Anyons, ’t Hooft loops and a Generalized Connection in Three Dimensions

N. Itzhaki

Department of Physics, Princeton University, Princeton, NJ 08544, USA

Abstract

We consider a generalized connection in three dimensions and show that it emerges in Chern-Simons-Maxwell theories when one studies the disorder instanton operator. We generalized this construction to non-Abelian theories and find that the disorder operator (the ’t Hooft operator) is equivalent to a generalized Wilson loop in a representation that depends on the Chern-Simons term. We speculate about the effective action of the disorder operator and its application to the possible phases of the theory at the infra-red. We also show that fractional statistics can emerge in gauge theories without a Chern-Simons term if the generalized connection rather than the ordinary connection is used to couple charge particles.
1 Introduction

One of the most interesting aspects of physics in three dimensions is that the linking number is well defined. As a result fractional statistics can emerge in 3D. In gauge theories this is usually realized by adding a Chern-Simons term to the action. In this paper we show that 3D gauge theories are special in another, closely related, respect: they admit a generalized connection and a generalized Wilson loop. As it turned out with the help of the generalized connection one can realize fractional statistics without introducing a Chern-Simons term. This provides a novel way to describe anyons.

The generalized Wilson line appears also in a somewhat unexpected place. In Maxwell theory the disorder operator is a singular instanton that can be thought of as the exponent of the dual scalar. The creation operator of such an instanton can be written explicitly in terms of the field strength. When one generalizes this to a Chern-Simons-Maxwell theory one finds that the disorder operator is now an open generalized Wilson line. Probably the most interesting aspect of this result is that it can be generalized to the non-Abelian SU($N$) case. In that case the disorder operator is a $Z_N$ instanton, known as the ’t Hooft point [1], that appears at the end point of a $Z_N$ Dirac particle. This operator plays a crucial role in determining the phase of the theory at the infra-red. We show that the ’t Hooft operator is equivalent to a generalized Wilson line in a representation that depends on the Chern-Simons term.

The paper is organized as follows. In section 2 we describe the generalized connection and discuss its physical meaning. The precise relation with anyons and fractional statistics is made in section 3. In section 4 we show that in Chern-Simons-Maxwell theory the disorder parameter is described by an open generalized Wilson line. The interesting fact that this construction can be generalized to the non-Abelian theory is discussed in section 5. Section 6 deals with the effective description of the disorder operator.

2 A generalized connection

In three dimensional gauge theories one can define a generalized connection in the following way

$$A_\mu^\theta \equiv A_\mu + \frac{\theta}{g^2} F_\mu, \text{ where } F_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho} F^{\nu\rho}. \quad (2.1)$$

$g$ is the coupling constant which in 3D has dimension 1/2 and $\theta$ is a dimensionless parameter that is real in Minkowski space-time and is imaginary with Euclidean signature. Under a
gauge transformation $A_\mu^\theta$ transforms like a connection for any value of $\theta$

$$A_\mu^\theta \to g^{-1}(x)A_\mu^\theta g(x) + ig^{-1}(x)\partial_\mu g(x). \quad (2.2)$$

We can consider, therefore, a new set of non-local gauge invariant operators that are natural generalizations of the Wilson loops

$$W^\theta = \text{Tr} P \exp(i \oint dx^\mu A_\mu^\theta). \quad (2.3)$$

One might wonder whether $W^\theta$ makes sense at the quantum level. The basic concern is that since $W^\theta$ involve both the gauge fields and their canonical conjugate fields, their order must be specify for the operator to be well defined. However, from the commutation relation between the gauge field and the field strength it follows that $A_i$ commutes with $F_i$ ($i = 1, 2$) and hence there is no quantum ambiguity in the definition of $W^\theta$. That also implies that, operators in which $\theta$ varies along the loop are also consistent. On the other hand $A_1^\theta$ and $A_2^\theta$ do not commute for $\theta \neq 0$. Therefore the generalized Wilson loop is a consistent operator at the quantum level only in the absence of intersections. This is reminiscent of ordinary Wilson loop in Chern-Simons theories where the fact that $A_1$ does not commute with $A_2$ implies that a Wilson loop is well defined only without intersections. As we shall see there are other similarities between the generalized Wilson lines in Maxwell theory and ordinary Wilson lines in CS theories.

Roughly speaking, in Abelian theories the physical meaning of an ordinary Wilson loop is that it creates an electric charge along the loop and it measures the magnetic flux that is passing through the surface bounded by the loop. What is the physical meaning of the generalized Wilson loop? With the help of Stokes’s theorem and the equations of motion, $\partial_\mu F_\mu^\nu = j^\nu$, the generalized Wilson line can be written as a surface integral,

$$W_P^\theta = \exp(i \oint_P dx^\mu A_\mu^\theta) = \exp i(\Phi_M + \theta Q), \quad (2.4)$$

where $\Phi_M$ is the total magnetic flux passing trough the surface and $Q$ is the total electric charge. We see that the generalized Wilson loop measures the total electric charge times $\theta$ plus the magnetic flux. Since $Q$ in an integer $W^\theta$ is a function only of $\theta \mod 2\pi$. That statement is correct only in the Abelian theory. To find what the generalized Wilson loop is creating we add $\log(W^\theta)$ as a source to the action and solve the equation of motion. In the case of

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\footnote{In non-Abelian theories one can consider a more generalized connection by taking $\theta$ to be an hermitian matrix. In $U(N)$ theories this has the effect of mixing between the $SU(N)$ and the $U(1)$ which we would like to avoid in this paper. We should however mention that such a connection might be interesting since it gives a new realization of non-Abelian anyons. In the present paper we consider only Abelian anyons (even in non-Abelian gauge theories).}

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Abelian case that is an easy task. Consider for simplicity a generalized Wilson line along a straight line in the \(x_0\) direction. We add to the Maxwell action \(\int_{-\infty}^{\infty} dx_0 A_0^\theta\) and solve the equations of motion

\[
A_0 = \frac{g^2}{2\pi} \log(r), \quad A_1 = \frac{\theta x_2}{2\pi r^2}, \quad A_2 = -\frac{\theta x_1}{2\pi r^2}, \quad r^2 = x_1^2 + x_2^2.
\] (2.5)

This background describes one unit of an electrically charge particle and an infinitely thin magnetic flux with total flux \(\theta\) both located at the origin. So the generalized Wilson loop creates a bound state of an electric charge and a magnetic flux. Such a bound state is known as an anyon.

Now consider the following generalized Wilson loop \(W^\tilde{\theta}\) that is winding around the origin and is probing the background (2.5).

\[
\exp(in(\theta + \tilde{\theta})),
\] (2.6)

where \(n\) is the winding number. It is easy to show that (2.6) holds for general loops \(P_1\) and \(P_2\) where \(n\) is the linking number. Note that the phase is not trivial for \(\tilde{\theta} = \theta\). Again the similarity with ordinary Wilson lines in Chern-Simons theory [2, 3] shows up. The origin of that similarity can be traced to the fact that

\[
\frac{k}{2\pi} \langle A_\mu(x)A_\nu(y) \rangle_{CS} = \frac{1}{g^2} \langle A_\mu(x), F_\nu(y) \rangle_{Maxwell} = i \varepsilon_{\mu\nu\rho} \frac{(x - y)_\rho}{4\pi (x - y)^3},
\] (2.7)

and hence upon integrating the exponent of (2.7) over \(P_1\) and \(P_2\) one gets the Gauss linking number

\[
\frac{i}{4\pi} \oint_{P_1} \oint_{P_2} dx^\mu dy^\nu \varepsilon_{\mu\nu\rho} \frac{(x - y)_\rho}{(x - y)^3} = n(P_1, P_2),
\] (2.8)

that is known to be an integer as long as \(P_1\) and \(P_2\) do not intersect. In the next section we elaborate further on that relation.
3 Anyons

As it is well known physics in three dimensions is special since it allows for the possibility of particles with any statistics [4]. The reason is that in three dimensions the number of times that a particle winds around another particle as the two evolve from past infinity to future infinity is well defined. Therefore, it is meaningful to view particles interchange as "half-winding" and to determine their statistics to be one-half of the phase that is picked up when one particle winds once around the other. A concrete way to realize non-trivial statistics is by attaching a magnetic flux, $\Phi$, to electrically charge particles. The Aharonov-Bohm effect then implies a shift, that is linear in $\Phi$, in the statistics of the particles. The question is how to attach a magnetic flux to the electrons. The simplest way considered so far is via a Chern-Simons theory. As the discussion in the previous section suggests exactly that effect can be achieved also in the context of pure Maxwell theory without a Chern-Simons term but with a generalized connection. Before we elaborate on this let us review the CS construction. The relevant Lagrangian is

$$\mathcal{L}(\alpha) = j^\mu A_\mu - \frac{\alpha}{2} \epsilon^{\nu\rho\sigma} A_\nu \partial_\rho A_\sigma,$$  

(3.1)

where $j^\mu$ is the current. The first term is the standard current gauge field coupling the second term is the Abelian Chern-Simons term. We do not write down the pure matter field term as it plays no role here. From the zero component of the equation of motion for $A_\mu$ one finds that the total number of charged particles $N$ and the total magnetic flux are related

$$N = \alpha \Phi.$$  

(3.2)

This implies that a magnetic flux is attached to every charged particle. The shift in the naive statistics of the particle is fixed by the Aharonov-Bohm effect to be

$$\Delta \phi = 1/2\alpha.$$  

(3.3)

Notice that because of the CS term in (3.1) the phase (3.3) is only one-half the Aharonov-Bohm phase.

We would like to show now that eq.(3.2) can be obtained without a Chern-Simons term but with the help of the generalized connection. Consider the following Lagrangian

$$\mathcal{L}(\theta) = -\frac{1}{4g^2} F^2 + j^\mu A_\mu^\theta,$$  

(3.4)

To be more precise the statistics is determined in that way only up to a minus sign. For example, there are two ways to realize no-phase: $1^2 = 1$ that corresponds to bosons, and $(-1)^2 = 1$ that corresponds to fermions. In other words, the fact that in pure Maxwell theory in three dimensions an ordinary Wilson loop does not pick up a topological phase while winding around another ordinary Wilson loop does not tell us whether the source for the Wilson loop is a charged Fermion or a charged Boson.
which describes Maxwell theory that couples to the current via the generalized connection rather than the usual connection. Note that this coupling is gauge invariant as long as \( j^\mu \) is a conserved current. Notice farther that the \( \epsilon_{\mu\rho\sigma} \) in the generalized connection leads to time reversal and parity violation which is crucial to realize fractional statistics. The equation of motions for \( A_\mu \) give

\[
\Box A_\mu = g^2 J_\mu + \theta \epsilon_{\mu\nu\rho} \partial_\nu J^\rho.
\]  

(3.5)

Consider now a static point like particle located at the origin. Such a particle is described by \( j^0 = \delta(x) \). Plugging this into eq.(3.5) and solving for the gauge fields one finds (2.5) which implies that the magnetic flux attached to \( N \) electrons is

\[
N = \frac{\Phi}{\theta},
\]

(3.6)

and that the shift in the statistics of the particle is

\[
\Delta \phi = \theta.
\]

(3.7)

Eqs.(3.6,3.7) differ from eqs. (3.2,3.3) by a factor of 2. The reason is that with the generalized connection construction a long range electric field is also generated (see eq.(2.5)) which couples to the current and gives exactly the same phase. Simply put in the CS case the phase is due to a charge particle winding around a magnetic flux while here we also have the contribution of a flux tube winding around the charged particle.

It is appropriate, therefore, to call the generalized connection an anyonic connection for it generates the parallel transport of an anyon. This can be phrased in terms of gauge invariant operators. The gauge invariant operator that creates an anyon at some point \( x \) propagates it along a certain path to point \( y \) where it is annihilated is

\[
\Psi(x) P \left\{ \exp \left( i \int_x^y dx^\mu A_\mu^\theta \right) \right\} \bar{\Psi}(y).
\]

(3.8)

Even though both \( \mathcal{L}(\alpha) \) and \( \mathcal{L}(\theta) \) lead to fractional statistics by the same mechanism of attaching a magnetic flux to the electrons the physics they describe is quite different. \( \mathcal{L}(\alpha) \) is a super-renormalizable theory while, at least naive power counting suggest that, \( \mathcal{L}(\theta) \) is a non-renormalizable theory that makes sense at the quantum level only with a cutoff. The infra-red is also quit different. In \( \mathcal{L}(\alpha) \) there are no dynamical degrees of freedom associated with the gauge fields. The sole function of the gauge fields is to attach a magnetic flux to the electrons. On the other hand there certainly are gauge degrees of freedom in \( \mathcal{L}(\theta) \). Moreover, in the absence of a CS term the gauge fields are massless and can widely fluctuate at the infra-red. If one’s goal is to suppress these fluctuations then one can, for example,
consider a CS theory that couples to the current via the generalized connection rather than the usual connection. This have the interesting effect that the previously singular magnetic flux tube is smeared. As a result the mean field theory approximation used to study anyons superconductivity [5] is very likely to be under a better control, possibly even away from the almost Fermions limit studied in details in [6].

4 Disorder operator in Maxwell-Chern-Simons theory

In Maxwell theory (without a Chern-Simons term) one can defined a local disorder operator

\[ \eta(x) = \exp \left( \frac{2\pi i}{g^2} \int_x^\infty dx^\mu F_\mu \right). \]  

(4.1)

To show that this is a local operator one needs to show that \( \eta(x) \) does not depend on the path from \( x \) to \( \infty \). In other words a closed loop should be trivial. With the help of Stokes’s theorem and the equation of motions we find that

\[ \exp \left( \frac{2\pi i}{g^2} \oint dx^\mu F_\mu \right) = \exp(2\pi iQ) = 1, \]  

(4.2)

where \( Q \) is the total charge passing through the surface. \( \eta(x) \) is the creation operator of a Dirac magnetic instanton at \( x \) with a Dirac particle (the three dimensional analog of the Dirac string) along the path from \( x \) to infinity. The statement that this is a local operator is nothing but the statement that with the appropriate quantization condition the Dirac particle is not a physical observable. In terms of the disorder operator the dual scalar is defined as

\[ \phi(x) = \frac{g^2}{2\pi i} \log(\eta(x)), \]  

(4.3)

which with the help of eq.(4.1) can be written in a more familiar form \( \partial_\mu \phi = F_\mu \).

Let us generalize the construction of the disorder operator to Chern-Simons-Maxwell theory. The Lagrangian is

\[ \mathcal{L} = -\frac{1}{4g^2} F^2 + \frac{k}{4\pi} F^\mu A_\mu. \]  

(4.4)

If we use the same definition for \( \eta(x) \) we find that since the equations of motions are modified by the CS term, \( \partial_\mu F^{\mu\nu} = j^\nu - \frac{k}{2\pi} F^\nu \), a closed loop gives

\[ \exp \left( \frac{2\pi i}{g^2} \oint dx^\mu F_\mu \right) = \exp \left( 2\pi i (Q - \frac{k}{2\pi} \Phi_B) \right) = \exp(-ik\Phi_B). \]  

(4.5)
As $\Phi_B$ is not quantized this is not a trivial operator and hence $\eta(x)$ is not a local operator. The following simple modification fixes that problem

$$\eta(x) = \exp\left(i \int_x^\infty dx^\mu (k A_\mu + \frac{2\pi}{g^2} F_\mu)\right). \quad (4.6)$$

One can easily verify that with the additional term the closed loop is trivial. Note that eq.(4.6) can be written as a generalized Wilson line

$$\eta(x) = \exp(ik \int_x^\infty dx^\mu A_\mu^{2\pi}), \quad (4.7)$$

which suggests that, as we shall see in the next section, a non-Abelian generalization should be fairly straightforward.

An interesting by product of that discussion is the following. Typically in gauge theories a charged field is not gauge invariant. Therefore, if we want to describe the creation operator of an electron at some point $x$ in a gauge invariant fashion we must attach a Wilson line to the field $\Psi(x) \exp(i \int_x^\infty dx^\mu A_\mu)$. That operator, however, is non-local as it depends on the path. So one can either work with a non gauge invariant operator or with a gauge invariant operator that is not unique. In a Chern-Simons theory with $k = 1$ one can define a gauge invariant operator that is also local $\Psi(x) \exp(i \int_x^\infty dx^\mu A_\mu^{2\pi})$. Note that this operator does not create only an electron but rather an electron-disorder bound state.

We close this section with a comment about the dual scalar in that case. Since (4.6) is a local operator it can be used to define the dual scalar. Following (4.3) we get

$$\partial_\mu \phi = F_\mu + \frac{g^2k}{2\pi} A_\mu. \quad (4.8)$$

The fact that for $k \neq 0$ the connection appears in that relation implies that $\phi$ is not gauge invariant. In fact under a gauge transformation $A \rightarrow A + d\Lambda$ the transformation of the dual scalar is $\phi(x) \rightarrow \phi(x) + \frac{g^2k}{2\pi} \Lambda(x)$. Therefore, without gauge fixing the only gauge invariant action that includes only $\phi$ is a constant. With partial gauge fixing it is possible that a dual scalar action can be constructed. For example in the Lorentz gauge $\partial_\mu A^\mu = 0$ the residual gauge symmetry is $\square \Lambda = 0$ and so $\tilde{\phi} = \square \phi$ is a scalar that is invariant under the residual gauge symmetry and it can be used to construct a non-trivial action.

5 Disorder operator in Non-Abelian theories

In this section we generalize the construction of the disorder operator to non-Abelian Yang-Mills-Chern-Simons theories. The construction in the previous section in the Abelian case...
relays on Stokes’s theorem and hence it cannot be generalized to non-Abelian theories. A different approach was used by Moore and Seiberg [7] to show that in pure CS theories a ’t Hooft loop is equivalent to a Wilson loop. Below we generalize their results and show that with a Yang-Mills term a ’t Hooft loop is equivalent to a generalized Wilson loop. First let us apply that approach to the Abelian case and rederive (4.6). The Maxwell-Chern-Simons action is invariant under a continuous gauge transformations. Singular gauge transformation, however, can define an operator. Consider a closed loop, $C$, that is parameterized by $\tau$ and perform a gauge transformation that near the loop takes the form $\Lambda(\phi) = c\phi$ where $\phi$ is the angular variable that winds the loop. The effect of such a singular gauge transformation is to create a magnetic flux tube along the loop with total magnetic flux $2\pi c$. As a result the action is not invariant under such a singular gauge transformation

$$S \to S + 2\pi c \oint_C d\alpha (kA_\mu + \frac{2\pi}{g^2} F_\mu).$$

(5.1)

Which implies that in the path integral this singular gauge transformation is equivalent to adding the operator

$$\exp \left( i2\pi c \oint_C d\alpha (kA_\mu + \frac{2\pi}{g^2} F_\mu) \right).$$

(5.2)

For $c = 1$ (or any other integer) the singular gauge transformation cannot be detected by any of the charged fields and hence $\eta(x)$ as defined in eq.(4.6) is a local operator.

Consider $SU(N)$ gauge theory (or $SU(N)/Z_N$ to be more precise) on $R^3$ and suppose that all the fields in the theory are in representations of the gauge group that is invariant under the center (for example all fields are in the adjoint representation). In that case all fields are invariant under a gauge transformation that near the loop takes the form

$$U(\phi) = \exp(i\frac{\phi}{N} T), \quad T = \text{diag}(1, 1, ..., 1, 1 - N),$$

(5.3)

since $U(2\pi) = e^{2\pi i/N}$ is in the center. We can, therefore, follow the same steps that led to (5.2) and find a trivial loop operator in $SU(N)/Z_N$ theories that in turn can be used to construct the disorder operator (which in that case is known as the ’t Hooft point [1]). The first step is to fix the normalization to be the standard ones, $\text{Tr} T_a T_b = \frac{1}{2} \delta_{ab}$. With that normalization the singular gauge transformation, eq. (5.3), takes the form $U(\phi) = \exp(i\phi \sqrt{2(N - 1)/N} H_{N-1})$ where $H_{N-1} = \sqrt{1/2(N - 1)N}\text{diag}(1, 1, ..., 1 - N)$ is the $N-1$ generator in the Cartan subalgebra (we are using the notation of [8]). The effect of such a singular gauge transformation is to create a magnetic flux tube along the loop that points in the $H_{N-1}$ direction. That in turn yields the following shift in the action

$$\delta S = k\sqrt{\frac{N - 1}{2N}} \oint_C d\alpha \text{Tr} \left( H_{N-1}(A_\mu + \frac{2\pi}{k} F_\mu) \right) = k\sqrt{\frac{N - 1}{2N}} \oint_C d\alpha \text{Tr}(H_{N-1}A^2_{\mu}/k).$$

(5.4)
Just like in the Abelian case we conclude that the singular gauge transformation along the loop is the exponent of $\delta S$. There is, however, an important difference. Eq. (5.2) is gauge invariant while the non-Abelian generalization, $\exp(i\delta S)$, is not since it points in a certain direction in the gauge group. To find the gauge invariant operator that is equivalent to the 't Hooft loop we need to perform integration over smooth gauge transformation along the loop. Here is where the fact that we can express eq.(5.4) in terms of the generalized connection is useful. The transformation of $A^{2\pi/k}_\mu$ under a gauge transformation $g(\tau)$ (recall that $\tau$ is the parameter along the loop) is $A^{2\pi/k}_\mu \rightarrow g(\tau)^{-1}A^{2\pi/k}_\mu g(\tau) + ig(\tau)^{-1}\partial_\mu g(\tau)$ and thus averaging $\exp(i\delta S)$ over the gauge group gives

$$
\int Dg(\tau) \exp \left( k\sqrt{\frac{N-1}{2N}} i \oint_C dx^\mu \text{Tr}[H_{N-1}(g(\tau)^{-1}A^{2\pi/k}_\mu g(\tau) + ig(\tau)^{-1}\partial_\mu g(\tau))] \right),
$$

which describes the quantum mechanics of the coadjoint orbit. Exactly that problem was studied in [9, 10] where it was found that

$$
\int Dg(\tau) \exp \left( i \oint_C dx^\mu \text{Tr}[l_iH_i(g^{-1}A_\mu g + ig^{-1}\partial_\mu g)] \right) = \frac{1}{d(r)} \text{Tr}_r \exp(i \oint_C dx^\mu A_\mu).
$$

$H_i$ are the Cartan subalgebra generators and the vector $l_i$ is the highest weight of the representation $r$ whose dimension is $d(r)$. Both sides of (5.6) make sense only if $l_i$ is the highest weigh of some representation. The simplest case is $SU(2)$ where there is only one generator in the Cartan subalgebra that can be chosen to be $\tau_3$. Then eq.(5.6) takes the form $W_J = \int Dg(\tau) \exp(iJ \oint dx^\mu \text{Tr}[\tau_3(g^{-1}A_\mu g + ig^{-1}\partial_\mu g)])$.

Before we apply (5.6) to the 't Hooft loop calculation we would like to provide an argument why eq.(5.6), that at first looks very surprising, is actually quit intuitive. For a detailed proof see [9, 10]. Consider a Wilson loop in $SU(2)$ Yang-Mills theory in the fundamental in arbitrary dimensions. One way to think about the Wilson loop is as the exponent of the phase that a quark picks up along the loop which is the right hand side of (5.6). An alternative way to think about the Wilson loop is the following. Let us cut a small sphere, $S^{d-2}$, around the loop so that now there is a boundary with topology $S^1 \times S^{d-2}$. Integration by part implies that the boundary term is

$$
\int_{S^1 \times S^{d-2}} \text{Tr}(A \wedge *F).
$$

The fact that we have a charged field in the fundamental representation along the loop means that $\int_{S^{d-2}} *F$ is "one" pointing in some direction in the gauge group, say, $\tau_3$. And so upon averaging over the gauge transformation on the loop the boundary term gives the left hand side of (5.6).
To complete the ’t Hooft loop operator calculations we notice that for \( k = 1 \) the \( \ell_i \) in eq.(5.5) agree with the highest weight in the fundamental representation and so we conclude that in that case a ’t Hooft loop is equivalent to a generalized Wilson loop in the fundamental representation

\[ T(C) = W^{2\pi}_{\mathbb{N}}(C). \quad (5.8) \]

For \( k > 1 \) we get

\[ T(C) = W^{2\pi}_{\text{sym} k}(C), \quad (5.9) \]

where by \( \text{sym} k \) we mean that the trace is taken in the symmetric piece of \( \mathbb{N} \otimes \mathbb{N} \otimes \ldots \otimes \mathbb{N} \).

To define the disorder operator we need, like in the Abelian case, to consider open ’t Hooft lines. We denote by \( t(x) \) the field that describes the disorder ’t Hooft point that is located at the end point of the open ’t Hooft line. Eq. (5.9) implies that \( t(x) \) must be in the \( \text{sym} k \) representation. One interesting aspect of that result is the following. We started by defining a ’t Hooft loop which is a singular gauge transformation that cannot be detected by any of the fields in the action. In terms of Wilson loops what this means is that any Wilson loop, ordinary or generalized, in a representation of the gauge group in which the center of the gauge group acts trivially cannot detect the ’t Hooft loop. That is to say that its expectation value is the same whether or not it is winding a ’t Hooft loop. The expectation value of Wilson loops in other representations that are sensitive to the center of the gauge group, for example the fundamental representation, depends on whether or not a ’t Hooft loop is present. But we do not worry about them since there are no dynamical fields in these representations. We are, therefore, tempted to claim that the ’t Hooft loop is a trivial operator in such theories and that the ’t Hooft point is a local operator. However, we have just showed that the ’t Hooft loop is equivalent to a generalized Wilson loop that is not necessarily in a representation that leaves the center intact. To be more precise for

\[ k \mod \mathbb{N} \neq 0, \quad (5.10) \]

a ’t Hooft loop is a generalized Wilson loop in a representation that is sensitive to the center. Therefore in such cases a ’t Hooft loop can be detected by another ’t Hooft loop. In fact, a ’t Hooft loop will pick up a non-trivial phase when going around another ’t Hooft loop. What this means is that \( t(x) \) is an anyon. It is an easy task to find its fractional statistics. By construction a ’t Hooft loop creates a singular gauge transformation with

\[ U(2\pi) = e^{2\pi i/N}. \]

A generalized (or ordinary) Wilson loop, in the \( \text{sym} k \) representation that winds once around that loop will pick up a phase \( (U(2\pi))^k = e^{2\pi ki/N} \) and hence the fractional
statistics associated with the ’t Hooft operator is

$$\delta \mod \pi = \pi \frac{k}{N},$$  \hspace{1cm} (5.11)

where the factor of 1/2 is due to the fact that the statistics is determined to be one half the phase we get by winding once and we mod by $\pi$ since when determining the statistics that way there is a sign ambiguity (see footnote 2).

In we take $g \rightarrow \infty$ the relevant piece of the action is the CS term and the generalized Wilson loop becomes just an ordinary Wilson loop. Thus eq.(5.11) should agree with Witten’s results on Wilson loop in pure CS theories [3]. For example he found that the fractional statistics of a Wilson loop in the fundamental representation is $(N^2 - 1)/2N(N + k)$. For $k = 1$ this should agree with (5.11) since for $k = 1$ the ’t Hooft loop is a generalized Wilson loop in the fundamental. Indeed it does. For $k = 2$ the ’t Hooft loop is a generalized Wilson loop in the symmetric piece of $N \otimes N$. For that representation Witten found $(N^2 + N - 2)/N(N + k)$ that for $k = 2$ agrees with eq.(5.11). To avoid fractional statistics at all (including Fermions) $k/2N$ must be an integer. This agrees with [7] where is was argued, based on the relation between pure CS and WZW, that for $SO(3) = SU(2)/Z_2$ theory to avoid the analog of a fractional statistics in WZW theories $k/4$ must be an integer.

There is an important case for which the ’t Hooft loop is not equivalent to a generalized Wilson loop in some representation. When $k = 0$, that is pure YM theory, eq.(5.5) makes no sense and we have to take a few steps back to realize that formally the ’t Hooft loop can be written in that case in the following way

$$T(C) = \int Dg(\tau) \exp \left( \frac{\pi}{2(N - 1)} \int_C dx^\mu Tr[H_{N-1}(g(\tau))^{-1} F_{\mu} g(\tau)] \right).$$  \hspace{1cm} (5.12)

The absence of a kinetic term for $g(\tau)$ makes it hard to put this integral in a useful form. Nevertheless in the next section we shall see that much can be said about the effective action of the ’t Hooft operator in that case.

6 Effective description

One might hope that the fact that the ’t Hooft loop is equivalent to a generalized Wilson loop can be used to make some precise statements, that can be supported by solid calculations, about the infra-red dynamics as a function of $N$ and $k$. In the lack of any concrete progress in that direction we discuss in that section some possible effective actions for the ’t Hooft loop and speculate about the infra-red dynamics. First we review and elaborate a bit on
't Hooft construction of the effective action for the pure Yang-Mills theory [1] and then we consider the $k \neq 0$ case.

6.1 $k = 0$

At the ultra-violet the disorder operators plays no significant role as the theory is well described in terms of the classical Yang-Mills action. At the infra red it is conceivable that the disorder operator takes over in the sense that the main features of the theory at the deep infra red are described in terms of an effective field theory of the disorder operator. This is the case in some exactly solvable two dimensional models.

In [1] 't Hooft suggested a simple effective action for the disorder parameter that yields condensation of the disorder parameter and admits confinement. Consider Yang Mills theory with no Chern Simons term and assume that other matter fields in the theory, if present, are in a representation of the gauge group that does not transform under the center, so that the 't Hooft operator is a local operator. In the absence of a CS term the 't Hooft field, $t(x)$, is a chargeless scalar (for $k = 0$ there is no fractional statistics). Nevertheless it is a complex field as (for $N > 2$) one can distinguish between a $Z_N$ instanton and an anti-instanton. For example a non-trivial operator in the theory is $\langle t(x) t^\dagger(y) \rangle$ that describes the instanton anti-instanton two point function. Since $\pi_1(SU(N)/Z_N) = Z_N$ if we have $N$ 't Hooft lines they can be annihilated and hence the instanton charge is conserved modulo $N$. Therefore, in the effective action for $t$ there must be a $t^N$ vertex. The simplest effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = \partial_\mu t \partial^\mu t^\dagger - m^2 tt^\dagger - V(tt^\dagger) - i\lambda(t^N - t^{iN}).$$

(6.1)

The action is invariant under the $Z_N$ symmetry $t \rightarrow e^{2\pi i/N} t$.

The sign of the mass term is a dynamical question whose answer depends on $N$ and the matter fields. Let us assume that it is such that $t$ condenses which in turn means that the $Z_N$ symmetry is broken. The condensation of the disorder parameter should lead to confinement in the order parameter via the dual Meissner effect. If so then where are the closed strings associated with the electric flux tubes? Since a discrete symmetry is broken there are domain wall solutions that interpolate between two vacua and since we are in three dimensions the domain wall is a string. The way to see that these strings are indeed the electric flux tube associated with the insertion of a Wilson loop is the following. In the Hamiltonian formalism a Wilson loop in the fundamental does not commute with $t(x)$ if the Wilson loop winds around $x$. That is simply the Hamiltonian formalism analog of the statement that when a Wilson loop winds around a 't Hooft loop it picks up a $Z_N$ phase.
That phase can be interpreted, in the Hamiltonian formalism, as shifting $t$ by a $Z_N$ phase inside the Wilson loop relative to the outside of the loop which is exactly what the domain wall solution does. The glue ball spectrum can be calculated from fluctuations of the domain wall strings.

The way these domain wall closed strings emerge from the gauge theory is a bit different than in the string theory case. Indeed, the closed strings are dual to Wilson loops as was argued on general grounds in [11] and demonstrated in the context of the AdS/CFT duality in [12, 13], but here the strings are not the fundamental degrees of freedom in the dual description. Here the dual description is in terms of the ’t Hooft operator and the strings emerge as domain wall solutions. In particular, they propagate in the same number of dimensions as the gauge fields.

What happens when we introduce quarks? We expect to see now open strings with a quark at one end and an anti quark at the other end. How to see this at the level of the operators was explained by ’t Hooft. Here we suggest a way to think about this at the level of the effective action.

When we have fields in the fundamental (or any other representation that transforms non-trivially under the center of the gauge group) the ’t Hooft loop can be detected by the quarks which means that $t(x)$ is no longer a local operator but it depends on the path. For example $\langle t(x)t^\dagger(y) \rangle$ does not seem to make sense any more without specifying the path from $x$ to $y$. Does this means that in the presence of quarks there is no local effective description in terms of the ’t Hooft operator? In other words, should we conclude that the effective action $L_{\text{eff}}$ cannot be modify in a local fashion that captures the effect of the quarks?

Not necessarily. The fact that an operator like $\langle t(x)t^\dagger(y) \rangle$ depends on the path between $x$ and $y$ and not only on the end points is very familiar from gauge theories. There, a meaningful electron positron operator will take the form $\phi(x)\exp(i\int_x^y dx^\mu A_\mu)\phi^\dagger(y)$. Clearly, that operator depends on the path. What this is suggesting is that from the point of view of the disorder operator the quarks are like gauge fields. That is, the dual effect of adding quarks to Yang Mills is to add a gauge symmetry to $L_{\text{eff}}$. That gauge symmetry should not be confused with the original $SU(N)$ gauge symmetry. In fact, as we show below, a $U(1)$ gauge symmetry is sufficient. On general grounds we expect the coupling of $t(x)$ to the new gauge field to increase as we decrease the quark’s mass. The reason being that if we have a very heavy quark then the fact that it is sensitive to the ’t Hooft loop have a small effect in the path integral. Hence $\langle t(x)t^\dagger(y) \rangle$ depends weakly on the path. In the gauge theory analogy that means that the coupling constant is small and hence the effect of the Wilson line between the electron and the positron is also small.
$t$ is a complex scalar, thus the simplest modification is to leave it as it is and to couple it to a $U(1)$ gauge field, $\tilde{A}_\mu$. We also want to add a Maxwell term for the gauge field, so that the modified action takes the form

$$L_{\text{eff}} = -\frac{1}{4} F^2 + D_\mu t D^\mu t^\dagger - m^2 t t^\dagger - V(t N - t^\dagger N), \quad D_\mu = \partial_\mu + ig\tilde{A}_\mu. \quad (6.2)$$

The last term in the action is not invariant under the gauge transformation. Thus the gauge transformation is only an approximation that becomes better and better as we flow to the infra-red since (for $N > 2$) the last term is an irrelevant term. As we now show the approximate gauge symmetry is sufficient to yield the correct infra red dynamics.

If we set $\lambda$ to zero then there is a dual Nielsen Olesen solution where $t$ gets expectation value that at infinity goes to

$$t(\theta) = t_0 e^{i\theta}. \quad (6.3)$$

The reason why a soliton with such a boundary condition costs only a finite amount of energy is that this boundary condition can be obtained as a gauge transformation from $t(x) = t_0$ and so there is no divergence coming from spatial infinity. When $\lambda > 0$ a configuration with boundary condition (6.3) will cost an infinite amount of energy since the last term creates a sine-Gordon like potential in the $\theta$ direction. To wind once at infinity with a finite energy we have to consider a more general asymptotic behavior

$$t(\theta, r) = t_0 e^{if(\theta, r)}, \quad \text{with} \quad f(0, r) = 0, \quad f(2\pi, r) = 2\pi. \quad (6.4)$$

To minimize the effect of the sine-Gordon like potential $f(\theta)$ has to ”jumps” from one minimum of $t N = t_0 N$ to the other. So that all together there are $N$ ”jumps”. For any finite $r$ these jumps are continuous in $\theta$. In the limit $r \to \infty$ the jumps are discontinuous in $\theta$. Normally, that would cost a large amount of energy from the kinetic term of $t$ but since for any $f(\theta)$ eq.(6.3) can be obtained as a gauge transformation of $t = t_0$ we need not worry about infinities coming from the kinetic terms. We should be a bit more careful here. The gauge symmetry is only an approximated gauge symmetry. That means that acting with a gauge transformation on $t = t_0$ the contribution coming from all terms in (6.2) but the last one is zero no matter what $f$ is. The contribution of the last term depends on $f$ and with the $f$ describes above it is finite. Simply put, one can construct a finite energy configuration that winds one at infinity by appropriate choice of $f(\theta, r)$. The actual solution, which is quite hard to find as it depends both on $r$ and $\theta$, will have lower energy than this finite energy configuration and so the soliton solution we are after must exist. Since that solution makes $N$ ”jumps” it should be associated with a baryon. The baryon mass scales like $\sim 1/\tilde{g}^2$ and as anticipated it decreases as we increase the dual coupling constant.
One might attempt to describe a single quark by taking only \(1/N\) the winding of eq. (6.4),
\[
t(\theta, r) = t_0 e^{i f(\theta, r)}, \quad \text{with} \quad f(0, r) = 0, \quad f(2\pi, r) = 2\pi/N.
\]
That is, however, not a consistent boundary condition as \(t\) does not go back to itself, \(t(0) \neq t(2\pi)\). This is expected since in the phase where the ’t Hooft operator condenses one should not find an isolated quark. On the other hand if there is another solution with the exact same jump in \(t\) then the two can be glued together to cancel the discontinuity. This is exactly what the domain-wall string solution does and hence we conclude that the domain-wall string can end on a quark. A heuristic form of the solution that describes an open string with a quark on one end and an anti-quark on the other end is

\[
t = t_0 e^{i f(\theta_1, r)/N} \quad \text{with} \quad t = t_0 e^{2\pi i/N} \quad \text{and} \quad t = t_0 e^{i(2\pi - f(\theta_2, r))/N}
\]

Note that whether we have a quark or an anti-quark at the, say, left end is a matter of notation. However, if we have a quark on one side we must have an anti-quark on the other side. We see that also measons can be described in terms of soliton solutions. In particular, one should be able to find a solution that corresponds to the Regge trajectory where the string tension is balanced by the centrifugal force. Note that unlike in the Skyrme model [14, 15], where the baryons are solitons with respect to the measons fields, here the measons and baryons are on equal footing, both are described as soliton solutions.

If we add more then one quark the natural modification seems to involve several \(U(1)\) gauge fields, one for each quark so that the gauge group is \(U(1)^{N_f}\) and measons with different flavor can be constructed as solitons. The idea that fields in the fundamental “transform” into gauge fields in the dual description is reminiscent of Seiberg duality [16] where an \(SU(N_c)\) gauge theory with \(N_f\) quarks in the fundamental is dual at the infra-red to a gauge theory with \(SU(N_f - N_c)\) (with a small modification this is correct also in three dimensions [17]) which implies that the dual gauge group grows with \(N_f\) much like in the case considered here.

6.2 \(k \neq 0\)

It is tempting to apply a similar approach to Chern-Simons-Yang-Mills theory. However, as we will now see the effective description in terms of the disorder operator in that case is
much more complicated and it is less clear if it yields reasonable results.

For $k \neq 0$ the gauge field acquire classically a mass $\sim g^2 |k|$ [18]. Since the three-dimensional analog of $\Lambda_{QCD}$ is $g^2 N$, for $k \gg N$ the mass of the gauge fields is larger than $\Lambda_{QCD}$ and the classical mass gap can be trusted to conclude that the theory is in a Higgs like phase. The interesting question is what is happening when $k$ is of the order or smaller than $N$. In that case we enter the strong coupling region before reaching the scale sets by the photon mass and the ’t Hooft operator is expected to play an important role. The fact that for $k \neq 0$ the ’t Hooft loop is charged under the gauge group and it also has fractional statistics suggests that, at the level of effective description, thinks are much more complicated than in the pure Yang-Mills case. In particular, there is no hope to find a local effective action of $t(x)$ without including gauge fields as well.

For $k = 1$ we saw in the previous section that $t$ is in the fundamental representation of the gauge group. We also saw that it has fractional statistics $\pi/N$. The simplest local effective action one can write that yield the correct fractional statistics

$$L_{eff} = CS + D_\mu t D^\mu t^\dagger - V(tt^\dagger), \quad (6.6)$$

where $CS$ is the Chern-Simons term with $k = 1$ that makes sure that $t$ has the correct fractional statistics. Note that eventhough $t$ has fractional statistics it can condenses as it is clear from (6.6). The reason being that the fractional statistics comes from attaching a magnetic flux to charge particles. But there is no charge density when $\langle t(x) \rangle = \text{const}$. In other words, condensation of $t$ gives an anyonic plasma rather than anyonic gas.

This effective action is the simplest also "conceptually". At the ultra-violet we describe the system via the classical Chern-Simons-Yang-Mills action. Semi-classically the CS term is relevant compared to the YM term and so by assumption it is kept in $L_{eff}$. The YM term is replaced, much like in the pure Yang-Mills case, by the disorder terms. Therefore, by assuming that form of the effective action in a sense we have already assumed that noting dramatic is going to happen in the infra-red. That is, the theory is in the Higgs phase. Indeed, since $t$ is charged its condensation implies screening. To be more precise the smallest mass for the gauge field one finds when $t$ condences scales like $\sim |t_0|^2$ [19], and so we are in the Higgs phase.

For $k > 1$ we found that $t$ is in the sym$k$ representation of the gauge group so it has charge $k$ under the center of the gauge group. This suggests that condensation of $t$ leads to a phase in which all Wilson loops in the $N^l$ representation for $l < k$ admit an area law while a Wilson loop in the $N^k$ representation is screened. For example, if $k = N = 2$ then $t$ is in the adjoint and its condensation implies screening of adjoint charges, but much like in [20],
instantonic effects induces an area law for a Wilson loop in the fundamental.

7 Discussion

In some theories in two dimensions the order disorder map is a powerful tool to study non-perturbative aspects. In the 2D Ising model for example, the simplest description, in which it is most transparent that the theory is dual to a free fermion, is in terms of a field that involve both the order and the disorder parameter. In that respect it is certainly encouraging that in Chern-Simons-Yang-Mills theory the disorder operator, the ’t Hooft loop, is equivalent to a generalized Wilson line that can be described in terms of the order parameter (the connection) and its derivatives. So far, however, we did not find a way to invert this observation into a precise progress in the understanding of non-perturbative aspects of gauge theories in 3D. For example, it would be very nice to actually show that the ’t Hooft operator condenses (or not). It might be that the findings in this paper should be combined with some other technique to yield concrete progress. One such possibility is to take \( N \) to be large. The motivation here is not so much the connection with string theory, as much as to tune the fractional statistics of the ’t Hooft operator to be almost Fermionic or almost Bosonic by an appropriate choice of \( k \). In the case of anyonic gas that was proven to be useful to improve the validity of the mean field theory approximation [6]. It should be interesting to see whether it is helpful in the present case as well.

A different approach that is worthwhile to explore is to add supersymmetry. The disorder operator discussed here is most relevant for mirror symmetry in three dimensions [21]. Its crucial role in mirror symmetry was emphasized in [22] for \( U(1) \) and \( SU(N) \) broken down to \( U(1)^{N-1} \). In fact, in the \( U(1) \) case it can be used to make some non-trivial relations between the mirror theories away from the deep infra-red [23] (for recent papers see [24, 25]). We hope that the results of the present paper will shead some light on mirror symmetry when the gauge group is not broken. In particular, with enough supersymmetry there are scalars in the adjoint. In that case, one can consider a more general connection that contains covariant derivatives of the scalars ontop of the dual field strength in such a way that some supersymmetry is preserved. It seems reasonable that such an operator will play a role in the non-Abelian mirror symmetry transformation.

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