Abstract

On fast travel through spherically symmetric spacetimes

(Bellus, Cabrera, Palme, and Donald, 2003)

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I. INTRODUCTION

In familiar settings, gravity has a tendency to slow the transfer of information from one place to another. The linearized version of this effect is known as the Shapiro time delay and has been the subject of many precision tests of general relativity [1]. The locally measured speed of light remains constant, but the curvature of spacetime nevertheless requires a signal travelling between two “locations” $x, y$ to take longer than would be required to signal between the corresponding locations in Minkowski space.

We are interested here in whether this delay can take the form of an advance, so that the curved spacetime is in some sense ‘faster’ than Minkowski space. Several such senses have been used in the literature and we will introduce more below. This issue has a long history and allows for interesting speculations. The question is often asked whether a technologically advanced civilization might alter the spacetime geometry to take a form that is maximally convenient for their transportation and communication needs and what bounds exist in principle on their ability to do so. Another motivation comes from cosmology, as any spacetime of use to such an advanced civilization might provide for what is in effect a ‘variable speed of light cosmology’ [2, 3, 4, 5, 6, 7, 8, 9] within the context of traditional Einstein-Hilbert gravity.

Of course, the Einstein equations themselves impose no restrictions on the geometry as they merely relate curvature to the matter content of the universe. Thus, any spacetime may be constructed so long as one includes a sufficiently exotic matter source. Well-known examples of spacetimes widely considered to be ‘fast’ include those of Alcubierre [10] and Krasnikov [11] as well as wormhole solutions (see e.g. [12]). The negative mass Schwarzschild solution also allows fast signaling. In fact, all known solutions which are readily agreed to be fast contain matter that violates the weak energy condition. This condition [13] requires the stress-energy tensor to satisfy

$$T_{\mu\nu}t^\mu t^\nu \geq 0$$

(1.1)

for any timelike vector $t^\mu$. In the absence of a negative cosmological constant, forms of matter violating (1.1) are widely believed not to exist, or at least to be severely limited.

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1 A somewhat weaker condition known as the weak null energy condition is sometimes used in which the timelike vector $t^\mu$ is replaced by an arbitrary null vector $k^\mu$. However, we consider only the stronger version based on timelike vectors.
by fundamental principles. See e.g. [14, 15, 16, 17, 18] for a summary of the current understanding of the limitations on negative energy fluxes from quantum field theoretic effects\(^2\) and [20] for some discussion of the relationship between the ‘negative energies’ of stringy orientifolds and the weak energy condition.

There are substantial theorems to the effect that fast travel is not possible without violating this energy condition. Such theorems include the results of Hawking [21] on the formation of closed timelike curves and those of Olum [22], Visser, Basset, and Liberati [23], and Gao and Wald [24] which relate more directly to ‘fast travel’. These theorems can be quite powerful and each involves a somewhat different concept of ‘fast travel’. Visser, Basset, and Liberati [23] focus on a perturbative description about flat space. Gao and Wald essentially discuss signaling between points in various asymptotic flat space. Olum takes a different approach and derives a rather more abstract theorem showing that without violations of (1.1), a certain de-focusing property cannot arise. This property is expected to be related to localized regions of fast travel, and in particular the idea that there is a ‘fastest’ path to follow.

However, only the perturbative results of [23] provide actual bounds on signaling times between locations within the interior of a spacetime. There is thus a sizable gap in the literature in terms of what one might call ‘concrete’ results referring to the interior of a spacetime in the non-perturbative context. As discussed in [23], the basic difficulty is to find a setting in which one may ask a well-defined question. One would like to ask whether one spacetime is ‘faster’ than another, but this would require some way to identify standard ‘locations’ in the two spacetimes between which one wishes to travel. The diffeomorphism invariance of general relativity is well-known to make such notions extremely difficult to define.

We begin to fill this gap below by using the restricted context of spherically symmetric static spacetimes to ask well-defined questions. In section II, we derive a non-perturbative version of [23] within this context. Roughly speaking, it states that in terms of the time \(T\) measured by an observer at infinity, a signal between any two orbits \(x, y\) of the Killing field takes longer to travel than it would between the corresponding worldlines in Minkowski space\(^3\).

\(^2\) Reference [19] provides a somewhat different perspective on these results.

\(^3\) The spacetime is mapped to Minkowski space by mapping each sphere of symmetry to one with corre-
Because it refers to the time measured by an observer at infinity, this result has much of the ‘asymptotic’ flavor discussed above. We therefore find it rather unsatisfying. To be precise: if an advanced civilization wished to send signals quickly in order to compete in some way against a neighboring civilization, this theorem would provide useful guidance. Assuming spherically symmetry and a static spacetime, they should make their home as close to flat Minkowski space as possible. But one might imagine that a fast signaling time was desired for other reasons\(^4\). Perhaps it is desired (\(e.g.,\) for reasons of social coherence) to exchange signals on a timescale that seems short to the participants involved; \(i.e.,\) as measured by the proper time along the orbits \(x, y\) of the Killing field? One might imagine that organizations and individuals living in distant parts of the civilization wish to exchange goods or information without receiving undue delays (such as waiting a year to receive a much desired letter or package) due to limitations imposed by the speed of light.

Let us therefore suggest the following two questions to provide a framework for our discussion.

**Question 1:** Given a static region \(\mathcal{V}\) of spacetime containing a sphere of area \(4\pi R^2\), consider the proper time \(2\tau_x\) along the orbit \(x\) of the Killing field required for a signal to propagate from \(x\) to another orbit \(y\) on the sphere and then return. Let \(2\tau_{\text{max}}\) be the maximum such signaling time between two such orbits. What spacetime satisfying the dominant energy condition and having such has a static region minimizes \(\tau_{\text{max}}(R)\) and what is this minimum?

**Question 2:** Consider a spherically symmetric asymptotically flat spacetime of total mass \(M\) and which is vacuum outside some sphere of area \(4\pi R^2\). Let us take \(\tau_R = T/\sqrt{1-2M/R}\) to be the Killing time normalized to measure proper time at the chosen sphere. What interior solution satisfying the dominant energy condition allows a causal signal to propagate between two given orbits \(x\) and \(y\) of the time translation Killing field in the smallest amount of time \(\tau_R\), and what is this shortest signaling time?

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\(^4\) After all, the density of advanced civilizations in our galaxy appears to be rather small, so one might not expect competition to be extreme.
In the case that the infimum of $\tau_R, \tau_{\max}$ is not realized by a smooth geometry, we take a sequence of smooth geometries approaching the infimum to yield answers to these questions. A number of related questions also come to mind, but we will not discuss them here.

Question 1 is of intrinsic interest to an advanced civilization wishing to create a ‘maximally convenient’ home. Question 2 is of interest because it provides a setting in which, due to Birkhoff’s theorem\(^5\) (see e.g. [13]), one feels confident that the sphere being discussed is in some sense ‘the same sphere’ regardless of how the interior is filled. It also acknowledges the constraint that, while the civilization may be able to modify there spacetime in the interior of their sphere, they may have less control over the exterior region of the spacetime.

Note that in both cases we have required the dominant energy condition. Recall that this condition consists of the weak energy condition together with the requirement that $T^\mu{}_{\nu} T^\nu{}_{\mu}$, if non-zero, should be a future directed timelike vector. We choose it here because it is the strongest of the usual energy conditions that is expected to hold for all reasonable forms of matter [13], unless one allows a negative cosmological constant. We will have more to say about the interplay between these questions and the choice of energy conditions in section IV.

The theorem of section II does provide some information of interest to both questions. For example, it gives a lower bound on the signaling time as measured in both Question 1 (in the spherically symmetric setting) and Question 2, but it in no way guarantees that the bound can be saturated. Because this information is incomplete, we are motivated to explore the issue further by considering several particular spacetimes in detail in section III. While we are unable to answer either Question 1 or Question 2 in full, we identify features that may be of use in future investigations. In particular, we find that at least in regimes far from that containing a black hole one can come close to saturating the bound of section II. In addition, it appears that the dominant energy condition is significantly more constraining than is the weak energy condition. These conclusions are discussed briefly in section IV.

\(^5\) Which states that the spacetime outside of the stated sphere will necessarily the Schwarzschild spacetime with some mass $M$. 
II. A BOUND ON FAST SPACETIMES

In this section we prove a theorem showing that, as viewed from infinity, static spherically symmetric spacetimes are never ‘faster’ than Minkowski space. We begin with the following Lemma:

**Lemma 1:** Consider an asymptotically flat spherically symmetric spacetime which is static for \( r > r_0 \), satisfies \( m(r_0) \geq 0 \) as defined below, and satisfies the *weak* energy condition. In such a spacetime, no clock with \( r > r_0 \) runs faster than a clock at infinity. That is, if the Killing time \( T \) is normalized at infinity, the proper time \( \tau \) of any static clock increases no faster than \( T \).

Recall that spherically symmetric static metrics take the general form

\[
ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2
\]

in Schwarzschild coordinates. Our Lemma therefore is just the statement that the metric function \( f \) satisfies \( f < 1 \). The metric function \( h(r) \) is related to the spherically symmetric mass function \( m(r) \) by \( h = (1 - 2m/r)^{-1} \). Note that the above restriction on \( m(r_0) \) is fulfilled whenever the region \( r > r_0 \) is part of a spacetime which satisfies the conditions of the positive mass theorem [25, 26, 27, 28, 29]. Note that we allow spacetimes with, e.g. a central black hole with horizon at \( r_0 \).

The proof is straightforward. Consider any spacetime satisfying the premises stated above. The metric components \( g_{tt} = -f(r) \) and \( g_{rr} = h(r) \) solve the Einstein equations with sources given by the stress-energy tensor \( T^\mu_\nu = \text{diag}(-\rho, P_r, P_\theta, P_\phi) \). These equations are equivalent to

\[
\partial_r m = 4\pi r^2 \rho, \quad \text{(2.2)}
\]
\[
\frac{\partial_r f}{2f} = \frac{m + 4\pi r^3 P_r}{r(r - 2m)}, \quad \text{and} \quad \text{(2.3)}
\]
\[
\partial_r P_r = -\left(\frac{\partial_r f}{2f} + \frac{3}{r}\right)P_r - \frac{\partial_r f}{2f} \rho + \frac{2}{r}P_\theta. \quad \text{(2.4)}
\]

The last of these (2.4) encapsulates stress-energy conservation in a spherically symmetric background.

Now consider the density profile \( \rho_0(r) \) and the pressure profile \( P_{r0}(r) \) in our spacetime. Since asymptotic flatness requires \( f = 1 \) at infinity, \( f(r) \) is determined by integrating (2.3)
inward from infinity. As a result, for a fixed density profile, reducing the radial pressure at any \( r \) increases \( f \) at every smaller value of \( r \). Now recall that the given spacetime must satisfy the weak energy condition, which requires \( \rho_0 \geq -P_{\text{tot}} \). Thus, if we introduce a new spacetime with the same density profile \( \rho_0(r) \) but a new pressure profile \( \tilde{P}_r = -\rho_0 \) saturating the above bound, the corresponding \( \tilde{f} \) satisfies \( \tilde{f}(r) > f(r) \) at each \( r \). Note that our new spacetime is described by the same function \( h(r) \) as the original. Now, since \( h = (1 - 2m/r)^{-1} \), we have

\[
\frac{\partial_r h}{h} = \frac{m + 4\pi r^3 \rho}{r(r - 2m)}.
\]

Since \( \tilde{P}_r = -\rho_0 \), comparison with (2.3) shows that we have \( \partial_r \ln h = -\partial_r \ln \tilde{f} \); i.e., \( \tilde{f} h = \text{constant} \). Evaluating this in the asymptotic region we find \( \tilde{f} = 1/h \). But, using the timelike vector \( \partial_t \) in (1.1) yields \( \rho \geq 0 \) so that \( m \geq 0 \) from (2.2) and \( h = (1 - 2m/r)^{-1} > 1 \). Thus \( f < \tilde{f} < 1 \), proving Lemma 1. In fact, our result is somewhat stronger as we only used \( \rho \geq -P_r \) (and not the entire weak energy condition).

With the aid of Lemma 1, it is now easy to prove the following theorem:

**Theorem 1.** Consider a smooth, spherically symmetric, spacetime satisfying the weak energy condition and static for \( r > r_0 \) and \( m(r_0) \geq 0 \). Suppose the Killing time to be normalized at infinity and consider two orbits \( x \) and \( y \) of the time translation symmetry lying on symmetry spheres with areas \( 4\pi R_x^2 \) and \( 4\pi R_y^2 \) and separated by an angle \( \theta \) on the spheres. Then, as viewed from infinity, no signal staying within the static region can be sent between \( x \) and \( y \) faster than one could be sent if \( x \) and \( y \) lay on the corresponding sized spheres in Minkowski space with the same angular separation; i.e., the Killing time \( T \) required satisfies

\[
T \geq \sqrt{R_x^2 + R_y^2 - 2R_x R_y \cos \theta}.
\]

Using Lemma 1 and \( h(r) > 1 \) we find that the signaling time satisfies

\[
T = \int ds \sqrt{\frac{\dot{h}}{f} \dot{r}^2(s) + \frac{r^2}{f} \dot{\theta}^2(s)} \geq \int ds \sqrt{\dot{r}^2(s) + r^2 \dot{\theta}^2(s)} \geq \sqrt{R_x^2 + R_y^2 - 2R_x R_y \cos \theta}. \tag{2.6}
\]

Thus, we can in some sense show that Minkowski space is the ‘fastest’ spherically symmetric static spacetime. However, this result has much of the ‘asymptotic’ flavor that we wished to avoid. In particular, the notion of how ‘fast’ the spacetime is has been referred to the observer at infinity.
Suppose we examine the implications of this theorem for Questions 1 and 2. It shows that the signaling time between two orbits $x$ and $y$ on the sphere of area $4\pi R^2$ satisfies $T \geq 2R \sin \theta/2$ where $\theta$ is the angular separation of $x$ and $y$. But, in terms of the proper time $\tau_R$ this is $\tau_R \geq 2R (\sin \theta/2) \sqrt{1 - 2M/R}$. From the perspective of observers on the shell this is a much weaker bound than they would find in Minkowski space. It is therefore useful to study the situation in more detail. We begin this below by investigating a number of examples. While we will not succeed in identifying a ‘fastest’ spacetime, we will learn much about the problem, and find some interesting interaction with the energy conditions.

III. SOME EXAMPLES OF ‘FASTER’ SPACETIMES

We now proceed to explore Questions 1 and 2 in more depth through a number of examples. We will make frequent use of a certain strategy to explore the linearized change in the travel time near each of our examples, so we present this method first in subsection III A. We then study three families of spacetimes in detail. All of these families satisfy the dominant energy condition, which is our primary regime of interest. The first family contains a Minkowski interior patched to the Schwarzschild exterior via a thin shell. The other two correspond to various ways of saturating of the energy conditions. Although the last two cases will prove to be faster than the first one, perturbative analysis show that there exist other spacetimes which are faster yet.

A. Linearization Strategy

We will use the same notation for the metric and stress-energy as in section II, though here we will be more concerned with the dominant energy condition as required by Questions 1 and 2. In our context this imposes

$$\rho \geq |P_r|, \quad \rho \geq |P_t|. \quad (3.1)$$

For simplicity, we take the points $x$ and $y$ between which our signal is exchanged to lie on the poles of the sphere at $r = R$. The fastest path connecting them must be a null geodesic with $\phi = const$ so that the integral

$$T = \int ds \sqrt{\frac{h}{f} \dot{r}^2(s) + \frac{r^2}{f} \dot{\theta}^2(s)} \quad (3.2)$$
provides the time of flight as measured by the Killing time $T$ normalized at infinity. We will also make use below of the change in (3.2) under small perturbations of the metric (2.1). The first order variation is
\[
\delta T = \int \frac{ds}{2\sqrt{f}} \sqrt{h\dot{r}^2 + r^2 \dot{\theta}^2} \left[ r^2 \delta h - (h \dot{r}^2 + r^2 \dot{\theta}^2) \frac{\delta f}{f} \right].
\] (3.3)

However, a more useful form is obtained by assuming a regular origin so that boundary conditions imply $\delta m(0) = 0$. The equations of motion (2.2)-(2.4) then imply three linear differential equations for the variations of $f$, $h$, $\rho$, $P_r$ and $P_\theta$ which can be used to rewrite (3.3) in terms of $\delta \rho$ and $\delta P_r$:
\[
\delta T = \int \frac{ds}{2\sqrt{f}} \sqrt{h\dot{r}^2 + r^2 \dot{\theta}^2} \left[ \frac{(2h \dot{r}^2 + r^2 \dot{\theta}^2) \delta h}{h} - (h \dot{r}^2 + r^2 \dot{\theta}^2) \left( \frac{\delta h(R)}{h(R)} + \frac{\delta f(R)}{f(R)} \right) + 8\pi (h \dot{r}^2 + r^2 \dot{\theta}^2) \int_r^R \frac{d\rho'}{\rho'} \left[ h(\delta \rho + \delta P_r) + (\rho + P_r) \delta h \right] \right],
\] (3.4)

where $\delta h = 2\frac{\dot{R}}{R} \delta m = 8\pi \int_0^r d\rho' \rho'^2 \delta \rho(\rho')$. We note for future reference that the derivation of (3.4) uses the inner boundary condition (at the origin) but does not require any outer boundary condition. All the examples we will study obey the equation of state $\rho + P_r = 0$ in the relevant region, so that, in order to maintain (3.1), perturbations should satisfy $\delta \rho + \delta P_r \geq 0$. The existence of perturbations that generate negative $\delta T$ will prove that a particular spacetime under study is not the fastest. To find such a perturbation, we consider variations having $\delta \rho + \delta P_r = 0$ in order to eliminate the positive contribution of the last term in (3.4). In our applications below, we will also have $\delta f(R) = 0$. Under these assumptions, the expression for $\delta T$ reduces to
\[
\delta T = \int \frac{ds}{2} \left( \frac{2h \dot{r}^2 + r^2 \dot{\theta}^2}{\sqrt{f} \sqrt{h\dot{r}^2 + r^2 \dot{\theta}^2}} \right) \frac{\delta h}{h} - \frac{1}{2} \int ds \sqrt{\frac{h\dot{r}^2 + r^2 \dot{\theta}^2}{f}} \frac{\delta h(R)}{h(R)},
\]
(3.5)

A spacetime with identically vanishing $\delta T$ would be an excellent candidate for the fastest spacetime. While we have not been able to find such a solution consistent with the positive energy condition, the result (3.4) is nevertheless quite useful in showing that the following simple cases do not minimize the travel time (3.2).
B. The empty shell spacetime

Let us begin with the simplest allowed spacetime: a flat region inside the sphere of radius $R$ and, in order to match a Schwarzschild exterior as required by Question 2, a thin shell at $r = R$ as determined by the discontinuity in the extrinsic curvature across this surface. The metric of the empty interior $r < R$ is

$$ ds^2 = -(1 - \frac{2M}{R}) dt^2 + dr^2 + r^2 d\Omega^2, $$

(3.6)
i.e., (2.1) with $f = 1 - 2M/R$ and $h = 1$. Here we have chosen the normalization of $t$ so that $g_{tt}$ is continuous at $r = R$ as required by the Israel junction conditions [30]. The surface stress-energy of the shell must also satisfy the dominant energy condition.

In general, the surface stress-energy tensor on a hypersurface $\Sigma$ is defined by the integral [30]

$$ S^\mu_\nu = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} T^\mu_\nu dn = \frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} G^\mu_\nu dn, $$

(3.7)
where $n$ is the proper distance measured along the normal to the hypersurface. For a general interior of the form (2.1) and our Schwarzschild exterior, this surface tensor is

$$ 8\pi S^t_t = -\frac{2}{R} \left( \sqrt{1 - \frac{2M}{R}} - \frac{1}{\sqrt{h}} \right), \quad S^r_r = 0, \quad 8\pi S^\theta_\theta = \frac{1 - \frac{M}{R}}{R\sqrt{1 - \frac{2M}{R}}} - \frac{1}{\sqrt{h}} \left( \frac{f'}{2f} + \frac{1}{R} \right), $$

(3.8)
where the metric components and their derivatives are evaluated by taking the limit $r \to R$ from below. We will continue to use this convention: any discontinuous function evaluated at $R$ is to be understood as the limit $r \to R$ from below.

In particular we find

$$ 8\pi S^t_t = -\frac{2}{R} \left( \sqrt{1 - \frac{2M}{R}} - 1 \right), \quad 8\pi S^\theta_\theta = \frac{1}{R} \left( \frac{1 - \frac{M}{R}}{\sqrt{1 - \frac{2M}{R}}} - 1 \right) $$

(3.9)
for the empty shell metric (3.6). One may then check that $S^t_t \geq |S^\theta_\theta|$ is satisfied exactly in the range $0 \leq \frac{M}{R} \leq \frac{12}{25}$.

It is clear that the fastest trajectory follows a radial path with $\dot{\theta} = 0$, so that, from (3.2), the travel time is

$$ T_{Empty} = \frac{2R}{\sqrt{1 - \frac{2M}{R}}}, $$

(3.10)
In terms of the proper time $\tau_R$ measured by a static observer at $r = R$ this is just $\tau_R^{Empty} = 2R$.

Although this spacetime is a natural one to study, it is not the fastest. This may be seen by considering the variation (3.5) of $\delta T$ under a perturbation $\delta \rho(r) = -\delta P_r(r) = -\delta P_t(r) = \delta \rho(0) > 0$, so that $\delta \rho(r) = \frac{8\pi}{3} \delta \rho(0) r^2$. Since the signal takes no time to cross the shell, it is sufficient to apply (3.5) at some $r$ just a bit less than $R$. We will not need the explicit form of the perturbation at the shell since, due to the continuity of $f$ at $r = R$ we have $\delta f(R) = 0$ so we may use equation (3.5) for the variation $\delta T$. The perturbed spacetime clearly satisfies the energy conditions in the interior and, since the original shell at $r = R$ does not saturate these condition$^6$, there is no danger that they will be violated at $r = R$ for small $\delta \rho$. For this perturbation one finds $\delta T = -\frac{8\pi}{9\sqrt{1 - \frac{M}{R}}}$ < 0 for a radial trajectory, so that the empty shell spacetime is not the fastest.

C. De Sitter space in a bottle

Since the only constraints in our problem are the energy conditions, one might expect these conditions to be saturated by our hypothetical fastest spacetime. The weak energy condition is saturated by taking $\rho = -P_t = -P_r > 0$, in which case stress-energy conservation requires $\rho(r)$ to be just some constant $\rho_0$. In the previous subsection we found the travel time to be reduced by perturbing our empty shell spacetime in this direction. Unfortunately, such a spacetime does match the boundary condition that $\rho = 0$ for $r > R$ as required by Question 2.

On the other hand, this discussion suggests that one might study the spacetime we call “De Sitter space in a bottle” in which we take $\rho = -P_t = -P_r = \rho_0 > 0$ for $r < R$ but add a shell at $r = R$ to satisfy stress-energy conservation. This shell effectively constitutes a ‘bottle’ whose stresses and gravitational self-attraction keeps the piece of de Sitter space with $r < R$ from expanding.

The metric takes the form

$$ds^2 = -(1 - \frac{2M}{R}) \frac{1}{1 - \frac{b^2 r^2}{R^2}} dt^2 + \frac{1}{1 - \frac{b^2 r^2}{R^2}} dr^2 + r^2 d\Omega^2,$$

(3.11)

$^6$ The case with $\frac{M}{R} = \frac{12}{25}$ does in fact saturate $S_t^r \geq |S_t^r|$ and requires more care. It may be treated as in section III C below.
where \( b^2 = \frac{8 \pi}{3} \rho_0 R^2 < 1 \). Here, \( t \) has again been normalized in the interior so that \( g_{tt} \) is continuous across \( r = R \). Comparing (3.11) and (3.8), one finds the surface stresses to be

\[
8 \pi S^t_i = \frac{-2}{R} \left( \sqrt{1 - \frac{2M}{R}} - \sqrt{1 - b^2} \right), \quad 8 \pi S^\theta_i = \frac{1}{R} \left( \frac{1 - M}{R} - \frac{1 - 2b^2}{\sqrt{1 - 2M/R}} \right). \tag{3.12}
\]

Imposing the dominant energy condition at the shell requires

\[
b^2 \leq b^2 \left( \frac{M}{R} \right) = \frac{3}{4} - \frac{S(M/R)}{32} - \frac{S(M/R)}{32}\sqrt{1 + \frac{16}{S(M/R)}}, \text{ where } S(M/R) = \frac{(3 - 5M/R)^2}{1 - 2M/R}, \tag{3.13}
\]

and \( 0 \leq\frac{M}{R} \leq \frac{12}{25} \), as in the previous example\(^7\).

As for the empty shell spacetime, the antipodal orbits \( x \) and \( y \) can be connected only by radial geodesics\(^8\). Thus, we again have \( \delta = 0 \) and the travel time is

\[
\tau_R = T(b) \sqrt{1 - \frac{2M}{R}} = \frac{T_{\text{Empty}}}{\tau_R^{\text{Empty}}} \sqrt{1 - \frac{b^2}{2b}} \ln \frac{1 + b}{1 - b}. \tag{3.14}
\]

Since the factor multiplying \( \tau_R^{\text{Empty}} \) is less than 1, this space is faster than the empty shell spacetime. As (3.14) is a monotonically decreasing function of \( b > 0 \), the smallest allowed time (for fixed \( M/R \)) occurs when (3.13) is saturated, i.e., when \( S^t_i = |S^\theta_i| \). In particular, the largest effect occurs for \( M/R = \frac{2}{5} \), when (3.13) gives the biggest allowed \( b \), \( b = \frac{19}{32} \left(1 - \sqrt{\frac{105}{364}} \right) \), and, therefore, the smallest value of (3.14), \( \tau_R^{\text{Empty}} / \tau_R \approx 0.987 \).

However, a perturbation analysis again shows that spacetimes outside this class are faster yet. Again we apply (3.5) to the region inside the shell. Let us denote the perturbed quantities with tildes. A perturbation \( \delta \rho(r) = -\delta P_r(r) = -\delta P_1(r) = \delta \rho(0) > 0 \), corresponding to \( \tilde{b} = \tilde{b}_+ + \delta \tilde{b} > \tilde{b}_+ \) would reduce the time by the amount \( \delta \rho T \equiv T(\tilde{b}) - T(\tilde{b}_+) < 0 \), but, it would also violate the energy condition \( \tilde{S}^t_i \geq |\tilde{S}^\theta_i| \). In order to respect the dominant energy condition at the shell, we instead use a sequence of perturbations \( \{ \delta_n \rho \} \) of the form

\[
\delta_n \rho(r) = -\delta_n P_r(r) = A_n(r - r_n) + \delta \rho(0) \quad \text{for} \quad R > r > r_n,
\]

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\(^7\) For configurations satisfying \( \rho = -P_r \) in the interior, the condition \( S^t_i \geq |S^\theta_i| \) can be rewritten as \( \sqrt{S(M/R)} - \sqrt{S(M/R)} \geq \frac{4 \pi R^2 \rho(R)}{\sqrt{1 - 2M/R}} \geq 0 \). In order to have \( M \geq m(R) \), the form of \( S \) requires \( 0 \leq M/R \leq \frac{12}{25} \).

\(^8\) Note that a non-radial geodesic would lead to an \( S^1 \) of such geodesics, and thus to a light cone with a caustic at finite affine parameter. As it is readily seen from the conformal diagram (see, e.g., [31]), this does not occur in de Sitter space.
\[ \delta_n \rho(r) = -\delta_n P_{\ell}(r) = \delta \rho(0) \quad \text{for} \quad r < r_n. \tag{3.15} \]

which differ from \( \delta \rho(0) \) in the region \( R > r > r_n \). Our goal will be to satisfy the energy conditions for large enough \( n \). Here \( A_n \) are constants and linearized stress-energy conservation together with the energy condition in the interior requires \( A_n = -\frac{4}{3} (\delta_n \rho + \delta_n P_{\ell}) < 0 \). Similarly, at the shell the dominant energy condition requires \[ \tilde{S}_\ell^\ell - \tilde{S}_\delta^\delta = \delta_n S_\ell^\ell - \delta_n S_\delta^\delta = \frac{1}{R^2 \sqrt{1 - \frac{b^2}{b_0^2}}} \left[ -4 \pi R^3 \delta_n \rho(R) - \frac{2 - b^2}{1 - \frac{b^2}{b_0^2}} \delta_n m(R) \right] > 0. \tag{3.16} \]

Let us choose \( r_n \) to converge to \( R \) and also require each \( \delta_n \rho \) to yield the same value \( \delta m(R) \) for the change in the mass function \( m(r) \) evaluated just inside the shell. Note that this is readily achieved by taking \( A_n \) to scale with \( (R - r_n)^{-2} \). In this case (3.16) is indeed satisfied for sufficiently large \( n \).

It is clear that at each point \( r \) in the interior \( \delta_n \rho(r) \) converges to \( \delta \rho(0) \). Thus, it makes sense to express the variation \( \delta_n T \) of \( T \) under \( \delta_n \rho \) in terms of the variation \( \delta \rho T \) obtained by the constant density perturbation associated with simply shifting \( b \). From (3.5) we find in the limit \[ \delta_n T \rightarrow \delta \rho T + \frac{\delta \rho m(R) - \delta m(R)}{\sqrt{1 - \frac{b^2}{b_0^2}}} \frac{1}{b \sqrt{1 - \frac{b^2}{b_0^2}}} \ln \frac{1 + b}{1 - b}, \tag{3.17} \]

where \( \delta \rho m(R) \) refers to the change in the mass \( m(r) \) evaluated just inside the shell under the constant density perturbation associated with changing the density uniformly by \( \delta \rho(0) \).

Since \( A \) is negative, the second term in (3.17) is positive. However, it is clear from the construction of \( \delta_n \rho \) that we are free to take \( \delta m(R) \) as close as desired to \( \delta \rho m(R) \) without changing \( \delta b \). As a result, this second (positive) term can be made negligible in comparison with the first (negative) term. We have therefore established the existence of small perturbations which preserve the positive energy conditions but reduce the travel time below that of the background “dS in a bottle” spacetime. A similar analysis applies to the empty shell spacetime in the extreme cases \( \frac{M}{R} = \frac{12}{25} \).

Because we imposed the dominant energy condition, spacetimes in this class were restricted to be much slower than would be guaranteed by Theorem 1. In contrast, note that we can do much better if we enforce only the weak energy condition. This will require \( S_\ell^\ell \geq 0 \) and thus \( b^2 \leq 2 M/R \), but this is the only requirement. Note that this is just the condition that \( m(R) \leq M \); \textit{i.e.}, that the mass contained in the region \( r < R \) is less than or equal to
the total mass $M$ of the spacetime. Denoting the bound set by Theorem 1 by $\tau_R^{\text{bound}}$ and comparing with (3.14) for $b^2 = 2M/R$, one finds

$$
\frac{\tau_R}{\tau_R^{\text{bound}}} = \frac{1}{2b} \ln \frac{1 + b}{1 - b}. \tag{3.18}
$$

So, for $b^2 = 2M/R \sim 1$, we find $\tau_R \gg \tau_R^{\text{bound}}$. Nevertheless, $\tau_R \to 0$ so that $\tau_R < \tau_R^{\text{Empty}}$. Thus, this example suggests that the dominant energy condition may be significantly more restrictive that the weak energy condition in investigating Questions 1 and 2.

D. Saturating the dominant energy condition

We now turn to our third example. We saw in the proof of Lemma 1 that it was advantageous to set $\rho = -P_r$ and take $\rho$ as large as possible. The same is true with our current boundary conditions. However, stress-energy conservation places bounds on how rapidly $P_r$ may change. In particular, we can rewrite (2.4) as

$$
\partial_r P_r = -\frac{\partial}{\partial r} f'(\rho + P_r) + \frac{2}{r}(\rho - P_r) - \frac{2}{r}(\rho - P_\theta). \tag{3.19}
$$

Maintaining $P_r = -\rho$ with a rapidly changing $\rho(r)$ may force $P_\theta$ to be very large and perhaps to violate the dominant energy condition $P_\theta < \rho$. In fact, if one has already imposed $P_r = -\rho$, taking $P_\theta = \rho$ allows $P_r$ to decrease at the fastest possible rate as one moves away from the boundary.

As a result, we are motivated to consider spacetimes with $\rho = -P_r = P_\theta$. Stress-energy conservation (3.19) then requires

$$
\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^4, \tag{3.20}
$$

for constants $\rho_0 \geq 0$ and $0 < r_0 < R$. To evade the divergence at $r = 0$, we excise the region $r < r_0$ and sew in a piece of another spacetime. For lack of a better choice, we once again use a piece of de Sitter space. We demand that $\rho$ is continuous at $r_0$ so that $m$ is $C^4$ and there is no additional shell of mass at this junction.

The mass function is

$$
m = \begin{cases} 
4\pi\rho_0 r_0^4 \left(\frac{1}{r}\right)^3 & \text{for } 0 \leq r \leq r_0 \\
-4\pi\rho_0 r_0^4 \left(\frac{1}{r}\right)^3 + \frac{16\pi}{3}\rho_0 r_0^3 = -\frac{r}{r} + a & \text{for } r_0 \leq r < R.
\end{cases} \tag{3.21}
$$
where \( r_0 = \frac{4a}{3\alpha} \) and \( \frac{4}{3\alpha}\rho_0 = \frac{3}{4}(\frac{\alpha}{4})^4 \). We refer to this case as the “dS/DEC” spacetime due to the saturation of the dominant energy condition for \( r > r_0 \) and the presence of the de Sitter region for \( r < r_0 \).

Let us introduce the dimensionless variables

\[
\hat{t} = \frac{2t}{T_{Empty}} = \frac{t}{R} \sqrt{1 - \frac{2M}{R}}, \quad \hat{r} = \frac{r}{R}, \quad \hat{a} = \frac{a}{R}, \quad \hat{c} = \frac{c}{R^2}, \quad \hat{m}(\hat{r}) = \frac{m(r)}{R}, \quad (3.22)
\]

in terms of which the metric takes the form

\[
d\hat{s}^2 = R^2 \left[ \frac{1 - \frac{2\hat{a}(\hat{r})}{\hat{r}}}{1 - \frac{2\hat{a}}{\hat{r}}} \, d\hat{t}^2 + \frac{1}{1 - \frac{2\hat{a}}{\hat{r}}} \, d\hat{r}^2 + \hat{r}^2 d\Omega^2 \right]. \quad (3.23)
\]

Note that for \( \hat{c} = \frac{3\alpha}{4} \), the de Sitter region fills all of \( r < R \). As a result, we require \( \hat{c} \leq \frac{3\alpha}{4} \).

The value \( \hat{r}_{BH} = \hat{a} + \sqrt{\hat{a}^2 - 4\hat{c}} \), which is real for \( \hat{c} \leq \frac{\hat{a}^2}{4} \), would correspond to the location of a Killing horizon, i.e., \( g_{tt}(\hat{r}_{BH}) \propto 1 - \frac{\hat{a}}{\hat{r}_{BH}} + \frac{\hat{a}^2}{\hat{r}_{BH}^2} = 0 \). But note that \( \hat{c} \leq \frac{\hat{a}^2}{4} \) yields \( \hat{r}_0 \leq \frac{2\hat{a}}{3} < \hat{a} < \hat{r}_{BH} \). Thus, to avoid the existence of a horizon\(^9\), we must have \( \hat{c} > \frac{\hat{a}^2}{4} \).

We also investigate any further restriction imposed by requiring the shell to satisfy the dominant energy condition. Again using (3.8), the relevant stresses are

\[
8\pi S^t_t = \frac{-2}{R} \left( \sqrt{1 - \frac{2M}{R}} - \sqrt{1 - 2\hat{a} + 2\hat{c}} \right), \quad 8\pi S^\phi_\phi = \frac{1}{R} \left( \frac{1 - \frac{M}{R}}{\sqrt{1 - \frac{2M}{R}}} - \frac{1 - \hat{a}}{\sqrt{1 - 2\hat{a} + 2\hat{c}}} \right). \quad (3.24)
\]

Condition \( S^t_t \geq |S^\phi_\phi| \) also constrains the values of \( (\hat{a}, \hat{c}) \) through

\[
\hat{c} \geq -\frac{3}{4} + \frac{5}{4} \hat{a} + \frac{1}{16} S(\frac{M}{R}) + \frac{1}{16} S(\frac{M}{R}) \sqrt{1 + \frac{8(\hat{a} - 1)}{S(\frac{M}{R})}}, \quad (3.25)
\]

where \( S(\frac{M}{R}) \) is again as in (3.13) and \( 0 \leq \frac{M}{R} \leq \frac{18}{29} \). A plot of the allowed regions in the \( \hat{a} \hat{c} \) plane for three different values of \( \frac{M}{R} \) is shown in figure 1. Curves of the form (3.25) move to the right in the \( \hat{a} \hat{c} \) plane for increasing \( \frac{M}{R} \leq \frac{2}{3} \), and back to the left for \( \frac{M}{R} > \frac{2}{3} \).

---

\(^9\) One could consider spacetimes with a black hole instead of a dS interior, but then there are no radial null geodesics connecting antipodal points on the sphere. We explored the behavior of selected non-radial null geodesics numerically in such a spacetime but in each case found \( T > T_{Empty} \). For this reason we chose to concentrate on radial geodesics and on spacetimes that allow them.
FIG. 1: The allowed region in the $\hat{a}\hat{c}$ plane for configurations (3.23) is given by $\frac{\hat{a}^2}{4} < \hat{c} \leq \frac{3\hat{a}}{4}$ and condition (3.25). The case $\hat{c} = \frac{3\hat{a}}{4}$ represents the “dS in a bottle” spacetime of section III.C. The dash-dotted line is obtained by setting $\frac{M}{M} = \frac{4}{\hat{a}}$ in equation (3.25). For other values $\frac{M}{M}$, the allowed region becomes smaller, as shown by the dashed line which represents condition (3.25) for both $\frac{M}{M} < \frac{4}{\hat{a}}$ and $\frac{M}{M} = \frac{4}{\hat{a}} > \frac{4}{\hat{a}}$. The thin dotted line indicates the points $(\hat{a}_{\text{min}}, \hat{c}_{\text{min}})$ where the time (3.26) attains its minimum values in the allowed regions for each $\frac{M}{M}$.

For a radial trajectory, we can explicitly write down the expression for the time of flight

$$
\hat{T}(\hat{a}, \hat{c}) = \frac{T(\hat{a}, \hat{c})}{T_{\text{Empty}}} = \sqrt{1 - 2\hat{a} + 2\hat{c}} \left[ \frac{4}{\hat{a}} \left( \frac{2\hat{c}}{3\hat{a}} \right)^{\frac{3}{2}} \ln \left( \frac{1 + \sqrt{\frac{3\hat{a}^2}{8\hat{c}^2}}}{1 - \sqrt{\frac{3\hat{a}^2}{8\hat{c}^2}}} \right) + 1 - \frac{4\hat{c}}{3\hat{a}} + \hat{a} \ln \left( 1 - 2\hat{a} + 2\hat{c} \right) \right] - \frac{\sqrt{2\hat{c}(1 - \frac{\hat{a}^2}{4})}}{\sqrt{1 - \frac{\hat{a}^2}{4}}} \left( \arctan \frac{1 - \hat{a}}{\sqrt{2\hat{c}^2(1 - \frac{\hat{a}^2}{4})}} - \arctan \frac{4\hat{c}}{2\hat{a}^2} \left( 1 - \frac{\hat{a}^2}{4} \right) \right).
$$

(3.26)

This is complicated to study analytically. We have therefore used a simple C++ program to compute the minimum value of $\hat{T}$ for each $\frac{M}{M}$. The results are plotted in figure 2 and show a minimum at $M/R = 2/5$ at a value of approximately 0.939.

Note that $\hat{T}_{\text{min}}(\frac{M}{M})$ decreases monotonically for $0 < \frac{M}{M} < \frac{2}{5}$. Since, in this interval, the allowed region of parameters $(\hat{a}, \hat{c})$ grows monotonically with $\frac{M}{M}$, the minimum of $\hat{T}$ for each $\frac{M}{M}$ must be attained on the boundary of the allowed region. This means that the minimum occurs where $S_{\hat{c}}' \geq |S_{\hat{c}}|$ is saturated. A similar analysis applies for $\frac{2}{5} < \frac{M}{M} < \frac{12}{17}$. 

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FIG. 2: The minimum time of flight (3.26) in “dS/DEC” configurations as a function of \( \frac{M}{R} \) is represented by the lower curve. For comparison, the upper curve shows the minimum time of flight in the (slower) “dS in a bottle” configurations as a function of \( \frac{M}{R} \).

Note that the uppermost curve (\( \hat{c} = \frac{3\hat{c}}{4} \)) in figure 1 represents the “dS in a bottle” spacetimes. Since it does not cross the middle curve showing the fastest “dS/DEC” spacetimes, we see that “dS in a bottle” is never the fastest case and we have indeed improved upon the results of section III C.

Perturbing around configurations \((c_{\text{min}}, a_{\text{min}})\) once again shows that the signaling time for this family of spacetimes can be reduced by perturbations outside of the family. Let us begin with the observation that we have already shown that the time of flight would decrease if we were allowed to move farther to the right in figure 1 for the same \(M, R\). This corresponds to a perturbation \(\delta_0\rho\) satisfying \(\delta_0\rho + \delta_0P_e = 0\) in the interior and preserving the dominant energy condition in the interior. However, it leads to a violation of the dominant energy condition at the shell. We therefore follow the strategy used in section III C of adapting this initial guess (which we call \(\delta_0\rho, \delta_0m, \delta_0T\)) to form a sequence of perturbations \((\delta_n\rho, \delta_nm, \delta_nT)\) which preserve the dominant energy condition at the shell for large enough \(n\).

This condition requires:

\[
\tilde{S}^\ell - \tilde{S}_\delta^\ell = \delta S^\ell - \delta S_\delta^\ell = \frac{1}{R^2 \sqrt{1 - 2a + 2c \delta m(R)}} \left[ \frac{1}{R \delta m'(R)} - \frac{2 - 5a + 6c}{1 - 2a + 2c} \delta m(R) \right] > 0. \quad (3.27)
\]

Each perturbation \(\delta_n\rho\) will be associated with a radius \(r_n\) such that \(\delta_n\rho = \delta_0\rho > 0\) for \(r < r_n\). We take the \(r_n\) to increase with \(n\) and to converge to \(R\). Choose some \(r_1\) and let \(\delta_1\rho\) be any such smooth perturbation which decreases for \(r_1 < r < R\). Such a \(\delta_1\rho\) will respect the dominant energy condition in the interior. For later use, we also require that the induced
change \( \delta_1 m(R) \) in the mass function just below the shell satisfy \( \delta_1 m(R) < \delta_0 m(R) \).

We now take \( \delta_n \rho \) to induce the same change in the mass just inside the shell for all \( n \): \( \delta_n m(R) = \delta_1 m(R) \). We also require \( \delta_n \rho \) to be a decreasing function of \( r \), and the sequence \( \{\delta_n \rho\} \) to have the property that \( \delta_n \rho^2 = \frac{i \phi}{4\pi r^2} \) become large and negative at \( r = R \) when \( n \) becomes large and \( r_n \to R \). Then (3.27) is clearly satisfied for large \( n \).

Since on the other hand \( \delta_n \rho(r) \to \delta_0 \rho(r) \) for \( r < R \), we find

\[
\delta_n T \to \delta_0 T + \left[ \delta_0 m(R) - \delta_1 m(R) \right] \frac{T}{R - 2a + 2c/R}.
\]

As in section III C, the first term is negative by construction, and the second term can be chosen to be arbitrarily small. Thus, we have demonstrated the existence of perturbations of the “dS/DEC” spacetime preserving the dominant energy condition and further reducing the signaling time between antipodal points.

IV. DISCUSSION

In this work we have investigated the possibility of fast travel in static spherically symmetric spacetimes. We derived a simple theorem to the effect that, when the signaling time is measured by an observer at infinity, a signal propagating through a spacetime satisfying the (timelike) weak energy condition never arrives at its destination sooner than would a corresponding signal in Minkowski space. This may be considered a non-perturbative generalization of [23]. Spherical symmetry and the static Killing field were essential in identifying a corresponding signal in Minkowski space.

However, we were not satisfied with this result and wished to investigate related questions concerning more local notions of signaling time. For example, it is of interest whether the observers who send and receive the signals find the propagation time to be less or greater than what they would expect based on their Minkowski space intuition. The theorem of section II does place a lower bound on this signaling time, but it is a bound that is arbitrarily small compared to the naive Minkowski signaling time\(^{10}\) when the signal propagates near the horizon of a black hole. We also wished to explore the consequences of requiring stronger energy conditions to hold.

\(^{10}\) i.e., a proper time of \( 2R \) for a light ray to propagate across a sphere of area \( 4\pi R^2 \).
For this reason we investigated several families of spacetimes in detail. We were most interested in cases where the dominant energy condition holds. With this restriction, we found that we could indeed construct positive energy spacetimes that improve upon the naive Minkowski time of $2R$, but only by factors of order one. Our fastest such spacetime improves this result by approximately 6%, Perturbative analysis tell us that spacetimes exist which are faster yet, but of course give us no idea of how much faster they might be. There thus remains a sizable gap\textsuperscript{11} between the fastest spacetime known to us and the bound we have derived. Discovering how this gap may be closed remains an open issue for future research, as does the exploration of other variants of Questions 1 and 2.

Perhaps the most interesting suggestion from our investigation is that imposing only the weak energy condition may allow much faster spacetimes. In particular, we found in section III C that we could construct spacetimes satisfying the weak energy condition which allowed signaling across our sphere in a proper time significantly faster than $2R$. For $2M/R \sim 1$ we found that while our signaling time was much larger than the bound of Theorem 1, it could be made arbitrarily short compared to the naive Minkowski bound.

Most of the work to date has considered the (null) weak energy condition because it leads to powerful analysis techniques based on the Raychaudhuri equation and focussing theorems. In our case\textsuperscript{11} we saw that the weak energy condition led directly to our lemma and our theorem in section II. One would expect that both of these results to generalize beyond the spherically symmetric context and to again require only the weak energy condition for their proof.

On the other hand, realistic spacetimes should also satisfy the dominant energy condition\textsuperscript{12}. Thus, our examples suggest that they should be subject to significantly stronger constraints. If this is indeed the case, new analysis tools more appropriate to the dominant energy condition will need to be constructed before one can conclusively identify the fastest DEC spacetime and the fastest allowed signaling time. We leave this task for future work.

\textsuperscript{11} When $\frac{2M}{R} \sim 1$. On the other hand, for $\frac{2M}{R} \ll 1$ the bound is of course close to the naive Minkowski estimate: $t_R^{\text{bound}} \approx 2R(1 + \mathcal{O}(M/R))$.

\textsuperscript{12} Unless one allows a negative cosmological constant.
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