Towards Noncommutative Integrable Systems

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\textbf{Abstract}

We present a powerful method to generate various equations which possess the Lax representations on noncommutative (1+1) and (1+2)-dimensional spaces. The generated equations contain noncommutative integrable equations obtained by using the bicomplex method and by reductions of the noncommutative (anti-)self-dual Yang-Mills equation. This suggests that the noncommutative Lax equations would be integrable and be derived from reductions of the noncommutative (anti-)self-dual Yang-Mills equation, which implies the noncommutative version of Richard Ward conjecture. The integrability and the relation to string theories are also discussed.

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1 Introduction

Non-Commutative (NC) gauge theories have been studied intensively for the last several years and succeeded in revealing various aspects of gauge theories in the presence of background magnetic fields [1]. Especially, NC solitons play crucial roles in the study of D-brane dynamics, such as tachyon condensation [2].

NC spaces are characterized by the noncommutativity of the coordinates:

\[ [x^i, x^j] = i\theta^{ij}, \]  

(1.1)

where \( \theta^{ij} \) are real constants. This relation looks like the canonical commutation relation in quantum mechanics and leads to “space-space uncertainty relation.” Hence the singularity which exists on commutative spaces could resolve on NC spaces. This is one of the distinguished features of NC theories and gives rise to various new physical objects. For example, even when the gauge group is \( U(1) \), instanton solutions still exist [3] because of the resolution of the small instanton singularities of the complete instanton moduli space [4].

NC gauge theories are naively realized from ordinary commutative theories just by replacing all products of the fields with star-products. In this context, NC theories are considered to be deformed theories from commutative ones and look very close to the commutative ones.

Under the deformation, the anti-self-dual \(^3\) (ASD) Yang-Mills equations could be considered to preserve the integrability in the same sense as in commutative cases [5, 6]. On the other hand, with regard to typical integrable equations such as the Kadomtsev-Petviashvili (KP) equation [7], the naive NC extension generally destroys the integrability. There is known to be a method, the bicomplex method, to yield NC integrable equations which have infinite number of conserved quantities [8, 9, 10].

On commutative spaces, there is a fascinating relationship between the ASD Yang-Mills equation and many other lower-dimensional integrable equations, that is, almost all integrable equations on (1 + 1) and (1 + 2)-dimensional spaces can be derived from the 4-dimensional ASD Yang-Mills equation by reductions. This is first conjectured by R. Ward [11] (Ward conjecture) and can be said to be positively confirmed now [12].

The reduced equations always possess the Lax representations [13] because the original ASD Yang-Mills equation also possesses the Lax representation in a wider sense. The Lax representations are common in many integrable equations. That is one of the reasons

\(^3\)There is essentially no difference between self-dual case and anti-self-dual case, which allows us to restrict ourselves to anti-self-dual case.
why Ward conjecture is reasonable and true. The equation which possesses the Lax representation is called the Lax equation.

The NC ASD Yang-Mills equation also possesses the Lax representation as the NC-deformed form. The reduced equations can also be NC Lax equations. Since the NC ASD Yang-Mills equation is integrable, it is expected that the reduced Lax equations are also integrable as in commutative case. This implies the NC version of Ward conjecture and would be expect to open the door to a new study area of integrable systems. However it is very hard to find NC Lax equations from reductions of the NC ASD Yang-Mills equation and so on. The successful examples seem to be found only in [14].

In this letter, we discuss NC extensions of wider class of integrable equations which are expected to preserve the integrability. First, we present a powerful method to generate various NC Lax equations. Then we discuss the relationship between the generated Lax equations and the NC integrable equations obtained from the bicomplex method and from reductions of the NC ASD Yang-Mills equation. All the results are consistent and some aspects of the integrability are proved, such as the linearizability of the NC Burgers equation, the existence of NC versions of KdV, KP and Burgers hierarchies and so on. These results strongly suggest that the NC Lax equations would be unique and integrable. Hence it is natural to propose the following conjecture which contains the NC version of Ward conjecture: many NC Lax equations would be integrable and be obtained from reductions of the NC ASD Yang-Mills equation. (See Fig. 1.) In the final section, more on the integrability and the relation to string theories are discussed.

![Diagram of NC Ward Conjecture](image-url)

Figure 1: NC Ward Conjecture
2 Noncommutative Lax Equations

2.1 The Lax-Pair Generating Technique

In commutative cases, Lax representations are common in many known integrable equations and fit well to the discussion of reductions of the ASD Yang-Mills equation. Here we look for the Lax representations on NC spaces. First we introduce how to find Lax representations on commutative spaces.

An integrable equation which possesses the Lax representation can be rewritten as the following equation:

\[ [L, T + \partial_t] = 0, \]

where \( \partial_t := \partial/\partial t \). This equation and the pair of operators \((L, T)\) are called the Lax equation and the Lax pair, respectively.

The NC version of the Lax equation (2.1), the NC Lax equation, is easily defined just by replacing the products of fields in \( L \) and \( T \) with the star products. The star product is defined for usual functions \( f(x) \) and \( g(x) \) by

\[ f \star g(x) := \exp \left( \frac{i}{2} \theta^{ij} \partial_i(x') \partial_j(x'') \right) f(x') g(x'') \bigg|_{x' = x'' = x}. \]

(2.2)

The star product has associativity: \( f \star (g \star h) = (f \star g) \star h \), and reduces to the ordinary product in the commutative limit \( \theta^{ij} \to 0 \). The modification of the product makes the ordinary coordinate “noncommutative,” which means:

\[ [x^i, x^j] := x^i \star x^j - x^j \star x^i = i\theta^{ij}. \]

We note that the coordinates and the functions themselves are the same as commutative ones and take c-number values. Hence the differentiations and the integrations of them are also the same as commutative ones. The effects of the NC deformation appear only in the products of the coordinates in the functions.

In this letter, we look for the NC Lax equation whose operator \( L \) is a differential operator. In order to make this study systematic, we set up the following problem:

**Problem**: For a given operator \( L \), find a corresponding operator \( T \) which satisfies the Lax equation (2.1).

This is in general very difficult to solve. However if we put an ansatz on the operator \( T \), then we can get the answer for wide class of Lax pairs including NC case. The ansatz for the operator \( T \) is of the following type:

**Ansatz for the operator \( T \)**:

\[ T = \partial_i^n L^m + T'. \]

(2.3)
Then the problem for $T$ reduces to that for $T'$. This ansatz is very simple, however, very strong to determine the unknown operator $T'$. In this way, we can get the Lax pair $(L, T)$, which is called, in this letter, the *Lax-pair generating technique*.

In order to explain it more concretely, let us consider the Korteweg-de-Vries (KdV) equation [15] on commutative $(1+1)$-dimensional space where the operator $L$ is given by $L_{	ext{KdV}} := \partial_x^2 + u(t, x)$.

The ansatz for the operator $T$ is given by

$$T = \partial_x L_{\text{KdV}} + T', \tag{2.4}$$

which corresponds to $n = m = 1$ and $\partial_t = \partial_x$ in the general ansatz (2.3). This factorization was first used to find wider class of Lax pairs in higher dimensional case [16].

The Lax equation (2.1) leads to the equation for the unknown operator $T'$:

$$[\partial_x^2 + u, T'] = u_x \partial_x^2 + u_t + uu_x, \tag{2.5}$$

where $u_x := \partial u / \partial x$ and so on. Here we would like to delete the term $u_x \partial_x^2$ in the RHS of (2.5) so that this equation finally reduces to a differential equation. Therefore the operator $T'$ could be taken as

$$T' = A \partial_x + B, \tag{2.6}$$

where $A, B$ are polynomials of $u, u_x, u_t, u_{xx}$, etc. Then the Lax equation becomes $f \partial_x^2 + g \partial_x + h = 0$. From $f = 0, g = 0$, we get\(^4\)

$$A = \frac{u}{2}, \quad B = -\frac{1}{4} u_x, \tag{2.7}$$

that is,

$$T = \partial_x^3 + \frac{3}{4} u_x + \frac{3}{2} u \partial_x. \tag{2.8}$$

Finally $h = 0$ yields the Lax equation, the KdV equation:

$$u_t + \frac{3}{2} uu_x + \frac{1}{4} u_{xxx} = 0. \tag{2.9}$$

In this way, we can generate a wide class of Lax equations including higher dimensional integrable equations [16]. For example, $L_{m\text{KdV}} := \partial_x^2 + v(t, x) \partial_x$ and $L_{\text{KP}} := \partial_x^2 + u(t, x, y) + \partial_y$ give rise to the modified KdV equation and the KP equation, respectively by the same ansatz (2.4) for $T$. If we take $L_{\text{BCS}} := \partial_x^2 + u(t, x, y)$ and the modified ansatz $T = \partial_y L_{\text{BCS}} + T'$, then we get the Bogoyavlenskii-Calogero-Schiff (BCS) equation [17].\(^5\)

Good news here is that this technique is also applicable to NC cases.

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\(^4\) In the present letter, we set the integral constants zero.

\(^5\) The multi-soliton solution is found in [18].
2.2 Some Results

We present some results by using the Lax-pair generating technique. First we focus on NC (1+2)-dimensional Lax equations where the time and space coordinates are denoted by $t$ and $(x, y)$, respectively. Let us suppose that the noncommutativity is basically introduced in the space directions, that is, $[x, y] = i\theta$.

- The NC KP equation [19]:

The Lax operator is given by

$$L_{\text{KP}} = \partial_x^2 + u(t, x, y) + \partial_y =: L'_{\text{KP}} + \partial_y.$$ (2.10)

The ansatz for the operator $T$ is the same as commutative case:

$$T = \partial_x L'_{\text{KP}} + T'.$$ (2.11)

Then we find

$$T' = \frac{1}{2} u \partial_x - \frac{1}{4} u_x - \frac{3}{4} \partial_x^{-1} u_y,$$ (2.12)

and the NC KP equation:

$$u_t + \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x \ast u + u \ast u_x) + \frac{3}{4} \partial_x^{-1} u_{yy} + \frac{3}{4} [u, \partial_x^{-1} u_y] = 0,$$ (2.13)

where $\partial_x^{-1} f(x) := \int^x dx' f(x')$, $u_{xxx} = \partial^3 u/\partial x^3$ and so on. This coincides with that in [19]. There is seen to be a nontrivial deformed term $[u, \partial_x^{-1} u_y]$ in the equation (2.13) which vanishes in the commutative limit. In [19], the multi-soliton solution is found by the first order to small $\theta$ expansion, which suggests that this equation would be considered as an integrable equation.

- The NC BCS equation:

This is obtained by following the same steps as in the commutative case. The new equation is

$$u_t + \frac{1}{4} u_{xyy} + \frac{1}{2} (u_y \ast u + u \ast u_y) + \frac{1}{4} u_x \ast (\partial_x^{-1} u_y)$$

$$+ \frac{1}{4} (\partial_x^{-1} u_y) \ast u_x + \frac{1}{4} [u, \partial_x^{-1} [u, \partial_x^{-1} u_y]] = 0,$$ (2.14)

whose Lax pair and the ansatz are

$$L_{\text{BCS}} = \partial_x^2 + u(t, x, y),$$

$$T = \partial_y L_{\text{BCS}} + T',$$

$$T' = \frac{1}{2} (\partial_x^{-1} u_y) \partial_x - \frac{1}{4} u_y - \frac{1}{4} \partial_x^{-1} [u, \partial_x^{-1} u_y].$$ (2.15)

This time, a non-trivial term is found even in the operator $T$. 

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We can generate many other NC Lax equations in the same way. (For explicit examples, see [22].) Moreover if we introduce the noncommutativity into time coordinate as $[t, x] = i\theta$, we can construct NC (1 + 1)-dimensional integrable equations. The introduction of noncommutativity in the time direction makes it hard or ambiguous to define the initial value problem and the integrability. We discuss this point for explicit examples as follows.

For example, the NC KdV equation is

$$u_t + \frac{3}{4}(u_x \star u + u \star u_x) + \frac{1}{4}u_{xxx} = 0,$$  \hspace{1cm} (2.16)

which coincides with that derived by using the bicomplex method [9] and by the reduction from NC KP equation (2.13) setting the fields $y$-independent: $\partial_y u = 0$. Here we reintroduce the noncommutativity as $[t, x] = i\theta$.\(^6\)

If we take the ansatz $T = \partial^6_x L_{KdV} + T'$, the $(n + 2)$-th NC KdV hierarchy equations are derived. (The 3-th NC KdV hierarchy equation is just the NC KdV equation.) This time, we can find the explicit form of $T'$ as $T' = \sum_{l=0}^{n} A_l \partial_x^{n-l}$ where $A_l$ are homogeneous polynomials of $u, u_x, u_{xx}$ and so on, whose degrees are $[A_l] = l + 2$. The important point is that these unknown polynomials are determined one by one as $A_0 = (n/2)u$ and so on, which guarantees the existence of the NC KdV hierarchy. Moreover it is interesting that the $2n$-th NC KdV hierarchy equations seem to be also trivial as commutative case. These results strongly suggest the possibility of NC extension of Sato theory [20], which will be discussed more in detailed [21]. The explicit and detailed discussions of the NC KdV hierarchy are found in [22].

As one of the new Lax equations, the NC Burgers equation is obtained:

$$u_t - \alpha u_{xx} + (1 - \alpha - \beta)u \star u_x + (1 + \alpha - \beta)u_x \star u = 0.$$  \hspace{1cm} (2.17)

We can linearize it by the following two kind of NC Cole-Hopf transformations [23, 24].\(^7\)

$$u = \psi^{-1} \star \psi_x,$$  \hspace{1cm} (2.18)

only when $1 + \alpha - \beta = 0$, and

$$u = -\psi_x \star \psi^{-1},$$  \hspace{1cm} (2.19)

\(^6\)We note that this reduction is formal and the noncommutativity here contains subtle points in the derivation from the (2+2)-dimensional NC ASD Yang-Mills equation by reduction because the coordinates $(t, x, y)$ originate partially from the parameters in the gauge group of the NC Yang-Mills theory [12]. We are grateful to T. Ivanova for pointing out this point to us.

\(^7\)After submitting the present letter and our paper [23], we were aware of the paper [24] by L. Martina and O. K. Pashaev on arXiv e-print server, which contains some overlaps with ours.
only when $1 - \alpha - \beta = 0$. The linearized equation is the NC diffusion equation

$$\psi_t = \alpha \psi_{xx}, \quad (2.20)$$

which is solvable via the Fourier transformation. Hence the NC Burgers equation is really integrable.

It is very interesting that an equation on NC $(1 + 1)$-dimensional spaces with infinite number of time derivatives can be mapped to the NC diffusion equation with the first derivative of time. This shows that the NC Burgers equation, an infinite differential equation, is actually integrable! In the exact solutions, non-trivial effects of the NC-deformation are also seen [23, 24].

The transformations (2.18) and (2.19) are analogy of the commutative Cole-Hopf transformation $u = \partial_x \log \psi$ [25]. This success makes us expect the possibility of NC extensions of the Hirota’s method [26] which is a strong and direct way to construct the exact multi-soliton solutions.

The $(n\text{-th})$ NC Burgers hierarchy equations are also obtained by taking the ansatz $T = \partial^2_x L_{\text{Burgers}} + T'$ [23]. The NC KP and BCS hierarchies could be also derived by similar ansatz. Hence the reductions of $(1 + 2)$-dimensional NC equations would give rise to $(1 + 1)$-dimensional NC equations, which is a part of the NC Sato theory [21].

Moreover, all of the NC integrable equations derived from the bicomplex method can also be obtained by our method. The bicomplex method guarantees the existence of infinite number of conserved topological quantities.

All these results suggest that NC Lax equations would possess the integrability.

Let us here comment on the multi-soliton solutions. First we note that if the field is holomorphic, that is, $f = f(x - vt) = f(z)$, then the star product reduces to the ordinary product:

$$f(x - vt) \star g(x - vt) = f(x - vt)g(x - vt). \quad (2.21)$$

Hence the commutative multi-soliton solutions where all the solitons move at the same velocity always satisfy the NC version of the equations. Of course, this does not mean that the equations possess the integrability. The integrability is characterized in this context by the existence of multi-soliton solution with non-trivial scattering processes.

The comprehensive list and the more detailed discussion are reported later soon.
3 Comments on the Noncommutative Ward Conjecture

In commutative case, it is well known that many integrable equations could be derived from symmetry reductions of the four-dimensional ASD Yang-Mills equation [12], which is first conjectured by R.Ward [11] (Ward conjecture).

Even in NC case, the corresponding discussions are possible and interesting. The NC ASD Yang-Mills equation also has Yang’s form [6, 27] and many other similar properties to commutative cases [5]. A simple reduction to three dimension yields the NC Bogomol’nyi equation which has the exact monopole solutions and can be rewritten as the non-Abelian Toda lattice equation [6, 28]. It is interesting that a discrete structure appears.

Moreover M.Legaré [14] succeeded in some reductions of the (2 + 2)-dimensional NC ASD Yang-Mills equation, which coincide with our results and those by using the bicomplex method. The NC Burgers equation is also derived from the ASD equation in NC $U(1)$ Yang-Mills theory, which is discussed in the revised version of [23]. In NC theories, the $U(1)$ part is important and the parameters in NC Burgers equation (2.17) actually come from the commutator part such as $[A_\mu, A_\nu]$ in the field strength of the original NC Yang-Mills theory, which becomes trivial in the commutative limit. These results strongly suggest that the NC deformation is unique and integrable and the Ward conjecture still holds on NC spaces.

In four-dimensional Yang-Mills theory, the NC deformation resolves the small instanton singularity of the (complete) instanton moduli space and gives rise to a new physical object, the $U(1)$ instanton. Hence the NC Ward conjecture would imply that the NC deformations of lower-dimensional integrable equations might contain new physical objects because of the deformations of the solution spaces in some case.

4 Conclusion and Discussion

In the present letter, we found a powerful method to find NC Lax equations which is expected to be integrable. The simple, but mysterious ansatz (2.3) plays an important role and actually gives rise to various new NC Lax equations. Finally we pointed out that some reductions of the NC ASD Yang-Mills equation gives rise to NC integrable equations including our results.

Now there are mainly three methods to yield NC integrable equations:

- Lax-pair generating technique
- Bicomplex method
Reduction of the ASD Yang-Mills equation

The interesting point is that all the results are consistent at least with the known NC Lax equations, which suggests the existence and the uniqueness of the NC deformations of integrable equations which preserve the integrability.

Though we can get many new NC Lax equations, there need to be more discussions so that such study should be fruitful as integrable systems. First, we have to clarify whether the NC Lax equations are really good equations in the sense of integrability, that is, the existence of many conserved quantities or of multi-soliton solutions, and so on. All of the previous studies including our works strongly suggest that this should be true. Second, we have to reveal the physical meaning of such equations. If such integrable theories can be embedded in string theories, there would be fruitful interactions between the both theories, just as between the (NC) ASD Yang-Mills equation and D0-D4 brane system (in the background of NS-NS $B$ field).

For NC $(1 + 1)$-dimensional equations, the initial value problem is even hard to define and the meaning of the integrability may be vague. However we have got such kind of integrable equations, for example, the NC Burgers equation which is linearizable. This is somewhat surprising and it is also interesting to reveal how the integrability is defined or how the initial value problem is solved with infinite number of time derivatives. The mechanism may relate the physical meaning of the introduction of the noncommutativity.

We believe that the systematic and co-supplement studies of them will pioneer a new area of integrable systems and perhaps string theories.

Note added

We were informed by O. Lechtenfeld that the $(2 + 2)$-dimensional NC ASD Yang-Mills equation and some reductions of it can be embedded [29, 30] in $N = 2$ string theory [31], which guarantees that such directions would have a physical meaning and might be helpful to understand new aspects of the corresponding string theory.

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