field is conjectured to take the form of the Born-Infeld action on the $AdS_5$ background in the large ’t Hooft coupling limit [8].

The effective action of the probe D3-brane should be obtained in principle by calculating (the logarithm of) the transition amplitude between the vacuum and the boundary state representing the probe D3-brane on the $AdS_5 \times S^5$ background in type IIB superstring. This has not yet been accomplished, because there does not yet exist a quantized theory of type IIB superstring on such a background. However, the above argument suggests that one can obtain the effective action of the probe D3-brane by calculating the classical on-shell action in type IIB supergravity, which is the classical counterpart of the transition amplitude.

In this paper, we show that this is indeed the case, at least for the ‘flat’ probe D3-brane. Here ‘flat’ means that we do not consider fluctuations transverse to the world-volume. The formalism suitable for this purpose is that of the Hamilton-Jacobi (H-J) equation in type
IIB supergravity, which is the classical limit of the Wheeler-De Witt equation and whose solutions are classical on-shell actions. We reduce type IIB supergravity on $S^5$, keeping the anti-symmetric tensor field and the Ramond-Ramond (R-R) fields, and obtain a five-dimensional gravity. Adopting the radial coordinate as time, we develop the ADM formalism for this five-dimensional gravity and derive the H-J equation. We solve the equation under the condition that the fields be constant on fixed-time surfaces. We show that the Born-Infeld action with the Wess-Zumino terms for the R-R fields is one of the solutions to the H-J equation. In general, the H-J equation has infinitely many solutions. Our solution to the H-J equation is the on-shell action for various near-horizon geometries of many D3-branes. In fact, the on-shell action for the supergravity solution representing the near-horizon limit of a stack of D3-branes in a $B_2$ field [9], which is conjectured to be dual to noncommutative Yang Mills and reduces to $AdS_5 \times S^5$ in the commutative limit, is reproduced by the solution to the H-J equation. It is conjectured that the solution to the H-J equation also includes the on-shell actions for general fluctuations around this supergravity solution. The solution to the H-J equation is the effective action of a probe D3-brane located in the backgrounds of the near-horizon geometries. The radial time corresponds to the position of the probe D3-brane and the vacuum expectation value of the Higgs field in the dual Yang Mills. Moreover, the solution to the H-J equation also reproduces the on-shell action for the supergravity solution of a stack of D3-branes in a $B_2$ field without the near-horizon limit. It is relevant to investigate whether this result for the region outside the near-horizon is universal or accidental and due to the special case in which only the ‘flat’ D3-brane is considered. This result should be related to the fact that the Laplacian for the transverse parts in the geometry generated by a stack of D3-branes is proportional to the flat space Laplacian [5] (see also Ref.[10]). The supersymmetry should also be essential in this result, since the R-R fields play crucial roles in our calculation, and therefore a counterpart to the nonrenormalization theorem in the dual Yang Mills should hold in supergravity. We can generalize our analysis to the cases of general $Dp$-branes.

Our results clarify a relation between the effective action in super Yang Mills in the Coulomb branch and the on-shell action in supergravity. They also lead to the question of whether the effective action in noncommutative Yang Mills or in super Yang Mills in different dimensions takes the form of the Born-Infeld action. If indeed it does take this
form, this provides strong evidence of the duality of noncommutative Yang Mills or super Yang Mills in different dimensions and supergravities on curved backgrounds. We hope to address this problem elsewhere. In general, when one studies the gauge/string duality based on the D-brane picture, it is necessary to work first in a region of coupling strengths in which the supergravity approximation is valid, since the quantization of strings on curved backgrounds has not yet been developed well. Therefore, we believe that the study presented in this paper represents a prototype for approaches to this problem.

Another motivation of our work is to understand more general holographic renormalization group flows. The authors of Ref.[11] analyzed the H-J equations around general AdS backgrounds and derived the holographic renormalization group equation for the dual gauge theories that are perturbed by the operators dual to the scalar fields in gravities. (For further developments, see Refs.[12, 13, 14, 15, 16, 17].) Our solution to the H-J equation can be interpreted as a potential that gives the renormalization group flows generated by the perturbations of the operators dual to the tensor fields. Furthermore, our study is expected to be useful for understanding the holographic renormalization group of noncommutative Yang Mills.

The organization of the paper is as follows. In section 2, we perform a reduction of type IIB supergravity on $S^5$ and obtain a five-dimensional gravity. The self-duality condition for the R-R 5-form is treated carefully. In section 3, we develop a canonical formalism for the five-dimensional gravity based on the ADM decomposition and derive the H-J equation. In section 4, we show that the D3-brane effective action is a solution to the H-J equation. In section 5, after reviewing the supergravity solution representing a stack of D3-branes in a $B_2$ field and its near-horizon limit, we show that the on-shell actions for the supergravity solution and its near-horizon limit are reproduced by the solution to the H-J equation. In section 6, we show that the solution to the H-J equation obtained in section 4 is the effective action of a probe D3-brane. Section 7 is devoted to summary and discussion. In particular, we comment on the extension of our results to the cases of general Dp-branes. The equations of motion in type IIB supergravity are listed in appendix A. Some useful formulae are gathered in appendix B. In appendix C, we elucidate the meaning of the momentum constraint and the Gauss law constraints obtained in section 3.
2 Reduction of type IIB supergravity on $S^5$

In this section, we reduce type IIB supergravity on $S^5$ and obtain a five-dimensional gravity. In this paper, we drop the fermionic degrees of freedom consistently. The bosonic part of type IIB supergravity is given by

$$I_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-G} \left[ e^{-2\Phi} \left( R_G + 4\partial_M \Phi \partial^M \Phi - \frac{1}{2} |H_3|^2 \right) - \frac{1}{2} |F_1|^2 - \frac{1}{2} |\tilde{F}_3|^2 - \frac{1}{4} |\tilde{F}_5|^2 \right]$$

$$+ \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3,$$  \hspace{1cm} (2.1)

where

$$H_3 = dB_2, \quad F_{p+2} = dC_{p+1} \quad (p = -1, 1, 3),$$

$$\tilde{F}_3 = F_3 + C_0 \wedge H_3,$$

$$\tilde{F}_5 = F_5 + C_2 \wedge H_3,$$  \hspace{1cm} (2.2)

the $X^M \ (M = 0, \ldots, 9)$ are ten-dimensional coordinates, and $C_{p+1}$ is the R-R $(p + 1)$-form. In the above equations, $|K_q|^2 = \frac{1}{q!} G_{M_1 N_1} \ldots G_{M_q N_q} K_{M_1 \ldots M_q N_1 \ldots N_q}$ for a $q$-form $K_q$. One must also impose the self-duality condition

$$*\tilde{F}_5 = \tilde{F}_5$$  \hspace{1cm} (2.3)

on the equations of motion derived from the above action. For completeness, we list all the equations of motion and the self-duality condition in type IIB supergravity explicitly in appendix A.

In order to perform a reduction on $S^5$, we split the ten-dimensional coordinates $X^M$ into two parts, as $X^M = (\xi^\alpha, \theta_i) \quad (\alpha = 0, \ldots, 4, \quad i = 1, \ldots, 5)$, where the $\xi^\alpha$ are five-dimensional coordinates and the $\theta_i$ parametrize $S^5$, and we adopt the following ansatz for the ten-dimensional metric, which preserves the five-dimensional general covariance:

$$ds_{10}^2 = G_{MN} \, dX^M dX^N$$

$$= h_{\alpha\beta}(\xi) \, d\xi^\alpha d\xi^\beta + e^{\rho(\xi)/2} \, d\Omega_5.$$  \hspace{1cm} (2.4)
Here $h_{\alpha\beta}$ is a five-dimensional metric. We also adopt the following ansatz for the other fields:

\[
\begin{align*}
\Phi &= \phi(\xi), \\
B_2 &= \frac{1}{2} B_{\alpha\beta}(\xi) \, d\xi^\alpha \wedge d\xi^\beta \equiv B, \\
C_0 &= \chi(\xi), \\
C_2 &= \frac{1}{2} C_{\alpha\beta}(\xi) \, d\xi^\alpha \wedge d\xi^\beta \equiv C
\end{align*}
\] (2.5)

and

\[
\begin{align*}
C_4 &= \frac{1}{4!} D_{\alpha\beta\gamma\delta}(\xi) \, d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma \wedge d\xi^\delta + \frac{1}{4!} k \, E_{\theta\theta_j\theta_k\theta_l}(\theta) \, d\theta_i \wedge d\theta_j \wedge d\theta_k \wedge d\theta_l \\
&\equiv D + k \, E,
\end{align*}
\] (2.6)

such that

\[
5 \, \partial_{[\theta} E_{\theta_j\theta_k\theta_l\theta_m]} = \varepsilon_{\theta\theta_j\theta_k\theta_l\theta_m},
\] (2.7)

where $\varepsilon_{\theta\theta_j\theta_k\theta_l\theta_m}$ is the totally anti-symmetric covariant tensor in $S^5$ and $k$ is a constant. We have also defined $B$, $C$, $D$ and $E$: $B$ and $C$ are 2-forms in the five dimensions, $D$ is a 4-form in these five dimensions $\xi^\alpha$, and $E$ is a 4-form in $S^5$. We set all the other fields to zero. We will check below that the ansatz (2.6) is consistent with the equations of motion and the self-duality condition.

From the ansatz (2.6), $\tilde{F}_5$ can be evaluated as

\[
\tilde{F}_5 = \tilde{G} + \frac{1}{5!} k \, \varepsilon_{\theta\theta_j\theta_k\theta_l\theta_m} \, d\theta_i \wedge d\theta_j \wedge d\theta_k \wedge d\theta_l \wedge d\theta_m,
\] (2.8)

where $\tilde{G}$ is defined by $\tilde{G} = G + C \wedge H$ with $H = \frac{1}{2} \partial_{[\alpha} B_{\beta\gamma]} \, d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma$ and $G = \frac{1}{4!} \partial_{[\alpha_1} D_{\alpha_2 \cdots \alpha_5]} \, d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_5}$.

We substitute these ansatizes into the equations of motion in type IIB supergravity (A.1)-(A.6). By using the formulae in appendix B, we obtain the following equations in the five dimensions:

\[
\begin{align*}
R_{\alpha\beta}^{(5)} + 2 \nabla_\alpha^{(5)} \nabla_\beta^{(5)} \phi - \frac{5}{4} \nabla_\alpha^{(5)} \nabla_\beta^{(5)} \rho - 5 \partial_\alpha \rho \partial_\beta \rho - \frac{1}{4} H_{\alpha\gamma\delta} H^{\gamma\delta} - \frac{1}{2} e^{2\phi} \partial_\alpha \chi \partial_\beta \chi - \frac{1}{4} e^{2\phi} \tilde{F}_{\alpha\gamma\delta} \tilde{F}^{\gamma\delta} \\
- &\frac{1}{96} e^{2\phi} \tilde{G}_{\alpha_1 \cdots \alpha_4} \tilde{G}^{\gamma_1 \cdots \gamma_4} + h_{\alpha\beta} \left( - \frac{1}{2} R^{(5)} - 2 \nabla_\gamma^{(5)} \nabla^{(5)} \phi + \frac{5}{4} \nabla^{(5)} \nabla^{(5)} \rho + 2 (\partial \phi)^2 + \frac{15}{16} (\partial \rho)^2 \\
- &\frac{5}{2} \partial_\gamma \phi \partial_\rho \rho + \frac{1}{4} |H|^2 + \frac{1}{4} e^{2\phi} (\partial \chi)^2 + \frac{1}{4} e^{2\phi} |\tilde{F}|^2 - \frac{1}{2} e^{-\rho/2} R^{(S^5)} \right) = 0,
\end{align*}
\]
\[ R^{(5)} + 4 \nabla^{(5)}_{\alpha} \nabla^{(5)}_{\alpha} \phi - \frac{5}{2} \nabla^{(5)}_{\alpha} \nabla^{(5)}_{\alpha} \rho - 4 (\partial \phi)^2 - \frac{15}{8} (\partial \rho)^2 + 5 \partial_{\alpha} \phi \partial^\alpha \rho - \frac{1}{2} |H|^2 + e^{-\rho/2} R^{(S^5)} = 0, \]

\[ -\frac{1}{2} e^{2\phi} (\partial \chi)^2 - \frac{1}{2} e^{2\phi} |F|^2 - \frac{1}{2} e^{2\phi} |\tilde{G}|^2 + \frac{3}{5} e^{-\rho/2} R^{(S^5)} = 0, \]

\[ \nabla^{(5)}_\gamma (e^{-2\phi + \frac{5}{4} \rho} H^{\gamma \alpha \beta}) + \nabla^{(5)}_\gamma (e^{\frac{5}{4} \rho} \chi F^{\gamma \alpha \beta}) + \frac{1}{6} e^{\frac{5}{4} \rho} F_{\gamma_1 \gamma_2 \gamma_3} \tilde{G}^{\alpha \beta \gamma_1 \gamma_2 \gamma_3} = 0, \]

\[ \nabla^{(5)}_\alpha (e^{\frac{5}{4} \rho} \partial^\alpha \chi) - \frac{1}{6} e^{\frac{5}{4} \rho} H_{\alpha \beta \gamma} \tilde{F}^{\alpha \beta \gamma} = 0, \]

\[ \nabla^{(5)}_\gamma (e^{\frac{5}{4} \rho} \tilde{F}^{\gamma \alpha \beta}) - \frac{1}{6} e^{\frac{5}{4} \rho} H_{\gamma_1 \gamma_2 \gamma_3} \tilde{G}^{\alpha \beta \gamma_1 \gamma_2 \gamma_3} = 0, \]

\[ \nabla^{(5)}_\gamma (e^{\frac{5}{4} \rho} \tilde{G}^{\gamma \alpha_1 \ldots \alpha_4}) = 0, \]

\[ (2.9) \]

where

\[ H = \frac{1}{2} \partial_{(\alpha} B_{\beta \gamma)} d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma, \]

\[ F = \frac{1}{2} \partial_{(\alpha} C_{\beta \gamma)} d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma, \]

\[ G = \frac{1}{4!} \partial_{(\alpha_1} D_{\alpha_2 \ldots \alpha_5)} d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_5}, \]

\[ \tilde{F} = F + \chi \wedge H, \]

\[ \tilde{G} = G + C \wedge H. \]  \hspace{1cm} (2.10)

In the above equations, \(|L_q|^2 = h^{\alpha_1 \beta_1} \ldots h^{\alpha_q \beta_q} L_{\alpha_1 \ldots \alpha_q} L_{\beta_1 \ldots \beta_q}^{\beta_q} \) for a \( q \)-form \( L_q \), \((\partial \phi)^2 = h^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi \) and so on. On the other hand, the self-duality condition (2.3) gives the relation

\[ (\tilde{F}_5)_{\theta_1 \theta_2 \theta_3 \theta_4 \theta_5} = \frac{1}{5!} \varepsilon_{\theta_1 \theta_2 \theta_3 \theta_4 \theta_5} \alpha_1 \ldots \alpha_5 \ (\tilde{F}_5)_{\alpha_1 \ldots \alpha_5} \]

\[ = -\frac{e^{5p/4}}{\sqrt{-h}} \tilde{G}_{01234} \varepsilon_{\theta_1 \theta_2 \theta_3 \theta_4 \theta_5}. \]  \hspace{1cm} (2.11)

By comparing (2.8) and (2.11), we obtain

\[ k = -\frac{e^{5p/4}}{\sqrt{-h}} \tilde{G}_{01234}. \]  \hspace{1cm} (2.12)
The last equation in (2.9) implies that the right-hand side of (2.12) is constant, so that we have verified that the ansatz (2.6) is consistent with the equations of motion and the self-duality condition.

One can easily verify that the equations (2.9) can be derived from

\[
I_5 = \frac{1}{2\kappa_5^2} \int d^5\xi \sqrt{-h} \left[ e^{-2\phi + \frac{3}{2}\rho} \left( R^{(5)} + 4\partial_\alpha \phi \partial^\alpha \phi + \frac{5}{4} \partial_\alpha \rho \partial^\alpha \rho - 5\partial_\alpha \phi \partial^\alpha \rho - \frac{1}{2} |H|^2 \right) 
- \frac{1}{2} e^{\frac{3}{2}\phi} \left( \partial_\alpha \chi \partial^\alpha \chi + |\tilde{F}|^2 + |\tilde{G}|^2 \right) + e^{-2\phi + \frac{3}{2}\rho} R^{(S^5)} \right],
\]

(2.13)

where

\[
\frac{1}{2\kappa_5^2} = \frac{\text{volume of } S^5}{2\kappa_{10}^2}, \quad R^{(S^5)} = 20.
\]

Note that by substituting (2.4) and (2.5) into the ten-dimensional action (2.1), one can obtain the above action, except for $|\tilde{G}|^2$. We have thus reduced type IIB supergravity on $S^5$ and obtained the five-dimensional system. This reduction is a consistent truncation in the sense that every solution of (2.13) can be lifted to a solution of type IIB supergravity in ten dimensions. In the remainder of this paper, we set $2\kappa_5^2 = 1$.

3 ADM formalism and the H-J equation

In this section, we develop the ADM formalism for the five-dimensional system described by (2.13) and derive the H-J equation. First, we rename the five-dimensional coordinates as follows:

\[
\xi^\mu = x^\mu \quad (\mu = 0, \cdots, 3), \quad \xi^4 = r.
\]

Adopting $r$ as the time, we carry out the ADM decomposition for the five-dimensional metric

\[
ds_5^2 = h_{\alpha\beta} d\xi^\alpha d\xi^\beta = \left( n^2 + g^{\mu\nu} n_\mu n_\nu \right) dr^2 + 2n_\mu dr dx^\mu + g_{\mu\nu} dx^\mu dx^\nu,
\]

(3.1)

where $n$ and $n_\mu$ are the lapse function and the shift function, respectively. Henceforth $\mu$ and $\nu$ run from 0 to 3.
In what follows, we consider a boundary surface specified by \( r = \text{const.} \) and impose the Dirichlet condition for the fields on the boundary. Here we need to add the Gibbons-Hawking term \([19]\) to (2.13), which is defined on the boundary and ensures that the Dirichlet condition can be imposed consistently \([12, 16, 17]\). Then, the five-dimensional action (2.13) with the Gibbons-Hawking term on the boundary can be expressed in terms of the ADM variables as

\[
I_5 = \int dr d^4x \sqrt{-g} \left[ e^{-2\phi + \frac{\phi}{2} + \frac{5}{4}\rho} \left( - (K_{\mu\nu})^2 + K^2 \right) 
+ \frac{1}{n} \left( -4 (\partial_r \phi - n^\mu \partial_\mu \phi) + \frac{5}{2} (\partial_r \rho - n^\mu \partial_\mu \rho) \right) K 
+ \frac{1}{n^2} \left( 4 (\partial_r \phi - n^\mu \partial_\mu \phi)^2 + \frac{5}{4} (\partial_r \rho - n^\mu \partial_\mu \rho)^2 
- 5 (\partial_r \phi - n^\mu \partial_\mu \phi) (\partial_r \rho - n^\mu \partial_\mu \rho) - \frac{1}{4} (H_{\mu\nu} - n^\lambda H_{\lambda\mu\nu})^2 \right) 
+ \frac{1}{n^2} e^{\frac{\phi}{2}} \left( - \frac{1}{2} (\partial_r \chi - n^\mu \partial_\mu \chi)^2 - \frac{1}{4} (\tilde{F}_{\mu\nu} - n^\lambda \tilde{F}_{\lambda\mu\nu})^2 
- \frac{1}{48} (\tilde{G}_{\mu\nu\rho\lambda} - n^\sigma \tilde{G}_{\sigma\mu\nu\rho\lambda})^2 \right) + \mathcal{L} \right],
\]

where

\[
\mathcal{L} = e^{-2\phi + \frac{\phi}{2} + \frac{5}{4}\rho} \left( R_g + 4 \nabla_\mu \nabla^\mu \phi - \frac{5}{2} \nabla_\mu \nabla^\mu \rho - 4 \partial_\mu \phi \partial^\mu \phi - \frac{15}{8} \partial_\mu \rho \partial^\mu \rho + 5 \partial_\mu \phi \partial^\mu \rho - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) 
+ e^{\frac{\phi}{2}} \left( - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{1}{12} \tilde{F}_{\mu\nu\lambda} \tilde{F}^{\mu\nu\lambda} \right) + e^{-2\phi + \frac{\phi}{2}} R^{(S^5)},
\]

and \( K_{\mu\nu} \) is the extrinsic curvature on the four-dimensional manifold given by

\[
K_{\mu\nu} = \frac{1}{2n} (\partial_r g_{\mu\nu} - \nabla_\mu n_\nu - \nabla_\nu n_\mu), \quad K = g^{\mu\nu} K_{\mu\nu}.
\]

Furthermore, by introducing the canonical momenta, we rewrite the above expression as

\[
I_5 = \int dr d^4x \sqrt{-g} \left( \pi^{\mu\nu} \partial_\mu g_{\nu\rho} + \pi_\phi \partial_\mu \phi + \pi_\rho \partial_\mu \rho + \pi^{\mu\nu}_B \partial_\mu B_{\nu\rho} 
+ \pi_\chi \partial_\mu \chi + \pi^{\mu\nu}_C \partial_\mu C_{\nu\rho} + \pi^{\mu\nu}_D \partial_\mu D_{\nu\rho} 
- n H - n^\mu H^\mu - B_{\tau\mu} Z_{\tau\mu}^\nu - C_{\tau\mu} Z_{\tau\mu}^\nu - D_{\tau\mu\lambda} Z_{\tau\mu\lambda}^{\mu\nu} \right),
\]

with

\[
H = -e^{\frac{\phi}{2}} \left( (\pi^{\mu\nu})^2 + \frac{1}{2} \pi_\phi^2 + \frac{1}{2} \pi^{\mu}_\phi \pi_\phi + \frac{4}{3} \pi_\rho^2 + \pi_\phi \pi_\rho + \left( \pi^{\mu\nu}_B - \chi \pi^{\mu\nu}_C - 6 C_{\lambda\rho} \pi^{\mu\nu\lambda\rho} \right)^2 \right).
\]
In fact, by varying (3.5) with respect to $\pi$ and by substituting these relations into (3.5), we reproduce (3.2).

The first of these is the Hamiltonian constraint, the second is the momentum constraint and
and

The constraints coming from the $U(1)$ gauge symmetries for $B$, $C$ and $D$.

In the remainder of this section, we derive the H-J equation. Let $\tilde{g}_{\mu\nu}(x, r), \tilde{\phi}(x, r), \tilde{\rho}(x, r), B_{\mu\nu}(x, r), \tilde{\chi}(x, r), C_{\mu\nu}(x, r)$ and $D_{\mu\nu\lambda}(x, r)$ be a classical solution of (3.2) with the boundary...
conditions
\[\bar{g}_{\mu\nu}(x, r = r_0) = g_{\mu\nu}(x), \quad \bar{\phi}(x, r = r_0) = \phi(x), \quad \bar{\rho}(x, r = r_0) = \rho(x),\]
\[\bar{B}_{\mu\nu}(x, r = r_0) = B_{\mu\nu}(x), \quad \bar{\chi}(x, r = r_0) = \chi(x), \quad \bar{C}_{\mu\nu}(x, r = r_0) = C_{\mu\nu}(x),\]
\[\bar{D}_{\mu\nu\lambda\rho}(x, r = r_0) = D_{\mu\nu\lambda\rho}(x).\]  
(3.13)

We also define \(\pi_{\mu\nu}(x), \cdots, \pi^{\mu\nu\lambda\rho}_{D}(x)\) by
\[\pi_{\mu\nu}(x) = \bar{\pi}_{\mu\nu}(x, r = r_0), \quad \pi_{\phi}(x) = \bar{\pi}_{\phi}(x, r = r_0), \quad \pi_{\rho}(x) = \bar{\pi}_{\rho}(x, r = r_0),\]
\[\pi_{B}(x) = \bar{\pi}_{B}(x, r = r_0), \quad \pi_{\chi}(x) = \bar{\pi}_{\chi}(x, r = r_0), \quad \pi_{C}(x) = \bar{\pi}_{C}(x, r = r_0),\]
\[\pi_{D}(x) = \bar{\pi}_{D}(x, r = r_0),\]  
(3.14)

where the right-hand sides of these equations are calculated using the relations (3.11) for the classical solution.

By substituting the solution into (3.2) with the boundary specified by \(r = r_0\), we obtain the on-shell action \(S\), which is in general a functional of \(g_{\mu\nu}(x), \cdots, D_{\mu\nu\lambda\rho}(x)\) and \(r_0\). The standard argument employed in the H-J formalism leads to the relations (for example, see Ref.[12])
\[\pi^{\mu\nu}_{D}(x) = \bar{\pi}^{\mu\nu\lambda\rho}_{D}(x, r = r_0),\]  
(3.15)

and
\[\frac{\partial S}{\partial r_0} = 0.\]  
(3.16)

The last equation is characteristic of gravitational systems; that is, the on-shell action does not depend on the boundary time explicitly.

The quantities \(\bar{g}_{\mu\nu}(x, r = r_0), \cdots, \bar{D}_{\mu\nu\lambda\rho}(x, r = r_0)\) and \(\pi^{\mu\nu}(x, r = r_0), \cdots, \pi^{\mu\nu\lambda\rho}(x, r = r_0)\) must satisfy the constraints (3.12). Therefore we see from (3.13), (3.14) and (3.15) that the constraints give functional differential equations for \(S\). The momentum constraint and the Gauss law constraints imply that \(S\) must be invariant under the diffeomorphism in four dimensions and the \(U(1)\) gauge transformations. We give a proof of this in appendix C. On
the Hamiltonian constraint gives a non-trivial equation that determines the form of \( S \). We call this equation the H-J equation and solve it in the next section.

### 4 D3-brane effective action as a solution to the H-J equation

We assume that the fields are constant on the fixed-time surface. Let \( S_0 \) be a solution to the H-J equation under this assumption. Then, we see from (3.3), (3.6) and (3.15) that \( S_0 \) satisfies the equation

\[
-e^{\frac{1}{2} \phi - \frac{2}{5} \rho} \left( \frac{1}{\sqrt{-g} \delta g_{\mu \nu}} \right)^2 \frac{1}{2} g_{\mu \nu} \frac{1}{\sqrt{-g} \delta g_{\mu \nu}} + \frac{1}{\sqrt{-g} \delta \phi} \frac{1}{\sqrt{-g} \delta \phi} + \frac{1}{2} \left( \frac{1}{\sqrt{-g} \delta \phi} \right)^2 + \frac{4}{5} \left( \frac{1}{\sqrt{-g} \delta \rho} \right)^2 \\
+ \frac{1}{\sqrt{-g} \delta \phi} \frac{1}{\sqrt{-g} \delta \phi} + \frac{1}{\sqrt{-g} \delta B_{\mu \nu}} - \frac{1}{\sqrt{-g} \delta C_{\mu \nu}} - 6C_{\lambda \rho} \frac{1}{\sqrt{-g} \delta D_{\mu \nu \lambda \rho}} \right)^2 \\
-e^{\frac{1}{2} \phi - \frac{2}{5} \rho} \left( \frac{1}{2} \frac{1}{\sqrt{-g} \delta S_0} + \frac{1}{\sqrt{-g} \delta C_{\mu \nu}} \right)^2 + 12 \left( \frac{1}{\sqrt{-g} \delta D_{\mu \nu \lambda \rho}} \right)^2 \\
= e^{\frac{1}{2} \phi - \frac{2}{5} \rho} R(S^5). \tag{4.1}
\]

We show that the form

\[
S_0 = S_c + S_{BI} + S_{WZ} + \sigma \tag{4.2}
\]

is a solution to (4.1), with

\[
S_c = \alpha \int d^4 x \sqrt{-g} e^{-2\phi + \rho}, \\
S_{BI} = \beta \int d^4 x e^{-\phi} \sqrt{-\det(g_{\mu \nu} + F_{\mu \nu})}, \\
S_{WZ} = \gamma \left( \int D + \int C \wedge F + \frac{1}{2} \int \chi F \wedge F \right) \\
= \gamma \int d^4 x \sqrt{-g} \varepsilon^{\mu \nu \lambda \rho} \left( \frac{1}{24} D_{\mu \nu \lambda \rho} + \frac{1}{4} C_{\mu \nu} F_{\lambda \rho} + \frac{1}{8} \chi F_{\mu \nu} F_{\lambda \rho} \right), \tag{4.3}
\]

where \( F_{\mu \nu} = B_{\mu \nu} + F_{\mu \nu} \), \( F_{\mu \nu} \) is an arbitrary constant anti-symmetric tensor, and \( \sigma \) is an arbitrary constant. Noting that

\[
\frac{1}{\sqrt{-g} \delta B_{\mu \nu}} - \chi \frac{1}{\sqrt{-g} \delta C_{\mu \nu}} - 6C_{\lambda \rho} \frac{1}{\sqrt{-g} \delta D_{\mu \nu \lambda \rho}} = \frac{1}{\sqrt{-g} \delta B_{\mu \nu}} \]


and
\[
\frac{1}{\sqrt{-g}} \frac{\delta S_{WZ}}{\delta g_{\mu\nu}} = 0,
\]

one can see that the left-hand side of (4.1) can be decomposed into the four parts
\[
\text{L.H.S. of (4.1)} = -e^{2\phi}\left( (1) + (2) + (3) \right) - e^{-\frac{3}{2}\phi} \times (4),
\]

with
\[
(1) = \left( \frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta g_{\mu\nu}} \right)^2 + \frac{1}{2} g_{\mu\nu} \frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta g_{\mu\nu}} + \frac{1}{2} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta \phi} \right)^2 + \frac{4}{5} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta \rho} \right)^2,
\]
\[
(2) = 2 g_{\mu\nu} g_{\mu\rho} \frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta g_{\rho\nu}} \frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta g_{\rho\nu}} + \frac{1}{2} g_{\mu\nu} \frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta \phi} + \frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta \rho} + \frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta \phi} \frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta \rho},
\]
\[
(3) = \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta \phi} \right)^2 + \frac{1}{2} g_{\mu\nu} \frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta g_{\mu\nu}} + \frac{1}{2} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta \phi} \right)^2 + \frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta \rho},
\]
\[
(4) = \frac{1}{2} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{WZ}}{\delta \chi} \right)^2 + \frac{1}{2} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{WZ}}{\delta C_{\mu\nu}} \right)^2 + 12 \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{WZ}}{\delta D_{\mu\nu\lambda\rho}} \right)^2.
\]

(4.4)

(1) and (4) are easily calculated as
\[
(1) = -\frac{1}{5} \alpha e^{-2\phi+2\rho},
\]
\[
(4) = -\frac{1}{2} \gamma^2 \left( 1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} (F_{\mu\nu} F^{\mu\nu})^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} F^{\mu\nu} F^{\mu\nu} \right).
\]

(4.6)

In order to calculate (2) and (3), we introduce the 4 × 4 matrices G and B:
\[
(G)_{\mu\nu} = g_{\mu\nu}, \quad (B)_{\mu\nu} = F_{\mu\nu}.
\]

Then, we have
\[
\frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta g_{\mu\nu}} = \frac{1}{2} \alpha e^{-2\phi+\rho} \left( \frac{1}{G} \right)^{\mu\nu}, \quad \frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta \phi} = -2\alpha e^{-2\phi+\rho}, \quad \frac{1}{\sqrt{-g}} \frac{\delta S_c}{\delta \rho} = \alpha e^{-2\phi+\rho},
\]
\[
\frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta g_{\mu\rho}} = \frac{1}{2} \beta e^{-\phi} \sqrt{\det(G+B)} \left( \frac{1}{G+B} \frac{1}{G} - \frac{1}{G} \right)^{\mu\nu},
\]
\[
\frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta B_{\mu\rho}} = \frac{1}{2} \beta e^{-\phi} \sqrt{\det(G+B)} \left( \frac{1}{G+B} \frac{1}{B} - \frac{1}{B} \right)^{\mu\nu},
\]
\[
\frac{1}{\sqrt{-g}} \frac{\delta S_{BI}}{\delta \phi} = -\beta e^{-\phi} \sqrt{\det(G+B)}.
\]

(4.7)
Using this notation, we can express each term in (2) and (3) in terms of the trace of the
4 × 4 matrix and calculate (2) and (3) as follows:

\[(2) = \alpha \beta e^{-3\phi + \rho} \sqrt{\frac{\det (G + B)}{\det G}} \left( \frac{1}{2} \text{tr} \left( \frac{1}{G} \frac{1}{G + B} \frac{1}{G} \frac{1}{G - B} \right) - 1 \right) \]
\[+ \frac{1}{2} \text{tr} \left( \frac{1}{G + B} \frac{1}{G} \frac{1}{G - B} \right) + 2 - 1 \]
\[= 0, \]
\[(3) = \frac{1}{4} \beta^2 e^{-2\phi} \frac{\det (G + B)}{\det G} \left( \text{tr} \left( \frac{1}{G + B} G \frac{1}{G - B} G \frac{1}{G + B} G \frac{1}{G - B} \right) - \text{tr} \left( \frac{1}{G + B} G \frac{1}{G - B} G \right) \right) \]
\[= \frac{1}{2} \beta^2 e^{-2\phi} \frac{\det (G + B)}{\det G} \]
\[= \frac{1}{2} \beta^2 e^{-2\phi} \left( 1 + \frac{1}{2} F_{\mu \nu} F^{\mu \nu} + \frac{1}{8} (F_{\mu \nu} F^{\mu \nu})^2 - \frac{1}{4} F_{\mu \nu} F^{\nu \lambda} F_{\lambda \rho} F^{\rho \mu} \right). \tag{4.8} \]

From (4.1), (4.4), (4.6) and (4.8), we conclude that \( S_0 \) satisfies the H-J equation (4.1) if
\[\alpha^2 = 5 R^{(S_0)} = 100 \quad \text{and} \quad \beta^2 = \gamma^2. \tag{4.9} \]

5 Supergravity solution of D3-branes in \( B_2 \) field

In this section, we first review the supergravity solution of \( N \) D3-branes in a constant \( B_2 \) field background [18] and its near-horizon limit [9]. Next, we find that the on-shell actions for this supergravity solution and its near-horizon limit are included in our solution to the H-J equation obtained in the previous section.

The supergravity solution of \( N \) D3-branes with only \((B_2)_{23}\) non-vanishing that preserves 16 supersymmetries [18] is also a solution of the five-dimensional gravity (3.2) given by

\[ds^2 = f^{-1/2}[-(dx^0)^2 + (dx^1)^2 + h((dx^2)^2 + (dx^3)^2)] + f^{1/2}dr^2,\]
\[f = 1 + \frac{\alpha'^2 R^4}{r^4}, \quad h^{-1} = \sin^2 \theta f^{-1} + \cos^2 \theta,\]
\[e^{2\phi} = g^2 h, \quad e^{\rho/2} = r^2 f^{1/2}, \quad B_{23} = \tan \theta f^{-1} h,\]
\[C_{01} = \frac{1}{g} \sin \theta f^{-1}, \quad D_{0123} = \frac{\cos \theta}{g} f^{-1} h,\]
\[\cos \theta R^4 = 4\pi g N. \tag{5.1}\]
The ten-dimensional geometry of this solution is asymptotic to flat space for \( r \to \infty \), while it has a horizon at \( r = 0 \) and behaves like \( AdS_5 \times S^5 \) near \( r = 0 \). When \( \theta = 0 \), the solution reduces to the ordinary D3-brane solution.

In order to decouple the asymptotic region with the \( B \) field remaining non-trivial, the parameters should be rescaled as

\[
\alpha' \to 0, \quad \tan \theta = \frac{\tilde{b}}{\alpha'}, \quad x^{0,1} \to x^{0,1}, \quad \frac{\tilde{b}}{\alpha'} x^{2,3} \to x^{2,3},
\]

\[
r = \alpha' R^2 u, \quad g = \frac{\alpha'}{\hat{g}},
\]

(5.2)

where \( \tilde{b}, u, \hat{g} \) and the new coordinates \( x^\mu \) are fixed. This scaling corresponds to the Seiberg-Witten limit [20]. Then, the supergravity solution (5.1) reduces to the following form [9]:

\[
ds_5^2 = \alpha' R^2 \left[ u^2((-dx^0)^2 + (dx^1)^2) + \frac{u^2}{1 + a^4 u^4}((dx^2)^2 + (dx^3)^2) + \frac{du^2}{u^2} \right],
\]

\[
e^{2\phi} = \frac{\hat{g}^2}{1 + a^4 u^4}, \quad e^{\varphi/2} = \alpha' R^2, \quad B_{23} = \frac{\alpha'}{\hat{b}} \frac{a^4 u^4}{1 + a^4 u^4},
\]

\[
C_{01} = \frac{\alpha'}{\hat{g} \hat{b}} a^4 u^4, \quad D_{0123} = \frac{\alpha'^2 R^4}{\hat{g}} \frac{u^4}{1 + a^4 u^4},
\]

\[
a^2 = \tilde{b} R^2, \quad R^4 = 4\pi \hat{g} N.
\]

(5.3)

Note that (5.3) is still a solution of the five-dimensional gravity (3.2) with the identification \( u = r \). This solution is conjectured to be the gravity dual of noncommutative Yang Mills in which \( \theta_{23} = \tilde{b} \). When \( \tilde{b} = 0 \), the ten-dimensional geometry of this solution is identical to \( AdS_5 \times S^5 \), which is dual to the ordinary \( \mathcal{N} = 4 \) super Yang Mills.

Let us show that the on-shell actions for (5.1) and (5.3) are reproduced by the solution (4.2) to the H-J equation. First, by using (3.11), we calculate the values of the canonical momenta for (5.1) and (5.3) on the boundaries specified by \( r = r_0 \) and \( u = u_0 \), respectively. For (5.1), we obtain

\[
-\pi_{00} = \pi_{11} = \frac{f_0^{-1/2}}{g^2 h_0} \left( 5 r_0^4 f_0 + \frac{1}{2} r_0^5 \partial_{r_0} f_0 \right),
\]

\[
\pi_{22} = \pi_{33} = \frac{f_0^{-1/2}}{g^2} \left( 5 r_0^4 f_0 + \frac{1}{2} r_0^5 \partial_{r_0} f_0 - \frac{1}{2} \sin^2 \theta r_0^5 \partial_{r_0} f_0 f_0^{-1} h_0 \right),
\]

\[
\pi_\phi = \frac{1}{g^2 h_0} (-20 r_0^4 f_0 - r_0^5 \partial_{r_0} f_0), \quad \pi_\rho = \frac{10}{g^2 h_0} r_0^4 f_0, \quad \pi_{B23} = 0,
\]

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\[ \pi_\chi = 0, \quad \pi_{C01} = \frac{\sin \theta}{2g} r_0^2 f_0^{-1} \partial_\rho f_0, \quad \pi_{D0123} = \frac{\cos \theta}{24g} r_0^2 h_0^{-1} \partial_\rho f_0, \quad \] (5.4)

where

\[ f_0 = 1 + \frac{\alpha^2 R^4}{r_0^4}, \quad h_0^{-1} = \sin^2 \theta f_0^{-1} + \cos^2 \theta. \]

For (5.3), we obtain

\[-\pi_{00} = \pi_{11} = \frac{3\alpha^3 R^6}{g^2} u_0^2 (1 + a^4 u_0^4), \quad \pi_{22} = \pi_{33} = \frac{\alpha^3 R^6}{g^2} u_0^2 \left( 3 + \frac{2a^4 u_0^4}{1 + a^4 u_0^4} \right), \]

\[ \pi_\phi = -\frac{16\alpha^2 R^4}{g^2} (1 + a^4 u_0^4), \quad \pi_\rho = \frac{10\alpha^2 R^4}{g^2} (1 + a^4 u_0^4), \quad \pi_{B23} = 0, \]

\[ \pi_\chi = 0, \quad \pi_{C01} = -2\alpha^2 R^4 \frac{\alpha'}{gb} a^4 u_0^4, \quad \pi_{D0123} = -\frac{\alpha^4 R^8}{6g} \frac{u_0^4}{1 + a^4 u_0^4}. \quad \] (5.5)

Note that the right-hand sides of (5.4) reduce to the right-hand sides of (5.5) in the near-horizon limit (5.2). On the other hand, the solution (4.2) to the H-J equation gives the following canonical momenta:

\[ \pi_{\mu \nu} = g_{\mu \lambda} g_{\nu \rho} \frac{1}{\sqrt{-g}} \frac{\delta S_0}{\delta g_{\lambda \rho}} \]

\[ = \frac{1}{2} (\alpha e^{-2\phi + \rho} + \beta e^{-\phi} A) g_{\mu \nu} + \frac{\beta e^{-\phi}}{2A} \left( -\mathcal{F}_{\mu \lambda} \mathcal{F}^{\lambda \nu} - \frac{1}{2} \mathcal{F}_{\mu \lambda} \mathcal{F}_{\rho \sigma} \mathcal{F}^{\rho \sigma} + \mathcal{F}_{\mu \lambda} \mathcal{F}^{\lambda \rho} \mathcal{F}_{\rho \sigma} \mathcal{F}^{\sigma \nu} \right), \]

\[ \pi_\phi = \frac{1}{\sqrt{-g}} \frac{\delta S_0}{\delta \phi} = -2\alpha e^{-2\phi + \rho} - \beta e^{-\phi} A, \]

\[ \pi_\rho = \frac{1}{\sqrt{-g}} \frac{\delta S_0}{\delta \rho} = \alpha e^{-2\phi + \rho}, \]

\[ \pi_{B\mu \nu} = g_{\mu \lambda} g_{\nu \rho} \frac{1}{\sqrt{-g}} \frac{\delta S_0}{\delta B_{\lambda \rho}} \]

\[ = \frac{\beta e^{-\phi}}{2A} \left( \mathcal{F}_{\mu \nu} + \frac{1}{2} \mathcal{F}_{\mu \lambda} \mathcal{F}_{\nu \rho} \mathcal{F}^{\lambda \rho} + \mathcal{F}_{\mu \lambda} \mathcal{F}^{\lambda \rho} \mathcal{F}_{\rho \nu} \right) + \frac{\gamma}{4} \varepsilon_{\nu \lambda} (C_{\lambda \rho} + \chi \mathcal{F}_{\lambda \rho}), \]

\[ \pi_\chi = \frac{1}{\sqrt{-g}} \frac{\delta S_0}{\delta \chi} = \frac{\gamma}{8} \varepsilon_{\mu \nu \lambda \rho} \mathcal{F}_{\mu \lambda} \mathcal{F}_{\rho \nu}, \]

\[ \pi_{C\mu \nu} = g_{\mu \lambda} g_{\nu \rho} \frac{1}{\sqrt{-g}} \frac{\delta S_0}{\delta C_{\lambda \rho}} = \frac{\gamma}{4} \varepsilon_{\mu \nu \lambda} \mathcal{F}_{\lambda \rho}, \]

\[ \pi_{D\mu \nu \lambda \rho} = g_{\mu \nu} g_{\nu \sigma} g_{\lambda \chi} g_{\rho \rho'} \frac{1}{\sqrt{-g}} \frac{\delta S_0}{\delta D_{\mu \nu \lambda \rho}} = \frac{\gamma}{24} \varepsilon_{\mu \nu \lambda \rho}, \quad \] (5.6)

where

\[ A \equiv \frac{\det(G + B)}{\det G} = \sqrt{1 + \frac{1}{2} \mathcal{F}_{\mu \nu} \mathcal{F}^{\mu \nu} + \frac{1}{8} (\mathcal{F}_{\mu \nu} \mathcal{F}^{\mu \nu})^2 - \frac{1}{4} \mathcal{F}_{\mu \nu} \mathcal{F}^{\lambda \rho} \mathcal{F}_{\lambda \rho} \mathcal{F}^{\mu \nu}}. \]
We substitute the values of the fields in (5.1) and (5.3) into the right-hand sides of (5.6), setting
\[ F_{\mu\nu} = 0. \]
Noting that \( A = h_0^{-1/2}/\cos \theta \) for (5.1) and \( A = \sqrt{1 + a^4 u_0^4} \) for (5.3) on the boundaries, it can be easily verified that the right-hand sides of (5.6) reproduce the right-hand sides of (5.4) if
\[ \alpha = 10 \quad \text{and} \quad \beta = -\gamma = -\frac{4\alpha'^2 R^4 \cos \theta}{g}, \] (5.7)
and the right-hands of (5.5) if
\[ \alpha = 10 \quad \text{and} \quad \beta = -\gamma = -\frac{4\alpha'^2 R^4}{\hat{g}}. \] (5.8)
These conditions are consistent with (4.9), and (5.7) reduces to (5.8) in the near-horizon limit (5.2).

Next, we compare the value of the on-shell action with that of \( S_0 \) directly. We substitute the values of the fields with \( r = r_0 \) in (5.1) and the values of the fields with \( u = u_0 \) in (5.3) into (4.3), respectively. For (5.1), we obtain
\[ S_c = \frac{\alpha V_4 r_0^4}{g^2}, \]
\[ S_{BI} = \beta \int d^4 x \sqrt{-g} e^{-\phi} A = \frac{\beta V_4}{g \cos \theta} f_0^{-1}, \]
\[ S_{WZ} = \gamma \int d^4 x (D_{0123} + C_{01} B_{23}) = \frac{\gamma V_4}{g \cos \theta} f_0^{-1}, \] (5.9)
where \( V_4 = \int d^4 x \). For (5.3), we obtain
\[ S_c = \frac{\alpha V_4 \alpha'^4 R^8}{\hat{g}^2} u_0^4, \quad S_{BI} = \frac{\beta V_4 \alpha'^2 R^4}{\hat{g}} u_0^4, \quad S_{WZ} = \frac{\gamma V_4 \alpha'^2 R^4}{\hat{g}} u_0^4. \] (5.10)
Here, we set
\[ \sigma = 0. \]
Then, it follows from (4.2), (5.7), (5.8), (5.9) and (5.10) that
\[ S_0 = S_c + S_{BI} + S_{WZ} = \begin{cases} \frac{10V_4 r_0^4}{g^2} & \text{for (5.1)} \\ \frac{10V_4 \alpha'^4 R^8}{\hat{g}^2} u_0^4 & \text{for (5.3)}. \end{cases} \] (5.11)
In (5.9) and (5.11), the quantities for (5.1) reduce to those for (5.3) in the near-horizon limit (5.2). We calculate the values of the on-shell actions for (5.1) and (5.3) by substituting (5.4) by $r_0$ replaced by $r$ and (5.5) with $u_0$ replaced by $u$ into (3.5), respectively. Noting that the constraints in (3.5) are satisfied on shell, we reproduce the value of $S_0$ for (5.1) as follows:

$$I_{on-shell}^{5} = \int_{0}^{r_0} dr d^4 x \sqrt{-g} (\pi^{\mu\nu} \partial_\mu g_{\nu\rho} + \pi_\phi \partial_\mu \phi + \pi_\rho \partial_\mu \rho + \pi_B \partial_\mu B_{\rho}$$

$$+ \pi_c \partial_\mu c + \pi_C \partial_\mu C_{\rho} + \pi_D \partial_\mu D_{\rho\lambda})$$

$$= \frac{40V_4}{g^2} \int_{0}^{r_0} dr \ r^3$$

$$= \frac{10V_4}{g^2} r_0^4.$$  \hspace{1cm} (5.12)

We reproduce the value of $S_0$ for (5.3) in the same way. Thus, we have shown that the on-shell actions for the supergravity solution (5.1) and its near-horizon limit (5.3) are reproduced by our solution (4.2) with $F_{\mu\nu} = 0$ and $\sigma = 0$ when $\alpha$ and $\beta$ take the values in (5.7) and (5.8), respectively.

6 Effective action of a probe D3-brane

In this section, we show that the solution (4.2) to the H-J equation obtained in section 4 is the effective action of a probe D3-brane. In section 4, we obtained $S_0$ as a solution to the H-J equation (4.1). $S_0$ is a functional of the boundary values of the fields and an on-shell action for a set of solutions of the five-dimensional gravity (2.13). In section 5, we showed that the supergravity solutions (5.1) and (5.3) belong to this set. Intuitively, the solution to the H-J equation corresponds to the effective action of a probe D3-brane located inside and outside the near-horizon region of a stack of D3-branes, and $u$ and $r$ correspond to the position of the probe D3-brane. In what follows, we give arguments that justify this interpretation.

The quantities $S_{BI}$ and $S_{WZ}$ in (4.2) together take the form of the D3-brane effective action if the metric, the dilaton, the anti-symmetric field and the R-R fields in those terms can be regarded as those induced in the D3-brane world-volume and $F_{\mu\nu}$ can be regarded as the $U(1)$ gauge field strength. Thus, if the above identifications are valid and $S_c$ can be ignored, our interpretation is justified.
First, let us recall the relation between the fields in the target space and the induced fields in the world-volume. Let \( \zeta^\mu (\mu = 0, \cdots, 3) \) be the coordinates of the D3-brane world-volume and \( \bar{\xi}^\alpha (\zeta) (\alpha = 0, \cdots, 4) \) be the embedding functions of the D3-brane in the five dimensions \( \xi^\alpha \). The induced metric on the D3-brane is defined by the pull back

\[
\bar{g}_{\mu\nu}(\zeta) = \frac{\partial \bar{\xi}^\alpha}{\partial \zeta^\mu} \frac{\partial \bar{\xi}^\beta}{\partial \zeta^\nu} h_{\alpha\beta}(\bar{\xi}).
\] (6.1)

The other induced fields in the world-volume are defined by the pull back in the same way. The effective action of the D3-brane is in general expressed in terms of these induced fields. When one considers the ‘flat’ D3-brane, \( \bar{\xi}^4(\zeta) \) is constant, so that the induced fields are equivalent to the fields in the target space, up to a diffeomorphism in the world-volume. In our calculation, the fixed-time surface corresponds to the world-volume of the probe D3-brane. Hence, the above situation is realized, and the static gauge \( \zeta^\mu = \bar{\xi}^\mu (= x^\mu) \) is adopted in our calculation, so that the induced fields coincide with the original fields in the target space.

Next, let us show that \( F_{\mu\nu} \) in the solution to the H-J equation corresponds to the \( U(1) \) gauge field strength. One can see that (5.1) and (5.3) remain solutions of the five-dimensional gravity even if the fields are deformed as

\[
\begin{align*}
B_{\mu\nu} &\rightarrow B_{\mu\nu} + b_{\mu\nu}, \\
C_{\mu\nu} &\rightarrow C_{\mu\nu} + c_{\mu\nu}, \\
D_{\mu\nu\lambda\rho} &\rightarrow D_{\mu\nu\lambda\rho} + d_{\mu\nu\lambda\rho} - 6 c_{[\mu\nu} B_{\lambda\rho]},
\end{align*}
\]

(6.2)

where \( b_{\mu\nu}, c_{\mu\nu} \) and \( d_{\mu\nu\lambda\rho} \) are constants. In fact, if we introduce \( \Lambda, \Sigma \) and \( \Xi \) in such a way that

\[
\begin{align*}
\begin{align*}
b_{\mu\nu} &= \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \\
c_{\mu\nu} &= \partial_\mu \Sigma_\nu - \partial_\nu \Sigma_\mu, \\
d_{\mu\nu\lambda\rho} &= 4 \partial_{[\mu} \Xi_{\nu\lambda\rho]},
\end{align*}
\]

(6.3)

the transformations (6.2) are identical to the transformations for the \( U(1) \) gauge symmetries in type IIB supergravity, as explained in appendix C. However, \( S_0 \) does not need to be invariant under (6.2), because the partial integrations in the arguments in appendix C fail.
for Λ, Σ and Ξ in (6.3). Instead, $F_{\mu\nu}$ and $\sigma$ in $S_0$ are transformed under (6.2) as

$$F_{\mu\nu} \rightarrow F_{\mu\nu} - b_{\mu\nu},$$

$$\sigma \rightarrow \sigma - \gamma V_4 \left( d_{0123} + 6 c_{[01}F_{23]} \right);$$ (6.4)

that is, $F_{\mu\nu}$ is equivalent with the constant shift of $B_{\mu\nu}$ caused by the gauge transformation. Therefore, it follows from a standard argument in string theory that $F_{\mu\nu}$ should be the $U(1)$ gauge field strength in the D3-brane world-volume.

Finally, in order to justify dropping $S_c$, let us see the dependence of each term in (4.2) on the dilaton field. By redefining the R-R fields in such a way that the action of type IIB supergravity is multiplied by an overall factor of $e^{-2\phi}$, one can see that $S_{BI}$ and $S_{WZ}$ are proportional to $e^{-\phi}$. As is well known, this fact indicates that these terms come from the disk diagram in string theory. On the other hand, $S_c$ is proportional to $e^{-2\phi}$. It is natural to consider $S_c$ to come from the sphere diagram in string theory, which corresponds to (the logarithm of) the vacuum transition amplitude or the vacuum bubble diagram. Therefore $S_c$ should be subtracted from the contribution to the effective action of the probe D3-brane. Thus $S_{BI} + S_{WZ}$ in the solution to the H-J equation is interpreted as the effective action of the probe D3-brane.

7 Summary and discussion

In this paper, we showed that the D3-brane effective action plus the cosmological term is a solution to the H-J equation in type IIB supergravity. This solution to the H-J equation reproduces the on-shell actions for the near-horizon geometries of a stack of D-branes and should correspond to the effective action in the dual Yang Mills with a nontrivial vacuum expectation value of the Higgs field. It also reproduces the on-shell action for the supergravity solution of a stack of D3-brane in a $B_2$ field without the near-horizon limit. Obtaining an interpretation of this result is an open problem. Our findings are expected to be a prototype for the calculations through which the correspondence between gauge theories and gravities is checked. They should also shed light on the holographic renormalization group flow generated by the perturbation of the operators dual to the tensor fields as well as on the holographic renormalization (group) in noncommutative Yang Mills.
We can apply similar calculations to the case of general D$p$-branes. In fact, we have already verified that the D$p$-brane effective action plus a cosmological term analogous to $S_c$ form a solution to the H-J equation in type IIA(IIB) supergravity reduced on $S^{8-p}$ for $p = 1, 2$. This result is natural from the viewpoint of Ref.[21], where it is shown that the effective action of a probe D$p$-brane in the near-horizon geometry generated by a stack of D$p$-branes, which takes the form of the Born-Infeld action, is determined only by the generalized conformal symmetry. It is relevant to investigate whether these solutions to the H-J equations reproduce the on-shell actions for the supergravity solutions of the D$p$-brane corresponding to noncommutative Yang Mills in $p + 1$ dimensions and our results can be extended to the cases $p > 3$. We hope to report studies of these problems in the near future.

Some comments are in order. In this paper, we solved the H-J equation under the ansatz that the fields are constant on the fixed-time surface. In other words, we solved the equation in the mini-superspace approximation. Our solution to the H-J equation corresponds to the lowest terms in the derivative expansion of the on-shell action [11], whose derivatives with respect to the fields give the beta functions and the anomalous dimensions. It is important to obtain the higher-order terms in the derivative expansion, going beyond this ansatz, and elucidate what in the Coulomb branch of the dual Yang Mills corresponds to these higher order terms. One can also study the structure of the holographic renormalization group in ordinary and noncommutative Yang Mills in terms of the higher-order terms. We considered only the ‘flat’ D-branes in this paper. In order to treat a D-brane fluctuating in the directions transverse to the world-volume, we have to develop a new formalism that generalizes the ADM formalism in the gravitational system and enables us to consider a ‘local’ time.

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Appendix A: Equations of motion in type IIB supergravity

In this appendix, we list explicitly the equations of motion and the constraints for type IIB supergravity. We have

\[ R^G_{\textit{MN}} + 2D_M D_N \Phi - \frac{1}{4} H^{(3)}_{\textit{ML}_1 L_2} H^{(3) L_1 L_2} - \frac{1}{2} e^{2\Phi} F^{(1)}_M F^{(1)}_N - \frac{1}{4} e^{2\Phi} \tilde{F}^{(3)}_{\textit{ML}_1 L_2} \tilde{F}^{(3) L_1 L_2} \]

\[ - \frac{1}{4 \cdot 4!} e^{2\Phi} \tilde{F}^{(5)}_{\textit{ML}_1 \ldots L_4} \tilde{F}^{(5) L_1 \ldots L_4} + G_{\textit{MN}} \left( -\frac{1}{2} R_G - 2D_L D^L \Phi + 2 \partial_L \Phi \partial^L \Phi + \frac{1}{4} (|H^3|^2 + e^{2\Phi} |F_1|^2 + e^{2\Phi} |\tilde{F}_3|^2) \right) \]

\[ = 0, \quad (A.1) \]

\[ R_G + 4 D_M D^M \Phi - 4 \partial_M \Phi \partial^M \Phi - \frac{1}{2} |H^3|^2 = 0, \quad (A.2) \]

\[ D_L (e^{-2\Phi} H^{LMN}) + D_L (C_0 \tilde{F}^{(3)LMN}) + \frac{1}{6} F^{(3)}_{L_1 L_2 L_3} \tilde{F}^{(5) MN L_1 L_2 L_3} = 0, \quad (A.3) \]

\[ D_L F^{(1)L} - \frac{1}{6} H^{(3)}_{L_1 L_2 L_3} \tilde{F}^{(5) L_1 L_2 L_3} = 0, \quad (A.4) \]

\[ D_L \tilde{F}^{(3)LMN} - \frac{1}{6} H^{(3)}_{L_1 L_2 L_3} \tilde{F}^{(5) MN L_1 L_2 L_3} = 0, \quad (A.5) \]

\[ D_L \tilde{F}^{(5)L M_1 M_2 M_3 M_4} + \frac{1}{36} e^{M_1 M_2 M_3 M_4 L_1 \ldots L_6} H^{(3)}_{L_1 L_2 L_3} F^{(3)}_{L_4 L_5 L_6} = 0, \quad (A.6) \]

where \( D_M \) represents the covariant derivative in ten dimensions. The self-duality condition for the five-form (2.3) is expressed explicitly as

\[ \tilde{F}^{(5) M_1 M_2 M_3 M_4 M_5} = \frac{1}{5!} e^{M_1 M_2 M_3 M_4 M_5 L_1 \ldots L_5} \tilde{F}^{(5)}_{L_1 L_2 L_3 L_4 L_5}. \quad (A.7) \]

Appendix B: Some useful formulae

Let us consider the following reduction of the ten-dimensional space-time on \( S^{8-p} \):

\[ ds_{10}^2 = G_{\textit{MN}} dX^M dX^N \]

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\[ h_{\alpha\beta}(\xi) \, d\xi^\alpha d\xi^\beta + e^{\rho(\xi)/2} \, d\Omega_{8-p}. \]  

(B.1)

Here, the \( \xi^\alpha \) are \((p+2)\)-dimensional coordinates, and \( S^{8-p} \) is parametrized by \( \theta_1, \ldots, \theta_{8-p} \).

The ten-dimensional curvatures are represented by the \((p+2)\)-dimensional curvatures and the \((8-p)\)-dimensional curvatures as

\[
R^G_{\alpha\beta} = R_{\alpha\beta}^{(p+2)} - \frac{8-p}{4} \left( \nabla^{(p+2)}_\alpha \nabla^{(p+2)}_\beta \rho + \frac{1}{4} \partial_\alpha \rho \, \partial_\beta \rho \right),
\]

\[
R^G_{\theta_i \theta_j} = R_{\theta_i \theta_j}^{(S^{8-p})} + \left( -\frac{1}{4} \nabla^{(p+2)}_\alpha \nabla^{(p+2)}_\alpha \rho - \frac{8-p}{16} \partial_\alpha \rho \, \partial^\alpha \rho \right) e^{p/2} g^{(S^{8-p})}_{\theta_i \theta_j},
\]

\[
R_G = R^{(p+2)} - \frac{8-p}{2} \nabla^{(p+2)}_\alpha \nabla^{(p+2)}_\alpha \rho - \frac{(8-p)(9-p)}{16} \partial_\alpha \rho \, \partial^\alpha \rho + e^{-p/2} R^{(S^{8-p})},
\]

(B.2)

where \( R^{(S^{8-p})} \) is the constant curvature of \( S^{8-p} \).

### Appendix C: Momentum constraint and Gauss law constraints

In this appendix, we elucidate the momentum constraint and the Gauss law constraints. First, note that the five-dimensional action (2.13) is invariant under the following \( U(1) \) transformations:

\[ \delta_{\text{gauge}} \, B = d\Lambda, \]

\[ \delta_{\text{gauge}} \, C = d\Sigma, \]

\[ \delta_{\text{gauge}} \, D = d\Xi - \Sigma \wedge H_3. \]  

(C.1)

The Gauss law constraints \( Z_B = 0, \, Z_C = 0 \) and \( Z_D = 0 \) imply that the following relations hold for arbitrary \( \Lambda, \Sigma \) and \( \Xi \), respectively:

\[
0 = \int d^4x \sqrt{-g} \left( \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \right) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta B_{\mu\nu}},
\]

\[
0 = \int d^4x \sqrt{-g} \left( \partial_\mu \Sigma_\nu - \partial_\nu \Sigma_\mu \right) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta C_{\mu\nu}} - (\Sigma_\mu H_{\nu\lambda\rho} - \Sigma_\nu H_{\lambda\rho\mu} + \Sigma_\lambda H_{\rho\mu\nu} - \Sigma_\rho H_{\mu\nu\lambda}) \frac{\delta S}{\delta D_{\mu\nu\lambda\rho}},
\]

\[
0 = \int d^4x \sqrt{-g} \left( \partial_\mu \Xi_\nu - \partial_\nu \Xi_\mu + \partial_\lambda \Xi_{\rho\mu} - \partial_\rho \Xi_{\mu\lambda} \right) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta D_{\mu\nu\lambda\rho}}. \]  

(C.2)
These relations indicate that $S$ is invariant under the $U(1)$ gauge transformations on the fixed-time surface. Finally, we examine the momentum constraint $H^\mu = 0$. For an arbitrary infinitesimal parameter $\varepsilon^\mu$, this constraint leads to the relation

$$0 = \int d^4x \sqrt{-g} \varepsilon_\mu \left( -2 \nabla_\nu \left( \frac{1}{\sqrt{-g}} \delta S \right) + \partial^\nu \phi \frac{1}{\sqrt{-g}} \delta S + \partial^\mu \rho \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \rho} + H^\mu_{\nu \lambda} \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta B_{\nu \lambda}} \right.$$

$$+ \partial^\nu \chi \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \chi} + F^\mu_{\nu \lambda} \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta C_{\nu \lambda}} + \left( G^\mu_{\nu \lambda \rho \sigma} + 4 C^\mu_{\nu} H_{\lambda \rho \sigma} \right) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta B_{\nu \lambda \rho \sigma}} \right)$$

$$= \int d^4x \sqrt{-g} \left( \frac{\delta S}{\delta g_{\mu \nu}} \frac{\delta S}{\delta \phi} \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \rho} + \frac{\delta S}{\delta \chi} \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \chi} \right.$$

$$+ \left( \delta \text{diff} B_{\mu \nu} + \partial_\mu (\varepsilon^\lambda B_{\nu \lambda}) - \partial_\nu (\varepsilon^\lambda B_{\mu \lambda}) \right) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta B_{\mu \nu}}$$

$$+ \left( \delta \text{diff} C_{\mu \nu} + \partial_\mu (\varepsilon^\lambda C_{\nu \lambda}) - \partial_\nu (\varepsilon^\lambda C_{\mu \lambda}) \right) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta C_{\mu \nu}}$$

$$+ \left( \delta \text{diff} D_{\mu \nu \lambda \rho} + \partial_\mu (\varepsilon^\sigma D_{\nu \lambda \rho \sigma}) - \partial_\nu (\varepsilon^\sigma D_{\mu \lambda \rho \sigma}) - \partial_\lambda (\varepsilon^\sigma D_{\rho \mu \lambda \sigma}) - \partial_\rho (\varepsilon^\sigma D_{\mu \nu \lambda \sigma}) \right)$$

$$- \varepsilon^\sigma \left( C_{\mu \sigma} H_{\nu \lambda \rho} - C_{\nu \sigma} H_{\lambda \rho \mu} + C_{\lambda \sigma} H_{\rho \mu \nu} - C_{\rho \sigma} H_{\mu \nu \lambda} \right) \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta D_{\mu \nu \lambda \rho}} \right), \quad (C.3)$$

where $\delta \text{diff}$ stands for the diffeomorphism transformation with respect to the parameter $\varepsilon^\mu$ on the fixed-time surface. By identifying $\varepsilon^\sigma B_{\mu \sigma}$, $\varepsilon^\sigma C_{\mu \sigma}$ and $\varepsilon^\sigma D_{\mu \nu \lambda \sigma}$ with $\Lambda_\mu$, $\Sigma_\mu$ and $\Xi_{\mu \nu \lambda}$, respectively, one can see from (C.2) and (C.3) that $S$ must be invariant under the diffeomorphism on the fixed-time surface.

References


[4] For a review,


