Einstein, Podolsky and Rosen
versus
Bohm and Bell

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Abstract

There is an opinion that the Bohm reformulation of the EPR paradox in terms of spin variables is equivalent to the original one. In this note we show that such an opinion is not justified. We apply to the original EPR problem the method which was used by Bell for the Bohm reformulation. He has shown that correlation function of two spins cannot be represented by classical correlation of separated bounded random processes. This Bell’s theorem has been interpreted as incompatibility of local realism with quantum mechanics. We show that, in contrast to Bell’s theorem for spin or polarization correlation functions, the correlation of positions (or momenta) of two particles, depending on the rotation angles, in the original EPR model always admits a representation in the form of classical correlation of separated random processes. In this sense there exists a local realistic representation for the original EPR model but there is no such a representation for the Bohm spin reformulation of the EPR paradox. It shows also that the phenomena of quantum nonlocality is based not only on the properties of entangled states but also on the using of particular bounded observables.
In 1935 Einstein, Podolsky and Rosen (EPR) advanced an argument about incompleteness of quantum mechanics [1]. They proposed a gedanken experiment involving a system of two particles spatially separated but correlated in position and momentum and argued that two non-commuting variables (position and momentum of a particle) can have simultaneous physical reality. They concluded that the description of physical reality given by Copenhagen’s interpretation, which does not permit such a simultaneous reality, is incomplete.

Though the EPR work dealt with continuous variables most of the further activity have concentrated almost exclusively on systems of discrete spin variables following to the Bohm [2] and Bell [3] works.

Bell’s theorem [3] states that there are quantum spin correlation functions that can not be represented as classical correlation functions of separated random variables. It has been interpreted as incompatibility of the requirement of locality with the statistical predictions of quantum mechanics [3]. For a recent discussion of Bell’s theorem see, for example [4] - [12] and references therein. It is now widely accepted, as a result of Bell’s theorem and related experiments, that ”Einstein’s local realism” must be rejected. For a discussion of the role of locality in the three dimensional space see, however, [11, 12].

The original EPR system involving continuous variables has been considered by Bell in [13]. He has mentioned that if one admits ”measurement” of arbitrary ”observables” on arbitrary states than it is easy to mimic his work on spin variables (just take a two-dimensional subspace and define an analogue of spin operators). The problem which he was discussing in [13] is narrower problem, restricted to measurement of positions only, on two non-interacting spinless particles in free space. Bell used the Wigner distribution approach to quantum mechanics. The original EPR state has a nonnegative Wigner distribution. Bell argues that it gives a local, classical model of hidden variables and therefore the EPR state should not violate local realism. He then considers a state with nonpositive Wigner distribution and demonstrates that this state violates local realism.

Bell’s proof of violation of local realism in phase space has been criticized in [14] because of the use of an unnormalizable Wigner distribution. Then in [15] it was demonstrated that the Wigner function of the EPR state, though positive definite, provides an evidence of the nonlocal character of this state if one measures a parity operator.

In this note we apply to the original EPR problem the method which was used by Bell in his well known paper [3]. He has shown that the correlation function of two spins cannot be represented by classical correlations of separated bounded random variables. This Bell’s theorem has been interpreted as incompatibility of local realism with quantum mechanics. We shall show that, in contrast to Bell’s theorem for spin correlation functions, the correlation function of positions (or momenta) of two particles always admits a representation in the form of classical correlation of separated random variables. This result looks rather surprising since one thinks that the Bohm-Bell reformulation of the EPR paradox is equivalent to the original one.

In the Bohm formulation one considers a pair of spin one-half particles formed in the singlet spin state and moving freely towards two detectors. If one neglects the space part of the wave function then one has the Hilbert space $C^2 \otimes C^2$ and the quantum mechanical correlation of two spins in the singlet state $\psi_{spin} \in C^2 \otimes C^2$ is

$$D_{spin}(a, b) = \langle \psi_{spin}|\sigma(a) \otimes \sigma(b)|\psi_{spin}\rangle = -a \cdot b$$

(1)
Here $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are two unit vectors in three-dimensional space $R^3$, 
$\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, $\sigma(a) = \sum_{i=1}^3 \sigma_i a_i$ and $\psi_{\text{spin}} = |01> - |10> / \sqrt{2}$.

Bell’s theorem states that the function $D_{\text{spin}}(a, b)$ Eq. (1) can not be represented in the form

$$\int \xi_1(a, \lambda)\xi_2(b, \lambda)d\rho(\lambda)$$

i.e.

$$D_{\text{spin}}(a, b) \neq \int \xi_1(a, \lambda)\xi_2(b, \lambda)d\rho(\lambda)$$

Here $\xi_1(a, \lambda)$ and $\xi_2(b, \lambda)$ are random fields on the sphere, $|\xi_1(a, \lambda)| \leq 1$, $|\xi_2(b, \lambda)| \leq 1$ and $d\rho(\lambda)$ is a positive probability measure, $\int d\rho(\lambda) = 1$. The parameters $\lambda$ are interpreted as hidden variables in a realist theory.

The proof of the theorem is based on Bell’s (or CHSH) inequalities. Let us stress that the main point in the proof is actually not the discretness of the classical or quantum spin variables but the bound for classical random fields $|\xi_n(a, \lambda)| \leq 1, \ n = 1, 2$.

Now let us apply the similar approach to the original EPR case. The Hilbert space of two one-dimensional particles is $L^2(R) \otimes L^2(R)$ and canonical coordinates and momenta are $q_1, q_2, p_1, p_2$ which obey the commutation relations

$$[q_m, p_n] = i\delta_{mn}, \ [q_m, q_n] = 0, \ [p_m, p_n] = 0, \ m, n = 1, 2$$

The EPR paradox can be described as follows. There is such a state of two particles that by measuring $p_1$ or $q_1$ of the first particle, we can predict with certainty and without interacting with the second particle, either the value of $p_2$ or the value of $q_2$ of the second particle. In the first case $p_2$ is an element of physical reality; in the second $q_2$ is. Then, these realities must exist in the second particle before any measurement on the first particle since it is assumed that the particle are separated by a space-like interval. However the realities can not be described by quantum mechanics because they are incompatible – coordinate and momenta do not commute. So that EPR conclude that quantum mechanics is not complete. Note that the EPR state actually is not a normalized state since it is represented by the delta-function, $\psi = \delta(x_1 - x_2 - a)$.

An important point in the EPR consideration is that one can choose what we measure – either the value of $p_1$ or the value of $q_1$. For a mathematical formulation of a free choise we introduce canonical transformations of our variables:

$$q_n(\alpha) = q_n \cos \alpha - p_n \sin \alpha, \ p_n(\alpha) = q_n \sin \alpha + p_n \cos \alpha; \ n = 1, 2$$

Then one gets

$$[q_m(\alpha), p_n(\alpha)] = i\delta_{mn}; \ n = 1, 2$$

In particular one has $q_n(0) = q_n, \ q_n(3\pi/2) = p_n, \ n = 1, 2$.

Now let us consider the correlation function

$$D(\alpha_1, \alpha_2) = \langle \psi | q_1(\alpha_1) \otimes q_2(\alpha_2) | \psi \rangle$$

The correlation function $D(\alpha_1, \alpha_2)$ (7) is an analogue of the Bell correlation function $D_{\text{spin}}(a, b)$ (1). Bell in [13] has suggested to consider the correlation function of just the free evolutions of the particles at different times (see below).
We are interested in the question whether the quantum mechanical correlation function (7) can be represented in the form

$$\langle \psi | q_1(\alpha_1) \otimes q_2(\alpha_2) | \psi \rangle = \int \xi_1(\alpha_1, \lambda) \xi_2(\alpha_2, \lambda) d\rho(\lambda)$$

(8)

Here $\xi_n(\alpha_n, \lambda), n = 1, 2$ are two real functions (random processes), possibly unbounded, and $d\rho(\lambda)$ is a positive probability measure, $\int d\rho(\lambda) = 1$. The parameters $\lambda$ are interpreted as hidden variables in a realist theory.

Let us prove that there are required functions $\xi_n(\alpha_n, \lambda)$ for an arbitrary state $\psi$. We rewrite the correlation function $D(\alpha_1, \alpha_2)$ (7) in the form

$$\langle \psi | q_1(\alpha_1) \otimes q_2(\alpha_2) | \psi \rangle = <q_1q_2> \cos \alpha_1 \cos \alpha_2 - <p_1q_2> \sin \alpha_1 \cos \alpha_2$$

$$- <q_1p_2> \cos \alpha_1 \sin \alpha_2 + <p_1p_2> \sin \alpha_1 \sin \alpha_2$$

(9)

Here we use the notations as

$$<q_1q_2> = \langle \psi | q_1q_2 | \psi \rangle$$

Now let us set

$$\xi_1(\alpha_1, \lambda) = f_1(\lambda) \cos \alpha_1 - g_1(\lambda) \sin \alpha_1,$$

$$\xi_2(\alpha_2, \lambda) = f_2(\lambda) \cos \alpha_2 - g_2(\lambda) \sin \alpha_2$$

Here real functions $f_n(\lambda), g_n(\lambda), n = 1, 2$ are such that

$$Ef_1f_2 = <q_1q_2>, \quad Eg_1f_2 = <p_1q_2>, \quad Ef_1g_2 = <q_1p_2>, \quad Eg_1g_2 = <p_1p_2>$$

(10)

We use for the expectation the notations as $Ef_1f_2 = \int f_1(\lambda)f_2(\lambda)d\rho(\lambda)$. To solve the system of equations (10) we take

$$f_n(\lambda) = \sum_{\mu=1}^{2} F_{n\mu} \eta_\mu(\lambda), \quad g_n(\lambda) = \sum_{\mu=1}^{2} G_{n\mu} \eta_\mu(\lambda)$$

where $F_{n\mu}, G_{n\mu}$ are constants and $E\eta_\mu \eta_\nu = \delta_{\mu\nu}$. We denote

$$<q_1q_2> = A, \quad <p_1q_2> = B, \quad <q_1p_2> = C, \quad <p_1p_2> = D.$$ 

A solution of Eqs (10) may be given for example by

$$f_1 = A\eta_1, \quad f_2 = \eta_1, \quad g_1 = B\eta_1 + (D - \frac{BC}{A})\eta_2, \quad g_2 = \frac{C}{A} \eta_1 + \eta_2$$

Hence the representation of the quantum correlation function in terms of the separated classical random processes (8) is proved.

The condition of reality of the functions $\xi_n(\alpha_n, \lambda)$ is important. It means that the range of $\xi_n(\alpha_n, \lambda)$ is the set of eigenvalues of the operator $q_n(\alpha_n)$. If we relax this condition then one can get a hidden variable representation just by using an expansion of unity:

$$\langle \psi | q_1(\alpha_1)q_2(\alpha_2) | \psi \rangle = \sum_{\lambda} \langle \psi | q_1(\alpha_1) | \lambda \rangle \langle \lambda | q_2(\alpha_2) | \psi \rangle$$
For a discussion of this point in the context of a noncommutative spectral theory see [12]. Similarly one can prove a representation

\[ \langle \psi | q_1(t_1) \otimes q_2(t_2) | \psi \rangle = \int \xi_1(t_1, \lambda) \xi_2(t_2, \lambda) d\rho(\lambda) \]  

(11)

where \( q_n(t) = q_n + p_n t, \ n = 1, 2 \) is a free quantum evolution of the particles. It is enough to take

\[ \xi_1(\alpha_1, \lambda) = f_1(\lambda) + g_1(\lambda)t_1, \ \xi_2(\alpha_2, \lambda) = f_2(\lambda) + g_2(\lambda)t_2. \]

To summarize, it is shown in the note that, in contrast to the Bell’s theorem for the spin or polarization variables, for the original EPR correlation functions which deal with positions and momenta one can get a local realistic representation in terms of separated random processes. The representation is obtained for any state including entangled states. Therefore the original EPR model does not lead to quantum nonlocality in the sense of Bell even for entangled states. One can get quantum nonlocality in the EPR situation only if we (rather artificially) restrict ourself in the measurements with a two dimensional subspace of the infinite dimensional Hilbert space corresponding to the position or momentum observables.

If we adopt the Bell approach to the local realism (i.e. through correlation functions with separated hidden variables) then one can say that the original EPR model admits a local realistic description in contrast to what was expected for the model. It follows also that the phenomena of quantum nonlocality in the sense of Bell depends not only on the properties of entangled states but also on particular observables which we want to measure (bounded spin-like or unbounded momentum and position observables). An interrelation of the roles of entangled states and the bounded observables in considerations of local realism and quantum nonlocality deserves a further study.

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References


