1. INTRODUCTION

We evaluate the decay rate of the unstopped accelerated proton.

A analytic expression of the decay rate for an accelerated proton

\[ \frac{d\psi}{dt} = \int_{\mathbb{R}} \frac{d^4 \chi}{(2\pi)^3} \langle \delta^{(4)}(\chi - \Delta \eta) \rangle \frac{d\tau}{\tau^2} \rho(\tau) \]

where \( \Delta \eta \) is the displacement of the uncharged point.

In 1978 Hanada introduced a coefficient, which is:

\[ C = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{d^4 \chi}{(2\pi)^3} \langle \delta^{(4)}(\chi - \Delta \eta) \rangle \frac{d\tau}{\tau^2} \rho(\tau) \]

The numerical integration of this expression over all space yields:

\[ C_0 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{d^4 \chi}{(2\pi)^3} \langle \delta^{(4)}(\chi - \Delta \eta) \rangle \frac{d\tau}{\tau^2} \rho(\tau) \]

where \( \langle \cdot \rangle \) denotes an average over all space-time points.

This result is consistent with the experimental data for the decay rate of the accelerated proton.
proton is independent of the frame not by numerical computation but by analytical computation. Moreover, we will perform the evaluation in four dimensional setting and treat neutrino a massive field. It is interesting that the decay rate can be analyzed analytically.

In Section II we will calculate the cross section of $\beta$ decay in the inertial frame. We will show that the decay rate can be obtained analytically in terms of a function which is an analog of Meijer’s $G$-function of two variables.

In Section III, we will perform the calculation in accelerating frame. We will be able to check that the resulting function is identical to the one obtained in inertial frame.

Section IV is devoted to the discussions.

In Appendix B, we list the explicit form of the function appeared in our main result.

II. INERTIAL FRAME

In this section, we analysis the $\beta$ decay of the accelerated proton in the inertial frame. In this frame, the accelerated proton decay resulting from the acceleration.

In the flat space-time, the solutions of the Dirac equation are simple but the calculation becomes complicated because of the vector current is on the hyperbola. We will mainly follow the notation appeared in Ref. [10, 11]

A. Accelerated Proton current

There may be several methods to represent the acceleration. One way is the followings. We represent the proton as a classical current. The position of the current should be on the world line of the accelerated particle, “hyperbola”. To do this, we introduce the Rindler coordinates. Rindler coordinates correspond to the world line of the uniformly accelerated observers. The inertial coordinates are shown by $(x^0, x^1, x^2, x^3)$ and the Rindler coordinates which show acceleration along the 3-axis are $(v, x^1, x^2, u)$ with $0 < u < \infty$ and $-\infty < v < \infty$. Two coordinates are related by

$$
\begin{align*}
x^0 &= u \sinh v, \\
x^1 &= x^1, \\
x^2 &= x^2, \\
x^3 &= u \cosh v.
\end{align*}
$$

Then the line element is described in the Rindler coordinates by

$$
d^s^2 = u^2 dv^2 - (dx^1)^2 - (dx^2)^2 - du^2.
$$

If some particle uniformly accelerated with a proper acceleration $a$, then $u = a^{-1} \equiv \text{const.}$ is its world line. As a result, the protons which are accelerated to the axis of $x^3$ on $x^1 = 0, x^2 = 0$ are represented as following classical current

$$
j^\mu = u \delta^\mu_0 \delta(x^1) \delta(x^2) \delta(u - a^{-1}),
$$

where $q$ is a small coupling constant and $u^\mu$ is the four-velocity $u^\mu = (a, 0, 0, 0)$ in Rindler coordinates. To deal with the proton-decay, nucleons $|n\rangle$ and protons $|p\rangle$ are represented as excited and unexcited states of the nucleon, respectively. For Hamiltonian $H$, neutron and proton mass $m_n$ and $m_p$, respectively, we set

$$
H|n\rangle = m_n|n\rangle, \quad H|p\rangle = m_p|p\rangle.
$$

We replace $q$ in last current by the Hermitian monopole

$$
\hat{q}(\tau) = e^{iH\tau} \hat{q}(0) e^{-iH\tau},
$$

where $\tau = v/a$ is proper time of the proton.

Then the current four-vector is

$$
\tilde{j}^\mu = \hat{q}(\tau) u^\mu \delta(x^1) \delta(x^2) \delta(u - a^{-1}).
$$

This is clear that the particle is on the hyperbola.

B. Fermionic field quantization

For electrons and neutrinos, we write the fermionic fields satisfying the Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi^{(\pm \sigma)}_{k,\sigma} = 0$

$$\Psi(x) = \sum_{\sigma = \pm} \int_{-\infty}^{\infty} d^3 k \left[ i_{k,\sigma} \psi^{(+ \sigma)}_{k,\sigma}(x) + d_{k,\sigma} \psi^{(- \sigma)}_{k,\sigma}(x) \right],
$$

where $x = (x^0, x^1, x^2, x^3), k = (\omega, k^1, k^2, k^3), x^\sigma = (x^1, x^2, x^3), k = (k^1, k^2, k^3), \dot{i}_{k,\sigma}$ and $d_{k,\sigma}$ are annihilation and creation operators of fermions and antifermions, respectively. $k$ and $\sigma$ are momentum and polarization, respectively. $\psi^{(+ \sigma)}_{k,\sigma}$ and $\psi^{(- \sigma)}_{k,\sigma}$ are positive and negative frequency solutions of the Dirac equation.

By solving the Dirac equation we obtain the orthonormal mode solutions [12].

$$
\psi^{(\pm \sigma)}_{k,\sigma}(x) = \frac{e^{\pm ik_\mu x^\mu}}{(2\pi)^{\frac{3}{2}}} u^\sigma_{\pm}(k),
$$

where

$$
u_{\pm}(k) = \frac{k_\mu \gamma^\mu \pm m}{\sqrt{\omega (\omega \pm m)}},
$$

and

$$
\begin{align*}
u_+ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\
u_- &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\end{align*}
$$

We choose the traditional inner product of the form

$$
\langle \psi^{(+ \sigma)}_{k,\sigma}(x), \psi^{(- \sigma')}_{k',\sigma'}(x) \rangle = \int d^3 \Sigma \gamma^\rho \psi^{(+ \sigma')}_{k',\sigma'}(x) \gamma^\rho \psi^{(- \sigma)}_{k,\sigma}(x) = \delta^\sigma(k - k') \delta_{\sigma, \sigma'} \delta_{\pm \pm} \delta_{\sigma' \pm}.
$$
and the solutions are normalized by this definition, where \( \vec{v} \equiv \psi^\dagger \gamma^0 \), \( d \Sigma_\mu \equiv n_\mu \Sigma \), \( n^\mu \) is a unit vector orthogonal to \( \Sigma \) and we have chosen \( \Sigma \) to be the hypersurface of constant \( x^0 \). The creation and annihilation operators satisfy the \( \delta \)-term equal-time commutation relations:

\[
\begin{align*}
\{ \hat{b}_{k\sigma}, \hat{\bar{b}}_{k'\sigma'} \} &= \{ \hat{d}_{k\sigma}, \hat{\bar{d}}_{k'\sigma'} \} = \delta^3(\mathbf{k} - \mathbf{k'})\delta_{\sigma\sigma'}, \nonumber \\
\{ \hat{b}_{k\sigma}, \hat{b}_{k'\sigma'} \} &= \{ \hat{d}_{k\sigma}, \hat{d}_{k'\sigma'} \} = 0,
\end{align*}
\]

Using electron \( \bar{\Psi}_e \) and neutrino \( \bar{\Psi}_\nu \) fields, we write the Fermi action by the nucleon current (2.6)

\[
S_f = \int d^4x \sqrt{-g} j_\mu(\bar{\Psi}_e \gamma^\mu \Psi_e + \bar{\Psi}_\nu \gamma^\mu \Psi_\nu),
\]

where the first term is used by inverse \( \beta \) decay.

You can see these general formation on any book of Quantum Field Theory in the inertial frame. We are ready to start the calculation of the cross section of the accelerated proton in the inertial frame.

### C. Calculation of Cross Section

We are now going to calculate the cross section of the \( \beta \) decay using the current and field in the subsection A.

\[
\frac{d^6 \sigma^{\beta \rightarrow n}}{d^3k_e d^3k_\nu} = \sum_{\sigma_e} \sum_{\sigma_\nu} |\mathcal{A}^{\beta \rightarrow n}|^2.
\]

Now in order to calculate the spin sums, we use the following standard formula:

\[
\sum_{\sigma_e} \sum_{\sigma_\nu} \left[ T^{(\pm \omega_e) \Gamma_1} \gamma_\mu \gamma_\nu T^{(\pm \omega_\nu) \Gamma_2} \gamma_\mu \gamma_\nu \right] = \text{Tr} \left[ T^{(\pm \omega_e) \Gamma_1} \gamma_\mu \gamma_\nu \right] \left[ T^{(\pm \omega_\nu) \Gamma_2} \gamma_\mu \gamma_\nu \right],
\]

where \( \alpha \) and \( \beta \) represent \( e \) or \( \nu \).

The completeness relations can be written as

\[
\sum_{\sigma_e} \gamma_{\sigma_e\alpha} (k_{\alpha}) \gamma_{\sigma_e\alpha} (k_{\alpha}) = \frac{1}{2\omega_\alpha}(k_{\alpha\beta} \gamma^\beta \pm m_\alpha),
\]

and we introduce \( s \) and \( \xi \) by

\[
\tau_1 = s + \frac{\xi}{2}, \quad \tau_2 = s - \frac{\xi}{2}
\]

The way of deriving the cross section is normal.

First, the vacuum transition amplitude of the proton decay is written by

\[
\mathcal{A}^{\beta \rightarrow n} = \langle n | \bar{c}^{\nu}_{\nu, \sigma_\nu} \bar{b}_{k_{\sigma}} | S_f \rangle | \nu \rangle.
\]

It is straightforward to compute \( \mathcal{A}^{\beta \rightarrow n} \) for the Fermi action \( S_f \) and we obtain

\[
\mathcal{A}^{\beta \rightarrow n} = G_F \int_{-\infty}^{\infty} d(x^0) d(x^3) \frac{e^{i\Delta m^2 t \frac{u_\mu}{v_\mu}}}{\sqrt{a^3(x^0)^2 + 1}} \times \delta(x^3 - \sqrt{(x^0)^2 + a^{-2}}) \langle \bar{c}_{\nu, \sigma_\nu} | \bar{b}_{k_{\sigma}} | 0 \rangle,
\]

where \( \Delta m \equiv m_\mu \pm m_\nu \), \( \tau = a^{-1} \sinh^{-1}(ax^0) \) is the nucleon’s proper time, \( G_F \equiv \langle \bar{\psi}_0 | \psi_0 | n \rangle \) is the Fermi constant. We can substitute the field and integrate by \( x^0 \).

The differential transition rate is

\[
\frac{d^6 \sigma^{\beta \rightarrow n}}{d^3k_e d^3k_\nu} = \sum_{\sigma_e} \sum_{\sigma_\nu} |\mathcal{A}^{\beta \rightarrow n}|^2.
\]

We obtain it by integration of proper times \( \tau_1 \) and \( \tau_2 \)

\[
\sum_{\sigma_e} \sum_{\sigma_\nu} \left[ -\omega^- \frac{\gamma_\mu \gamma_\nu}{\omega^+} \right] = \frac{1}{\omega^+ \omega^-} \left[ \frac{\omega^+ \omega^-}{\omega^+ \omega^-} \right] \left[ \frac{\omega^+ \omega^-}{\omega^+ \omega^-} \right] + \omega^- (\omega_\nu \omega_\nu + k^2_\nu \omega_\nu) \sinh (2as) - m_\nu \omega_\nu \cosh (2as)
\]

\[
\sinh (2as) - m_\nu \omega_\nu \cosh (2as)
\]

By using the spin sum

\[
\sum_{\sigma_e} \sum_{\sigma_\nu} \left[ -\omega^- \frac{\gamma_\mu \gamma_\nu}{\omega^+} \right] = \frac{1}{\omega^+ \omega^-} \left[ \frac{\omega^+ \omega^-}{\omega^+ \omega^-} \right] \left[ \frac{\omega^+ \omega^-}{\omega^+ \omega^-} \right] \left[ \frac{\omega^+ \omega^-}{\omega^+ \omega^-} \right] + \omega^- (\omega_\nu \omega_\nu + k^2_\nu \omega_\nu) \sinh (2as) - m_\nu \omega_\nu \cosh (2as)
\]

\[
\sum_{\sigma_e} \sum_{\sigma_\nu} \left[ -\omega^- \frac{\gamma_\mu \gamma_\nu}{\omega^+} \right] = \frac{1}{\omega^+ \omega^-} \left[ \frac{\omega^+ \omega^-}{\omega^+ \omega^-} \right] \left[ \frac{\omega^+ \omega^-}{\omega^+ \omega^-} \right] + \omega^- (\omega_\nu \omega_\nu + k^2_\nu \omega_\nu) \sinh (2as) - m_\nu \omega_\nu \cosh (2as)
\]

and the change of variables

\[
k^1_\alpha = k^1_\alpha, k^2_\alpha = k^2_\alpha, k^3_\alpha = -\omega_\alpha \sinh as + k^3_\alpha \cosh as,
\]
we can obtain the differential transition rate

\[
\frac{1}{T} \frac{d^3p_{in}^{+n}}{d^3k_e d^3k_{\nu}} = \frac{G_F^2}{(2\pi)^3} \frac{1}{\omega_{e\nu}} \int_{-\infty}^{\infty} d\xi e^{i\Delta m \xi \pm \frac{1}{a}} \left( \omega_{e\nu}^2 + k_e^2 k_{\nu}^2 - m_e m_\nu \cosh a\xi \right),
\]

where \( T \equiv \int_{-\infty}^{\infty} ds \). To integrate by \( \xi \) we redefine new variable by

\[
\lambda \equiv e^{\frac{a\xi}{T}}.
\]

And we use the following notations

\[
\tilde{k}_e \equiv \frac{k_e}{a}, \quad \tilde{\omega}_{e\nu} \equiv \frac{\omega_{e\nu}}{a}, \quad \tilde{m}_e \equiv \frac{m_e}{a}, \quad \tilde{\Delta} \equiv \frac{\Delta m}{a}. \tag{25}
\]

Then the cross section is given in the form:

\[
\Gamma^{+n, n}_{in} = \frac{1}{T} \int d^3p_{in}^{+n} = \frac{a^5 G_F^2}{2^3 3^2 \pi^2} \int_{-\infty}^{\infty} \frac{d\tilde{\omega}_{e\nu}}{\omega_{e\nu}} \int_{-\infty}^{\infty} d\lambda e^{i(\tilde{\omega}_{e\nu} + \tilde{\omega}_{e\nu})(\lambda - \frac{1}{ \tilde{\Delta} })} \times \lambda^{2(\tilde{\Delta} - 1)} \left[ \tilde{k}_e\tilde{k}_{\nu} - \frac{1}{2} \frac{\tilde{m}_e\tilde{m}_{\nu}}{\tilde{\omega}_{e\nu}} (\lambda^2 + \frac{1}{\lambda^2}) \right]. \tag{26}
\]

The integral of \( \lambda \) can be readily expressed as modified Bessel to find

\[
\Gamma^{+n, n}_{in} = \frac{2^2 a^5 G_F^2}{\pi^3 e^{\pi \Delta m}} \int_{0}^{\infty} d\tilde{\omega}_{e\nu} \tilde{k}_e \tilde{k}_{\nu} \left[ K_{2i(\Delta m)} \left( 2(\tilde{\omega}_{e\nu} + \tilde{\omega}_{e\nu}) \right) \right]
\]

\[
+ \frac{1}{2} \frac{\tilde{m}_e\tilde{m}_{\nu}}{\tilde{\omega}_{e\nu}} \left[ K_{2i(\Delta m + 2)} \left( 2(\tilde{\omega}_{e\nu} + \tilde{\omega}_{e\nu}) \right) \right] - \frac{1}{2} \frac{\tilde{m}_e\tilde{m}_{\nu}}{\tilde{\omega}_{e\nu}} \left[ K_{2i(\Delta m - 2)} \left( 2(\tilde{\omega}_{e\nu} + \tilde{\omega}_{e\nu}) \right) \right]. \tag{27}
\]

However, this form is difficult to integrate by \( \tilde{k}_e \) and \( \tilde{k}_{\nu} \). Therefore, we use the integration formula of modified Bessel

\[
K_{\mu}(z) = \frac{1}{2} \int_{C_1} ds \Gamma(-s) \Gamma(-s - \mu) \left( \frac{z}{2} \right)^{2s + \mu}. \tag{28}
\]

It is not hard to show this formula in complex plane by picking the residues of \( \Gamma(-s) \) and \( \Gamma(-s - \mu) \) (see Appendix A). To evaluate Eq. (27) we use this formulas and after some simple shift of variable we have

\[
\Gamma^{+n, n}_{in} = \frac{2a^5 G_F^2}{\pi^3 e^{\pi \Delta m}} \int_{0}^{\infty} d\tilde{\omega}_{e\nu} \int_{C_2} ds \int_{C_2} dt \left( \tilde{k}_e^2 + \tilde{m}_e^2 + \tilde{k}_{\nu}^2 + \tilde{m}_{\nu}^2 \right)^{2s + 2i\Delta m} \times \left[ \Gamma(-s - t + i\Delta m) + \frac{1}{2} \frac{\tilde{m}_e\tilde{m}_{\nu}}{\tilde{\omega}_{e\nu}} \left( \Gamma(-s - 1) \Gamma(-s - 2i\Delta m - 1) + \Gamma(-s - 1) \Gamma(-s - 2i\Delta m + 1) \right) \right]. \tag{29}
\]

where the contour \( C_1 \) must be chose so that all poles of \( \Gamma(-s) \Gamma(-s - \mu) \) are picked up and must be selected as \( k \) integration don’t infinity. But, this form of integration is still not simple for \( \tilde{k}_e \) and \( \tilde{k}_{\nu} \).

In order to perform the integration, we use the following expansion formula

\[
(A + B) z = \int_{C_2} \frac{dt}{2\pi i} \frac{\Gamma(-t) \Gamma(t - z)}{\Gamma(-z)} A^t z B^t \tag{30}
\]

to integrate easily by \( \tilde{k}_e \) and \( \tilde{k}_{\nu} \), where the contour \( C_2 \) is the path separating the poles of \( \Gamma(-t) \) from those of \( \Gamma(t - z) \) (see Appendix A).

Now we can rewrite Eq. (29) as

\[
\Gamma^{+n, n}_{in} = \frac{a^5 G_F^2}{2^3 \pi^3 e^{\pi \Delta m}} \int_{C_2} \frac{ds}{2\pi i} \int_{C_2} \frac{dt}{2\pi i} \frac{(\tilde{m}_e^2 + \tilde{m}_{\nu}^2)^2}{(\tilde{k}_e^2 + \tilde{m}_e^2)(\tilde{k}_{\nu}^2 + \tilde{m}_{\nu}^2)} \times \left[ \Gamma(-s - t + i\Delta m + 3) \right]^2 \left[ \Gamma(-s) \Gamma(-t) \Gamma(-s + 2) \Gamma(-t + 2) \right]
\]

\[
+ \text{Re} \left( \Gamma(-s - t + i\Delta m + 2) \Gamma(-s - t - i\Delta m + 4) \right) \left[ \Gamma(-s + \frac{1}{2}) \Gamma(-t + \frac{1}{2}) \Gamma(-s + \frac{3}{2}) \Gamma(-t + \frac{3}{2}) \right]. \tag{31}
\]
where the contour $C_s$ and $C_t$ is the path which picks up all poles of Gamma functions in $s$ and $t$ complex planes, respectively.

This is the two-dimensional analog of Meijer’s G-function [13]. The explicit form can be obtained by evaluating contour integral. We list the results in the Appendix B.

III. ACCELERATED FRAME

In this section, we are going to analyze the same physical phenomenon view of accelerated system.

A. Fermionic field quantization

The Dirac equation in curved space-time is written by

$$(\gamma^\mu \hat{\nabla}_\mu - m) \psi^{\omega}_{\text{wos}}(x) = 0, \quad (32)$$

where

$$x = (v, x^1, x^2, u), \quad w = (\omega, k^1, k^2), \quad \psi = (x^1, x^2, u),$$

$$(\epsilon^\mu)^{\rho} = u^{-1} \delta^\mu_\nu, \quad (\epsilon^\alpha)^{\beta} = \delta^\alpha_\beta,$$

$$\gamma^\mu \equiv (\gamma^0)^{\mu} \gamma^\nu,$$

$$\hat{\nabla}_\mu \equiv \partial_\mu + \frac{1}{8}[\gamma^\mu, \gamma^\nu](\epsilon^\alpha)^{\lambda} \nabla_\mu (\epsilon^\lambda). \quad (33)$$

In the Rindler coordinates, the equation becomes

$$i \frac{\partial \psi^{\omega}_{\text{wos}}(x)}{\partial v} = \left( \gamma^\mu mu^{\mu} - \frac{i\alpha^2}{2} - iuv \partial_1 \right) \psi^{\omega}_{\text{wos}}(x), \quad (34)$$

where $\alpha^i \equiv \gamma^0 \gamma^i$.

The fermionic field can be expanded as

$$\hat{\Psi}(x) = \sum_{\sigma = \pm} \int_0^\infty d\omega \int_{-\infty}^\infty d^2k \left[ \hat{J}_{\text{wos}} \psi^{\omega}_{\text{wos}} + \hat{\tilde{J}}_{\text{wos}} \tilde{\psi}^{\omega}_{\text{wos}} \right]. \quad (35)$$

We will solve the equation in the form

$$\psi^{\omega}_{\text{wos}} = \hat{J}_{\text{wos}}(x)e^{-i\omega u/m}, \quad (36)$$

for $-\infty < \omega < \infty$.

The function $\hat{J}_{\text{wos}}$ is eigenstate of Hamiltonian as

$$H \hat{J}_{\text{wos}} = \omega \hat{J}_{\text{wos}}, \quad (37)$$

where

$$H \equiv \sum_{\omega} \left[ mv_0^2 - \frac{i\omega^2}{2} - uv \partial_1 \right]. \quad (38)$$

We denote two-component spinors $\chi^\pm$ by

$$\hat{\chi}_{\text{wos}}(x) \equiv \left[ \chi_1(x, w), \chi_2(x, w) \right]. \quad (39)$$

then we find that the Dirac equation takes the form

$$\delta^{ij} u \partial_j (u \partial_i \chi_1) = \left[ m^2 u^2 + \frac{1}{4} - \hat{\omega}^2 \right] \chi_1 - \frac{i}{\hbar} \sigma^3 \chi_2, \quad (40a)$$

$$\delta^{ij} u \partial_j (u \partial_i \chi_2) = \left[ m^2 u^2 + \frac{1}{4} - \hat{\omega}^2 \right] \chi_2 - \frac{i}{\hbar} \sigma^3 \chi_1, \quad (40b)$$

where $i$ and $j$ run over 1, 2, 3. By now, for squared equations we used $f_{\omega \sigma}$, $\chi_1$ and $\chi_2$.

To simplify these equations, we introduce the functions $\phi^\pm \equiv \chi_1 \mp \chi_2$ and we can define $\xi^\pm$ and $\zeta^\pm$ through

$$\phi^\pm(x, w) \equiv \left[ \xi^\pm(x, w), \zeta^\pm(x, w) \right]. \quad (41)$$

Given this definition, we can separate the equation for $\xi$ and $\zeta$ in the form:

$$\delta^{ij} u \partial_j (u \partial_i \xi^\pm) = \left[ m^2 u^2 + \left( \frac{\hat{\omega} \pm \frac{1}{2}}{2} \right)^2 \right] \xi^\pm, \quad (42a)$$

$$\delta^{ij} u \partial_j (u \partial_i \zeta^\pm) = \left[ m^2 u^2 + \left( \frac{\hat{\omega} \mp \frac{1}{2}}{2} \right)^2 \right] \zeta^\pm, \quad (42b)$$

Here, we introduce $\ell = \ell^2 = (k^1)^2 + (k^2)^2 + m^2$ by which the equation can be written as

$$[\delta_{ij}^2 + \frac{1}{2} \gamma^2 - m^2] \xi^\pm = \frac{1}{m^2} \left[ - (m u \partial_3 u \partial_3 + \left( \frac{\hat{\omega} \pm \frac{1}{2}}{2} \right)^2 \right] \xi^\pm \equiv -\ell^2 \xi^\pm. \quad (43)$$

The solution can be written in the form

$$\xi^\pm = A^\pm(w) \chi(x^1, x^2, k^1, k^2, u). \quad (44)$$

This leads to the following decoupled equations

$$[\delta_{ij}^2 + \frac{1}{2} \gamma^2 - m^2] \chi(x^1, x^2, k^1, k^2) = 0, \quad (45)$$

$$\left[ \frac{\partial^2}{\partial (\ell u)^2} + \frac{1}{\ell u} \partial_1 \partial_{\ell u} - 1 - \left( \frac{\hat{\omega} \pm \frac{1}{2}}{\ell u} \right)^2 \right] M_\pm^\pm(u) = 0. \quad (46)$$

We can readily solve the equation in the form

$$\chi(x^1, x^2, k^1, k^2) = e^{i k \cdot x} A_k, \quad (47)$$

$$M_\pm^\pm(u) = K \frac{\hat{\omega} \mp \frac{1}{2}}{\ell u} M(u), \quad (48)$$

where the index $a$ run over 1 and 2.

Using the arbitrary function $A^\pm$ and $B^\pm$ of $k$, we find the most general solution can be written as

$$\xi^\pm = A^\pm(w) e^{i k \cdot x} K \frac{\hat{\omega} \mp \frac{1}{2}}{\ell u}, \quad (49a)$$

$$\zeta^\pm = B^\pm(w) e^{i k \cdot x} K \frac{\hat{\omega} \mp \frac{1}{2}}{\ell u}, \quad (49b)$$

and

$$\hat{\Psi}(x) = \sum_{\sigma = \pm} \int_0^\infty d\omega \int_{-\infty}^\infty d^2k \left[ \hat{J}_{\text{wos}} \psi^{\omega}_{\text{wos}} + \hat{\tilde{J}}_{\text{wos}} \tilde{\psi}^{\omega}_{\text{wos}} \right]. \quad (50)$$
which implies

\[
    f_{\omega \sigma} = \frac{1}{2} \begin{bmatrix}
        \xi^+ + \xi^- \\
        \xi^+ + \xi^- \\
        -\xi^+ + \xi^-
    \end{bmatrix} \left[ A^+ K_{\omega^+} \frac{1}{2} (\ell u) + A^- K_{\omega^+} \frac{1}{2} (\ell u) \right]
\]

\[
    = \frac{e^{ik_{\omega \sigma} a}}{2} \begin{bmatrix}
        A^+ K_{\omega^+} \frac{1}{2} (\ell u) + A^- K_{\omega^+} \frac{1}{2} (\ell u) \\
        B^+ K_{\omega^+} \frac{1}{2} (\ell u) + B^- K_{\omega^+} \frac{1}{2} (\ell u) \\
        -A^+ K_{\omega^+} \frac{1}{2} (\ell u) + A^- K_{\omega^+} \frac{1}{2} (\ell u)
    \end{bmatrix}.
\]

(50)

It is easy to see that \( f_{\omega \sigma} \) satisfies

\[
    \left[ i \alpha \cdot \partial_i - m \gamma^0 + \left( \frac{i \alpha}{2} + \bar{\omega} \right) \frac{1}{u} \right]
    \times \left[ i \alpha \cdot \partial_j - m \gamma^0 + \left( \frac{i \alpha}{2} - \bar{\omega} \right) \frac{1}{u} \right] \cdot f_{\omega \sigma} = 0. \quad (51)
\]

Therefore, the solutions of Dirac equation are simply given by

\[
    \tilde{f}_{\omega \sigma} \equiv \left[ i \alpha \cdot \partial_i - m \gamma^0 + \left( \frac{i \alpha}{2} - \bar{\omega} \right) \frac{1}{u} \right] f_{\omega \sigma}.
\]

(52)

If we set

\[
    A^+ = B^- = 0
\]

we can find a normalized \( A^- \) and \( B^+ \) by letting

\[
    A^- = B^+ = \left[ \frac{\cosh \omega \tau}{\pi \ell} \right]^\frac{1}{2} \equiv N,
\]

(53)

where

\[
    \tilde{\ell} \equiv \frac{\ell}{u},
\]

(55)

which is normalized with respect to the inner product

\[
    \langle \psi_{\omega \sigma}^{(w)}(x), \psi_{\omega \sigma'}^{(w)}(x) \rangle \equiv \int \mathcal{D} \Sigma \psi_{\omega \sigma}^{(w)}(x) \gamma_R \psi_{\omega \sigma'}^{(w)}(x)
\]

\[
    = \delta^3(w - w') \delta_{\omega \sigma'},
\]

(60)

where \( \bar{\tilde{\psi}} \equiv \psi \tilde{\gamma}^0 \) and \( \Sigma \) is set to be \( v = \text{const.} \). In this way, we find concrete form of fermionic field is written by

\[
    \psi_{\omega \sigma}^{(w)}(x) = \frac{e^{-i \omega \tau \sigma + i k_{\omega \sigma} x^\sigma}}{(2\pi)^\frac{3}{2}} u^{(w)}(u, w),
\]

(57)

where

\[
    u^{(w)}(u, w) = N \gamma^0 \left[ \tilde{k} \gamma^0 + \tilde{m} \right] K_{\omega^+} \frac{1}{2} (\ell u) + i \tilde{\omega} \gamma^0 K_{\omega^+} \frac{1}{2} (\ell u) u_{\sigma}.
\]

(58)

and

\[
    u_+ = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_- = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

(59)

The creation and annihilation operators should obey

\[
    \left\{ b_{\omega \sigma}, b_{\omega \sigma}^\dagger \right\} = \delta_{\omega \sigma}, \quad \left\{ d_{\omega \sigma}, d_{\omega \sigma}^\dagger \right\} = \delta^3(w - w') \delta_{\omega \sigma}.
\]

(60)

\[
    \left\{ b_{\omega \sigma}, d_{\omega \sigma'} \right\} = \left\{ d_{\omega \sigma}, b_{\omega \sigma'} \right\} = 0.
\]

B. Calculation of Cross Section

The process of \( \beta \) decay in the accelerated frame looks very different from that in rest frame. In this case, a proton is stable but the whole space is FDU thermal bath characterized by a temperature \( T = a f / 2 \pi \). Therefore, proton absorbs \( e^- \) and \( \bar{p} \) from FDU thermal bath and do emit \( e^+ \) and \( \bar{\nu} \). Three processes are possible through the processes:

(i) \( p^+ + e^- \rightarrow n + \nu \),

(ii) \( p^+ + \bar{p} \rightarrow n + e^+ \),

(iii) \( p^+ + e^- \rightarrow n \).

The transition rate is a combination of them.

Formally, we can calculate the cross sections by some of the inertial system but we have to deal with three processes.

The transition amplitudes are written by

\[
    \mathcal{A}_{(i)}^{\rho \to \rho'} = \langle \rho' | \psi_{\bar{\nu}_{\sigma'+ \sigma}} | \rho \rangle = \frac{G_F}{\alpha} \sqrt{a} \int_{-\infty}^{\infty} dt e^{i A m t} \langle \psi_{\bar{\nu}_{\sigma'+ \sigma}} | \psi \rangle \langle \psi | \psi_{\bar{\nu}_{\rho'+ \rho}} \rangle
\]

\[
    = \frac{G_F}{2\pi} \delta(\omega_{\rho'} - \omega_{\rho} - \Delta m) \bar{u}_{\sigma+'} \gamma^0 u_{\sigma'},
\]

(61a)

\[
    \mathcal{A}_{(ii)}^{\rho \to \rho'} = \frac{G_F}{2\pi} \delta(\omega_{\rho'} - \omega_{\rho} - \Delta m) \bar{u}_{\sigma+'} \gamma^0 u_{\sigma'},
\]

(61b)

\[
    \mathcal{A}_{(iii)}^{\rho \to \rho'} = \frac{G_F}{2\pi} \delta(\omega_{\rho'} - \omega_{\rho} - \Delta m) \bar{u}_{\sigma+'} \gamma^0 u_{\sigma'},
\]

(61c)

where \( \bar{\Sigma} \) is replaced \( \gamma^\rho \) in inertial frame with \( \gamma^\rho_{\mu} \) and \( u^\rho = (a, 0, 0, 0) \).

We assume that the observer is in the thermal bath. We attach the fermionic thermal factor for each process. Then the differential transition rate per absorbed and emitted particle energies for each processes are written
\[
\frac{1}{T} \frac{d^6P_{\rho \rightarrow n}^{\rho \rightarrow n}}{d\omega_{\rho} d\omega_{n} d^3 k_{\rho} d^3 k_{n}} = \frac{1}{T} \sum_{\sigma_{\rho}} \sum_{\sigma_{n}} \left| \mathcal{A}^{\rho \rightarrow n}_{(ii)} \right|^2 n_{F}(\omega_{\rho}) \left[ 1 - n_{F}(\omega_{n}) \right] \\
= \frac{G_{F}^{2}}{2^5 \pi^3} \left[ \frac{\gamma_{\rho} \gamma_{n}}{\gamma_{\rho} + \gamma_{n}} \right]^2 \delta(\omega_{\rho} - \omega_{n} - \Delta m) e^{\pm \Delta m c \cosh \omega_{\rho} \pm \pi c \cosh \omega_{n} \pi}, \quad (62a)
\]

\[
\frac{1}{T} \frac{d^6P_{\rho \rightarrow n}^{\rho \rightarrow n}}{d\omega_{\rho} d\omega_{n} d^3 k_{\rho} d^3 k_{n}} = \frac{1}{T} \sum_{\sigma_{\rho}} \sum_{\sigma_{n}} \left| \mathcal{A}^{\rho \rightarrow n}_{(ii)} \right|^2 n_{F}(\omega_{\rho}) \left[ 1 - n_{F}(\omega_{n}) \right] \\
= \frac{G_{F}^{2}}{2^5 \pi^3} \left[ \frac{\gamma_{\rho} \gamma_{n}}{\gamma_{\rho} + \gamma_{n}} \right]^2 \delta(\omega_{\rho} - \omega_{n} - \Delta m) e^{\pm \Delta m c \cosh \omega_{\rho} \pm \pi c \cosh \omega_{n} \pi}, \quad (62b)
\]

where

\[
n_{F}(\omega) \equiv \frac{1}{1 + e^{\omega/T}} \quad (63)
\]

is the fermionic thermal factor and \( T = 2\pi \delta(0) \) is total proper time of the proton.

We can obtain the completeness relations by

\[
\sum_{\sigma} \bar{u}_{\sigma}^{(n)}(u, w, \bar{w}) u_{\sigma}^{(n)}(u, w) = N^2 \gamma^0 \left[ 2 \tilde{\ell} \left( K_{\tilde{\ell}} \tilde{\ell} + \frac{1}{2} \tilde{\ell} \right) + \tilde{m} \tilde{\ell} \left( (\gamma^0 - \gamma^0) K_{\tilde{\ell}} \tilde{\ell} + (\gamma^0 + \gamma^0) K_{\tilde{\ell}} \tilde{\ell} \right) \right]. \quad (64)
\]

Using these relations, a direct calculation yields the spin sum for the process (i) as

\[
\sum_{\sigma_{\rho}} \sum_{\sigma_{n}} \left| \bar{u}_{\sigma_{\rho}}^{(n)}(u, w) u_{\sigma_{n}}^{(n)}(u, w) \right|^2 \\
= \frac{2}{\pi^2} \cosh \omega_{\rho} \cosh \omega_{n} \pi \left[ \tilde{\ell}_{\rho} \tilde{\ell}_{n} K_{\tilde{\ell}_{\rho}} + \frac{1}{2} K_{\tilde{\ell}_{n}} \tilde{\ell}_{\rho} \right]^2 + \tilde{m} \tilde{\ell} \left( K_{\tilde{\ell}_{\rho}}^2 + \frac{1}{2} K_{\tilde{\ell}_{n}}^2 - \frac{1}{2} \tilde{\ell} \tilde{\ell} \right). \quad (65)
\]

We can perform the analogous calculation for the processes (ii) and (iii).

According to Eq. (65), the differential transition rate of the process (i) is given by

\[
\frac{1}{T} \frac{d^6P_{\rho \rightarrow n}^{\rho \rightarrow n}}{d\omega_{\rho} d\omega_{n} d^3 k_{\rho} d^3 k_{n}} \\
= \frac{G_{F}^{2}}{2^5 \pi^3 e^{\Delta m}} \left[ \tilde{\ell}_{\rho} \tilde{\ell}_{n} K_{\tilde{\ell}_{\rho}} + \frac{1}{2} K_{\tilde{\ell}_{n}} \tilde{\ell}_{\rho} \right]^2 + \tilde{m} \tilde{\ell} \left( K_{\tilde{\ell}_{\rho}}^2 + \frac{1}{2} K_{\tilde{\ell}_{n}}^2 - \frac{1}{2} \tilde{\ell} \tilde{\ell} \right). \quad (66a)
\]

By using them, the cross sections for each processes can be obtained as

\[
\Gamma_{(i)}^{\rho \rightarrow n} = \frac{1}{T} \int d^6P_{\rho \rightarrow n}^{\rho \rightarrow n} \\
= \frac{2G_{F}^{2}}{\pi^2 e^{\Delta m}} \int_{\Delta m}^{\infty} d\omega_{\rho} \left[ \int_{0}^{\Delta m} d^2 k_{\rho} \tilde{\ell}_{\rho} \left( K_{\tilde{\ell}_{\rho}} + \frac{1}{2} \tilde{\ell}_{\rho} \right) \left| \tilde{\ell}_{\rho} \tilde{\ell}_{n} K_{\tilde{\ell}_{\rho}} + \frac{1}{2} K_{\tilde{\ell}_{n}} \tilde{\ell}_{\rho} \right|^2 \right] + \tilde{m} \tilde{\ell} \tilde{\ell} \left( K_{\tilde{\ell}_{\rho}}^2 + \frac{1}{2} K_{\tilde{\ell}_{n}}^2 - \frac{1}{2} \tilde{\ell} \tilde{\ell} \right). \quad (67a)
\]

\[
\Gamma_{(ii)}^{\rho \rightarrow n} = \frac{2G_{F}^{2}}{\pi^2 e^{\Delta m}} \int_{\Delta m}^{\infty} d\omega_{\rho} \left[ \int_{0}^{\Delta m} d^2 k_{\rho} \tilde{\ell}_{\rho} + K_{\tilde{\ell}_{\rho}} + \frac{1}{2} K_{\tilde{\ell}_{n}} \tilde{\ell}_{\rho} \left( K_{\tilde{\ell}_{\rho}} + \frac{1}{2} K_{\tilde{\ell}_{n}} \tilde{\ell}_{\rho} \right) \left| \tilde{\ell}_{\rho} \tilde{\ell}_{n} K_{\tilde{\ell}_{\rho}} + \frac{1}{2} K_{\tilde{\ell}_{n}} \tilde{\ell}_{\rho} \right|^2 \right] + \tilde{m} \tilde{\ell} \tilde{\ell} \left( K_{\tilde{\ell}_{\rho}}^2 + \frac{1}{2} K_{\tilde{\ell}_{n}}^2 - \frac{1}{2} \tilde{\ell} \tilde{\ell} \right). \quad (67b)
\]
\[ \Gamma^{\text{app}}_{(ii)} = \frac{2G^2_p}{\pi^2 e^{\Delta m}} \int_0^{\Delta m} d\tilde{\omega} \left[ \int_0^{\infty} d^3 \tilde{k}_e - \tilde{\omega}_e \cdot \left[ \frac{K_{i\tilde{\omega}_e} + \frac{1}{4} (\tilde{\omega}_e^2)}{K_{i(\tilde{\omega}_e + \Delta m)} + \frac{1}{4} (\tilde{\omega}_e^2)} \right] \right] + \tilde{m}_e \tilde{m}_e \text{Re} \left\{ \int_0^{\infty} d^3 \tilde{k}_e K_{i\tilde{\omega}_e}^2 + \frac{1}{4} (\tilde{\omega}_e^2) \int_0^{\infty} \text{d}^3 \tilde{k}_\nu \tilde{\omega}_\nu \left[ K_{i(\tilde{\omega}_\nu + \Delta m)} + \frac{1}{4} (\tilde{\omega}_\nu^2) \right] \right\} \right] \] (67c)

By summing them we can lump together the integration, and the total cross section becomes simply in the form

\[ \Gamma^{\text{app}}_{\text{acc}} = \Gamma^{\text{app}}_{(i)} + \Gamma^{\text{app}}_{(ii)} + \Gamma^{\text{app}}_{(III)} = \frac{2G^2_p}{\pi^2 e^{\Delta m}} \int_0^{\Delta m} d\tilde{\omega} \left[ \int_0^{\infty} d^3 \tilde{k}_e - \tilde{\omega}_e \cdot \left[ \frac{K_{i\tilde{\omega}_e} + \frac{1}{4} (\tilde{\omega}_e^2)}{K_{i(\tilde{\omega}_e + \Delta m)} + \frac{1}{4} (\tilde{\omega}_e^2)} \right] \right] + \tilde{m}_e \tilde{m}_e \text{Re} \left\{ \int_0^{\infty} d^3 \tilde{k}_e K_{i\tilde{\omega}_e}^2 + \frac{1}{4} (\tilde{\omega}_e^2) \int_0^{\infty} \text{d}^3 \tilde{k}_\nu \tilde{\omega}_\nu \left[ K_{i(\tilde{\omega}_\nu + \Delta m)} + \frac{1}{4} (\tilde{\omega}_\nu^2) \right] \right\} \right] \] (68)

It is hard to deal with these integral of modified Bessel's. Of course, we can use Eq. (28) again like the case of Section I. But we use more useful formula (see, for example, p219 of Ref. [13] or Ref. [14]).

\[ x^\sigma K_\nu(x)K_\mu(x) = \frac{\sqrt{\pi}}{2} G_{c2}^\nu \left( x^2 \left[ \frac{1}{2} (\nu + \mu + \sigma), \frac{1}{2} (\nu - \mu + \sigma), \frac{1}{2} (-\nu + \mu + \sigma), \frac{1}{2} (-\nu - \mu + \sigma) \right] \right), \] (69)

and by using the definition of \( G \) function, the integrand can be represented by power of \( t \) and we can easily integrate with respect to \( t \).

We find

\[ \Gamma^{\text{app}}_{\text{acc}} = \frac{G^2_p}{2\pi^2 e^{\Delta m}} \int_{C_1} \frac{ds}{2\pi i} \int_{C_2} \frac{dt}{2\pi i} \int_0^{\infty} d\tilde{\omega} \left( \frac{\tilde{m}_e^{2n+1} \tilde{m}_\nu^{2s+1}}{(2s+1)(2t+1)\Gamma(-s+1)\Gamma(-t+1)} \right) \times \left[ \frac{\Gamma(-s - 1/2) \Gamma(-t + 1/2)}{\Gamma(-s - i(\tilde{\omega} - \Delta m) + 1) \Gamma(-t + i\tilde{\omega} + 1)} \right] \]

\[ + \tilde{m}_e \tilde{m}_\nu \Gamma(-s - 1/2) \Gamma(-t - 1/2) \text{Re} \left\{ \Gamma(-s - i(\tilde{\omega} - \Delta m) + 1) \Gamma(-s - i(\tilde{\omega} - \Delta m)) \Gamma(-t - i\tilde{\omega} + 1) \Gamma(-t - i\tilde{\omega}) \right\} \] (70)

where all poles of complex \( s \) and \( t \) planes are picked up with \( C_1 \) and \( C_2 \), respectively, by definition of \( G \) function.

To integrate with respect to \( \tilde{\omega} \), we use the formula of Barnes [14]:

\[ \int_{-i\infty}^{i\infty} d\tilde{\omega} \left( \frac{\Gamma(a + \tilde{\omega}) \Gamma(b + \tilde{\omega}) \Gamma(c - \tilde{\omega}) \Gamma(d - \tilde{\omega})}{\Gamma(a + c) \Gamma(a + d) \Gamma(b + c) \Gamma(b + d)} \right) = 2\pi i \left[ \frac{\Gamma(a + c)}{\Gamma(a + b + c + d)} \right] \left[ \text{Re} a, \text{Re} b, \text{Re} c, \text{Re} d > 0 \right]. \] (71)

Eventually, we find that the total cross section in the Rindler frame is

\[ \Gamma^{\text{app}}_{\text{acc}} = \frac{a^2 c^2 G^2_p}{\pi^2 e^{\Delta m}} \int_{C_1} \frac{ds}{2\pi i} \int_{C_2} \frac{dt}{2\pi i} \left( \frac{\tilde{m}_e^{2s+1} \tilde{m}_\nu^{2s+1}}{(2s+1)(2t+1)\Gamma(-s - t + 1) \Gamma(-s + t + 1/2)} \right) \times \left[ \frac{\Gamma(-s - t + i\Delta m + 3/2) \Gamma(-s - t + 1/2)}{\Gamma(-s - 2) \Gamma(-t + 2)} \right] \]

\[ + \text{Re} \left\{ \frac{\Gamma(-s - t + i\Delta m + 2) \Gamma(-s - t - i\Delta m + 4)}{\Gamma(-s - t + 2) \Gamma(-s - t - 1/2) \Gamma(-s - t - 3/2) \Gamma(-t + 3/2) \Gamma(-t + 1/2) \Gamma(-t + 3/2)} \right\} \] (72)

Comparing this to the results in inertial frame (31), we find that resulting expression agrees perfectly. This result shows the existence of Unruh effect is inevitable.

IV. DISCUSSIONS

analytic expressions for both frame and found that they

We have analyzed the \( \beta \) decay and the inverse \( \beta \) decay of the accelerated proton in both frames. We found
FIG. 1: All residues of $\Gamma(-s)$ and $\Gamma(-s-\mu)$ are picked up.

agree with each other. If you see the calculation of G. E. A. Matsas and D. A. T. Vanzella in two-dimensional model [10, 11], you can realize that on four-dimensional model it became hopelessly complicated integral. So the main problem in this time is the complication of integral. To solve it, we used Barnes type representation as you can see from Eq. (28) and Eq. (69). We can demonstrate these formulae by picking up the poles in complex plane of the integral valuable. The sum of these readily becomes the infinity series which defines the special function. By using them, we accomplished perfectly analytical proof.

It is straightforward to apply our technique for two dimensional setup used in Ref. [10, 11], we can easily prove that the decay rates is independent of the frame.

APPENDIX A

In this appendix A, we prove both key equations Eq. (28) and Eq. (30).

Firstly we derive Eq. (28). Through the path $C_1$, all poles of $\Gamma(-s)$ and $\Gamma(-s-\mu)$ are picked up (see FIG 1).

$$\frac{1}{2} \int_{C_1} \frac{ds}{2\pi i} \frac{\Gamma(-s)\Gamma(-s-\mu)}{\Gamma(-s-\mu)} \left( \frac{z}{2} \right)^{2s+\mu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(\pi \mu - n) \left( \frac{z}{2} \right)^{2n+\mu}$$

$$= \frac{\pi}{2} \sum_{n=0}^{\infty} \sum_{\pm} \frac{(-1)^n}{n!} \sin(-n\pi + \pm\mu) (z)$$

$$= \frac{\pi}{2} I_{\mu}(z) + I_{-\mu}(z)$$

where $I_{\mu}(z)$ is modified Bessel function of the first kind.

The last form is the definition of modified Bessel function $K_{\mu}(z)$ for non-integer $\mu$. And you find the formula in case $\mu$ is integer $n$ by setting $\frac{\mu}{2} = n$ after partial differentiation of this formula by $\mu$.

Next we integrate Eq. (30) as FIG 2.

FIG. 2: The contour separates the poles of $\Gamma(-t)$ from those of $\Gamma(t-z)$.

$$\int_{C_2} \frac{dt}{2\pi i} \frac{\Gamma(-t)\Gamma(t-z)}{\Gamma(-z)} A^{-t+z} B^t$$

If you select the contour (i) you find

$$= A^z \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(-z+n) \left( \frac{B}{A} \right)^n$$

$$= A^z \left( 1 + \frac{B}{A} \right)^z$$

(A2)

So this integration is the expansion form of $(A+B)^z$ in $B < A$. Similarly if you select the contour (ii) then you obtain the expansion form of $(A+B)^z$ in $B > A$. This integral representation of expansion of $(A+B)^z$ includes both cases of $B < A$ and $B > A$ by selecting the contour (i) and (ii), respectively.

APPENDIX B

In this appendix B, we show the explicit form of the integral. There are poles of the power of 1, 2 and 3.

The integral can be simply calculated by the change of variables to

$$s \rightarrow s-t, \ t \rightarrow t.$$ (B1)

After this transformation we obtain
\[
\Gamma^{p\to n} = \frac{a^5 G_F^2 \bar{m}_e^3}{2^{5\pi/2} \pi \tau_{\Delta m}} \int_{C_i} \frac{dt}{2\pi i} \int_{C_i} \frac{ds}{2\pi i} \frac{GM^3}{\pi} \Gamma(-s + 3) \Gamma(-s + 2) \Gamma(-s + t) \Gamma(-s + t + 2) \\
\times \left[ \Gamma^2(-s + i\Delta m + 3) \right] \Gamma(-t) \Gamma(-t + 2) \Gamma(-s + t) \Gamma(-s + t + 2) \\
+ \text{Re} \left\{ \Gamma(-s + i\Delta m + 2) \Gamma(-s - i\Delta m + 4) \right\} \Gamma(-t + \frac{1}{2}) \Gamma(-t + \frac{3}{2}) \Gamma(-s + t + \frac{1}{2}) \Gamma(-s + t + \frac{3}{2}) \right].
\]

(B2)

where \(\psi(z) = \frac{d}{dz} \ln \Gamma(z)\) and \(C_i\) is the path separating the poles of, for example in first term, \(\Gamma(-t) \Gamma(-t + 2)\) from those of \(\Gamma(-s + t) \Gamma(-s + t + 2)\), and \(C_s\) is the path which picks up all the poles in \(s\) complex plane.

Firstly, we start by \(t\) integration because the existence of poles in \(t\) complex plane is independent of \(s\). If we integrate by \(t\), we obtain for \(\bar{m}_e > \bar{m}_\nu\):

\[
\Gamma^{p\to n} = \frac{a^5 G_F^2 \bar{m}_e^3}{2^{5\pi/2} \pi \tau_{\Delta m}} \int \frac{ds}{2\pi i} \frac{\bar{m}_\nu^2}{\bar{m}_e} \left\{ \left( \Gamma(-s) \Gamma(-s + 2) + \left( \frac{\bar{m}_\nu}{\bar{m}_e} \right)^2 \Gamma(-s + 1) \Gamma(-s + 3) \right) \Gamma(s + i\Delta m + 2) \Gamma(-s - i\Delta m + 4) \Gamma(-s + t + 1) \Gamma(-s + t + 3) \\
\times \left( \frac{\Gamma(-s) \Gamma(-s + 2)}{\Gamma(-s + 1) \Gamma(n + 1)} \right) \Gamma(-s + t + 1) \Gamma(-s + t + 3) \right\} \\
- \sum_{n=0}^{\infty} \left( \frac{\bar{m}_\nu}{\bar{m}_e} \right)^{2n+3} \left[ \psi(n + 1) + \psi(n + 2) - \psi(-s + n + 2) - \psi(-s + n + 3) - 2 \ln \frac{\bar{m}_\nu}{\bar{m}_e} \right] \\
\times \left\{ \Gamma(-s + i\Delta m + 2) \Gamma(-s - i\Delta m + 4) \right\} \frac{\Gamma(-s + n + 2) \Gamma(-s + n + 3)}{\Gamma(n + 1) \Gamma(n + 2)} \right].
\]

(B3)

The simplest case is for massless neutrino. If we set \(m_\nu = 0\), then only the first term remains non-vanishing. After a valuable transformation \(s \to s + \frac{3}{2}\) we have

\[
\Gamma^{p\to n} \quad \text{with} \quad m_\nu = 0 = \frac{a^5 G_F^2 \bar{m}_e^3}{2^{5\pi/2} \pi \tau_{\Delta m}} \int \frac{ds}{2\pi i} \frac{\Gamma(-s + i\Delta m + \frac{3}{2})}{\Gamma(-s + \frac{3}{2}) \Gamma(-s + \frac{3}{2})} \left( \frac{\bar{m}_\nu}{\bar{m}_e} \right)^{\bar{m}_\nu^2 \bar{m}_e} \\
= \frac{G_F^2 \bar{m}_e^3 a^2}{32 \pi \tau_{\Delta m}} G^{28}_F \left( \frac{\bar{m}_\nu}{\bar{m}_e} \right)^{\frac{1}{2}, \quad \frac{3}{2}, \quad \frac{2}{3}, \quad \frac{2}{3} + i\Delta m, \quad \frac{2}{3} - i\Delta m}.
\]

(B4)

This is exactly the cross section obtained by Vanzella and Matsas with \(c_\nu = 1\) and \(c_A = 0\) (see Eq. (4.19) in [15]).

We perform the integration of Eq. (B3) with respect to \(s\) and find the following expansions of the cross section.
\[
\Gamma^{p \to n} = \frac{a^n G_P^2}{\pi^2 e^{2\Delta m}} \times \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{1}{\pi^2 e^{2\Delta m}} \Gamma(n+1) - \psi(n+1) + \psi(n+2) - \psi(n+3) + \psi(n+4) - \ldots \right] \right\}
\]

where \( B(p, q) = \Gamma(p+q) \Gamma(p) \Gamma(q) \) and \( \gamma \) is Euler’s constant.

This is the final form of the cross section. A natural question is that we have obtained the result which is not manifestly symmetric with respect to \( \bar{m}_\nu \) and \( \bar{m}_\tau \) although the original expression (31) is manifestly symmetric. The resolution of this puzzle is that the integral is of discontinuous type. Namely, we can obtain the result just by interchanging \( \bar{m}_\nu \) and \( \bar{m}_\tau \) for \( \bar{m}_\nu < \bar{m}_\tau \). This can be checked directly by changing the order of integrations.