The renormalization of the effective Lagrangian with spontaneous symmetry breaking: the $SU(2)$ case

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We study the renormalization of the nonlinear effective $SU(2)$ Lagrangian up to $O(p^4)$ with spontaneous symmetry breaking. The Stueckelberg transformation, the background field gauge, the Schwinger proper time and heat kernel method, and the covariant short distance expansion technology, guarantee the gauge covariance and incorporate the Ward identities in our calculations. The renormalization group equations of the effective couplings are derived and analyzed. We find that the difference between the results gotten from the direct method and the renormalization group equation method can be quite large when the Higgs scalar is far below its decoupling limit.

I. INTRODUCTION

In our last paper [1], we discussed the renormalization of the nonlinear effective $U(1)$ Lagrangian. We learn from the case that in the framework of effective theory we can do the renormalization of the nonrenormalizable and nonlinear interactions order by order. In this paper, we use those related conceptions and methods to the non-Abelian cases. We will study the renormalization of the nonlinear effective $SU(2)$ Lagrangian $L^{eff}$ up to $O(p^4)$ and derive the renormalization group equations (RGE) of its effective couplings.

We will also numerically study the solutions of these RGE, and analyze the decoupling and nondecoupling effects of the Higgs boson to those effective couplings in the effective Lagrangian $L^{eff}$. We find that when the Higgs scalar is far below its decoupling limit, our results are significantly different from the results gotten by matching the full theory and

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effective directly at the one-loop level [2] (Hereby, we call this method the direct method, in contrast with the RGE method). The basic reason for this large difference is that the direct method ignores the contribution of the possible large tree level contributions of not too heavy Higgs, which can considerably affect the effective couplings through radiative corrections. While the RGE method has taken into account these important effects.

The paper is organized as following. In the section II, we briefly introduce the renormalizable $SU(2)$ Higgs model, and concentrate on its form in unitary gauge and the quartic divergence term. In the section III, the nonlinear effective $SU(2)$ Lagrangian $L_{\text{eff}}$ up to $O(p^4)$ is obtained by integrating-out the scalar Higgs boson at the tree level. We emphasize the importance of the quartic divergence terms. In the section VI, we perform the renormalization of the $L_{\text{eff}}$ up to $O(p^4)$ in the background field gauge, and by using the Schwinger proper time and heat kernel method, derive the renormalization group equations so as to sum the leading logarithm contributions of radiative corrections. Section V is devoted to study the numerical solutions of these RGEs in the Higgs scalar’s decoupling and nondecoupling limits. We end the paper with some discussions and conclusions.

## II. THE RENORMALIZABLE $SU(2)$ HIGGS MODEL

The partition functional of the renormalizable non-Abelian $SU(2)$ Higgs model [3] (Here we have not included the gauge fixing term and the ghost term.) can be expressed as

$$Z = \int DA^a_\mu D\phi D\phi^\dagger \exp \left( iS[A, \phi, \phi^\dagger] \right),$$  \hspace{1cm} (1)

where the action $S$ is determined by the following Lagrangian density

$$L = -\frac{1}{4g^2}W^a_\mu W^{a\mu} + (D\phi)^\dagger \cdot (D\phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2,$$ \hspace{1cm} (2)

and the definition of quantities in this Lagrangian is given below

$$W^a_\mu = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + f^{abc}W^b_\mu W^c_\nu;$$ \hspace{1cm} (3)

$$D_\mu \phi = \partial_\mu \phi - i W^a_\mu T^a \phi;$$ \hspace{1cm} (4)

$$\phi^\dagger = (\phi_1^\dagger, \phi_2^\dagger),$$ \hspace{1cm} (5)

where $T^a$ are the generators of the Lie algebra of $SU(2)$ gauge group.

The spontaneous symmetry breaking is induced by the positive mass square $\mu^2$ in the Higgs potential. The vacuum expectation value of Higgs field is given as $|\langle \phi \rangle| = v/\sqrt{2}$. 

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And by eating the corresponding Goldstone boson, the vector bosons $W$ obtain their mass.

The non-linear form of the Lagrangian given in Eqn. (2) is made by changing the variable $\phi$

$$\phi = \frac{1}{\sqrt{2}}(v + \rho)U, \quad U = \exp\left(\frac{i\xi a T^a}{v}\right), \quad v = 2\sqrt{\frac{\mu^2}{\lambda}},$$

where the field $U$ is the Goldstone boson as prescribed by the Goldstone theorem, and the $\rho$ is a massive scalar field. Then it reaches

$$L' = -\frac{1}{4g^2} W^a_{\mu\nu} W^{a\mu\nu} + \frac{(v + \rho)^2}{2} (DU)\cdot (DU) + \frac{1}{2} \partial \rho \cdot \partial \rho + \frac{1}{2} \mu^2 (v + \rho)^2 - \frac{\lambda}{16} (v + \rho)^4.$$

And the change of variables induces a determinant factor in the functional integral $Z$

$$Z = \int DW^a_{\mu\nu} D\rho D\xi^b \exp (iS'[W, \rho, \xi]) \det \left\{ 1 + \frac{1}{v\rho} \right\} \delta(x - y).$$

The determinant can be written in the exponential form, and correspondingly the Lagrangian density is modified to

$$L \rightarrow L' - i\delta(0) \ln \left\{ 1 + \frac{1}{v\rho} \right\}.$$

The determinant containing quartic divergences is indispensable and crucial to cancel exactly the quartic divergences brought into by the longitudinal part of vector boson, and is important in verifying the renormalizability of the Higgs model in the U-gauge [4].

III. THE NONLINEAR EFFECTIVE $SU(2)$ LAGRANGIAN $L^{\text{EFF}}$ UP TO $O(P^4)$

In the nonlinear effective $SU(2)$ Lagrangian $L^{\text{eff}}$, only the Goldstone and the vector bosons are included as the effective dynamic freedom at low energy region. The Lagrangian $L^{\text{eff}}$, if including all permitted operators composed by these light degrees of freedom (DOFs) and respecting the assumed Lorentz and gauge symmetries, is still renormalizable [5]. Two facts are important for the actual renormalization procedure: 1) The Wilsonian renormalization method [6] and the surface theorem [7] reveals that at the low energy region, only few operators play important parts to determine the behavior of the dynamic system at the low energy region, such a fact enables us to truncate the infinite divergence tower and to consider the renormalization of the effective Lagrangian order by order; 2) While the quartic divergence terms in the effective Lagrangian enable it possible to consistently throw away all quartic divergences.
The general effective $SU(2)$ Lagrangian $L^{eff}$ consistent with Lorenz spacetime symmetry, $SU(2)$ gauge symmetry, and the charge, parity, and the combined CP symmetries, can be formulated as

\[ L^{eff} = L_2 + L_4 + \cdots + L_{qd}, \]  
\[ L_2 = -\frac{\nu^2}{2} tr[V_{\mu}V^\mu], \]  
\[ L_4 = -\frac{1}{4g^2} W^{a\mu} W^{a\mu} - id_1 tr[W_{\mu\nu}V^\mu V^\nu] 
+ d_2 tr[V_{\mu}V_{\nu}]tr[V^\mu V^\nu] + d_3 tr[V_{\mu}V^\mu]tr[V_{\nu}V^\nu], \]  
\[ \cdots, \]  
\[ L_{qd} = i\delta(0) \left\{ e_1 \frac{tr[V_{\mu}V^\mu]}{m_1^2} + e_2 \frac{(tr[V_{\mu}V^\mu])^2}{m_1^4} + \cdots \right\}, \]

where the $L_2$ and $L_4$ represent the relevant and marginal operators in the Wilsonian renormalization method, respectively. The operators in the $L_2$ and $L_4$ also form the set of complete operators up to $O(p^4)$ in the usual momentum counting rule. The higher dimension (irrelevant) operators than $O(p^4)$ order are represented by the dots and omitted here. The auxiliary variable $V_{\mu}$ is defined as

\[ V_{\mu} = U^\dagger D_{\mu} U, \]

to simplify the representation. Due to the following relations of the $SU(2)$ gauge group

\[ tr[T^a T^b T^c T^d] = \frac{1}{8}(\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}), \]

the terms, like $tr[V_{\mu}V_{\nu}V^\mu V^\nu]$ and $tr[V_{\mu}V^\mu V_{\nu}V^\nu]$, can be linearly composed by $tr[V_{\mu}V_{\nu}]tr[V^\mu V^\nu]$ and $tr[V_{\mu}V^\mu]tr[V_{\nu}V^\nu]$. And since here we do not consider the term which breaks the charge, or parity, or both symmetries, therefore, the operators in Eqn. (12) are complete and linearly independent.

The effective couplings of $d_i$ form the parameter space of effective theory, and they effectively reflect the dynamics of the underlying theories and the ways of symmetry breaking. Different underlying theories and ways of symmetry breaking will fall into a special point in this effective parameter space. And the $d_i$ can also be called the anomalous couplings if seeing from the renormalizable $SU(2)$ gauge theory, they reflect the deviation from the requirement of renormalizability.

When the scalar Higgs is heavy and integrated out, the Higgs model given in Eq. (2) can be effectively described as a special parameter point of the effective Lagrangian given in Eq. (10). At the tree level, it suffices to integrate out the Higgs scalar boson by using
the equation of motion of it, which expresses it into the low energy dynamic DOFs and can be formulated as

\[ \rho = \frac{v}{m_0^2} (DU)^\dagger \cdot (DU) + \cdots, \]  

(16)

\[ m_0^2 = \frac{1}{2} \lambda v^2, \]  

(17)

where \( m_0 \) is the mass of Higgs bosons. The omitted terms contain at least four covariant partials and belong to higher order operators.

By substituting Eqn. (16) into Eqn. (9), at the matching scale (which is always taken at the scalar mass \( \mu = m_0 \)) the effective couplings at the tree level are determined as

\[ d_1(m_0) = 0, \quad d_2(m_0) = 0, \quad d_3(m_0) = \frac{v^2}{2m_0^2} = \frac{1}{\lambda v^2}, \cdots, \]  

(18)

In its decoupling limit \( m_0 \to \infty \), all these three effective couplings vanish. If a field does not participate in the process of symmetry breaking, we know it will not contribute to the anomalous couplings up to the \( O(p^4) \) order and its effects to the low energy dynamics will be simply suppressed by its squared mass according to the decoupling theorem [8].

**IV. THE RENORMALIZATION OF \( L^{\text{EFF}} \) AND ITS RENORMALIZATION GROUP EQUATIONS**

In the background field method (BFM) [9,10], the number of the Feynman diagrams for the loop corrections can be greatly decreased when compared with the standard Feynman diagram method. Another remarkable advantage is that, in the BFM, each step of calculation is manifestly gauge covariant with reference to the background gauge field, and the Ward identities — which are important to restrain the structure of divergences — have been incorporated in the calculation. The Schwinger proper time and heat kernel method [11] by itself is the Feynman integral. Combining with the covariant short distance Taylor expansion [12] in coordinate space, the divergent structures can be directly extracted out in the explicit gauge form and the loop calculation can be simplified to a considerable degree.

**A. The quadratic terms of the one-loop Lagrangian**

According to the spirit of the BFM, we split the Goldstone and vector bosons into classic and quantum parts, as given below
The Stueckelberg transformation [13] combines $\hat{W}$ and $\hat{U}$ into the Stueckelberg field $\hat{W}^s$

$$\hat{W}^s = \hat{U}^i \hat{W}^i U_i + i \hat{U}^i \partial_i \hat{U},$$

and eliminates the background Goldstone from the effective Lagrangian. After finishing the loop calculation, by performing the inverse Stueckelberg transformation (expanding the $\hat{W}^s$ in the $\hat{W}$ and $\hat{U}$), the effective Lagrangian can be restored to the form expressed by its low energy DOFs.

As one of the advantages of the BFM, we have the freedom to choose different gauge for the background and quantum fields, and such a freedom can help to further simplify the calculation. Then for the quantum fields, we can choose the covariant gauge fixing term as

$$\mathcal{L}_{GF} = \frac{-1}{2g^2} [(D^i \hat{W})^a + c_f f^{abc} \hat{W}^a b \cdot \hat{W}^c + f_{ws} \xi^a]^2,$$

where $c_f$ and $f_{ws}$ are determined by requiring the one-loop Lagrangian to have the standard form given in Eqn. (26—32), then it reads

$$c_f = \frac{1}{2} d_1 g^2, \quad f_{ws} = v g^2.$$

The partition functional $Z$ in the background field gauge can be expressed as

$$Z = \exp \left( i S_{ren}[\hat{W}] \right) = \exp \left( i S_{tree}[\hat{W}^s] + i \delta S_{tree}[\hat{W}^s] + i \delta_1 \text{loop}[\hat{W}^s] + \cdots \right) = \exp \left( i S_{tree}[\hat{W}] + i \delta S_{tree}[\hat{W}] \right) \int D\hat{W}_\mu D\bar{c} Dc D\xi \exp \left( i S[\hat{W}, \xi, \bar{c}, c; \hat{W}^s] \right),$$

where the tree effective Lagrangian $\mathcal{L}_{tree}$ is in the following form

$$\mathcal{L}_{tree} = \frac{\nu^2}{2} \hat{W}^s \cdot \hat{W}^s - \frac{1}{4g^2} \hat{W}^s a \hat{W}^s j \mu \nu a + d_1 \frac{1}{4} f^{abc} \hat{W}^a b \hat{W}^{\mu b} \hat{W}^{\nu c}$$

$$+ d_2 \frac{1}{4} \hat{W}^s a \cdot \hat{W}^s b \hat{W}^s a \cdot \hat{W}^s b + d_3 \frac{1}{4} \left( \hat{W}^s \cdot \hat{W}^s \right)^2 + \cdots$$

$$+ i \delta(0) \left[ e_1 \frac{\hat{W}^s \cdot \hat{W}^s}{m_1^2} + e_2 \frac{\left( \hat{W}^s \cdot \hat{W}^s \right)^2}{m_1^2} + \cdots \right].$$

And the corresponding counter terms $\delta \mathcal{L}_{tree}$ are defined as

$$\delta \mathcal{L}_{tree} = \delta Z_{v^2} \frac{\nu^2}{2} \hat{W}^s a \cdot \hat{W}^s a - \delta Z_{g^2} \frac{1}{4g^2} \hat{W}^s a \hat{W}^s j \mu \nu a + \delta Z_{d_1} d_1 \frac{1}{4} f^{abc} \hat{W}^a b \hat{W}^{\mu b} \hat{W}^{\nu c}$$

$$+ \delta Z_{d_2} d_2 \frac{1}{4} \hat{W}^s a \cdot \hat{W}^s b \hat{W}^s a \cdot \hat{W}^s b + \delta Z_{d_3} d_3 \frac{1}{4} \left( \hat{W}^s \cdot \hat{W}^s \right)^2 + \cdots$$

$$+ i \delta(0) \left[ \delta e_1 \frac{\hat{W}^s \cdot \hat{W}^s}{m_1^2} + \delta e_2 \frac{\left( \hat{W}^s \cdot \hat{W}^s \right)^2}{m_1^2} + \cdots \right].$$
where the renormalization constant of the Stueckelberg field $\tilde{W}^a$ can always be set to 1.

In the one-loop level, only the quadratic terms of quantum fields are related, and they can be cast into the following standard form

$$\mathcal{L}_{quad} = \frac{1}{2} \tilde{W}_\mu^a \square_{WW} \tilde{W}_\nu^b + \frac{1}{2} \xi^a \partial_\xi \xi^b + \bar{c}^a \partial_\xi c^b$$

$$+ \frac{1}{2} \tilde{W}_\mu^a \partial_\xi \xi^b + \frac{1}{2} \xi^a \tilde{W}_\nu^b$$

$$(26)$$

$$\square_{WW}^{\alpha\beta} = (D_{2,ab} + m_W^2 \delta_{ab}) g^{\alpha\beta} - \sigma^{\alpha\beta}_{WW},$$

$$(27)$$

$$\xi^a = \xi^a + X^\alpha \partial_\alpha \xi^a + X^\alpha \partial_\alpha \gamma^\alpha,$$

$$(28)$$

$$\tilde{X}^{\alpha,ab} = X_{03}^{\alpha,ab} + \partial_\alpha X_{03}^{\alpha,ab} + \sigma^{\alpha,ab},$$

$$(29)$$

$$\tilde{X}^{\nu,ab} = X_{03}^{\nu,ab} + X_{01}^{\nu,ab} + \sigma^{\nu,ab},$$

$$(30)$$

where $d_\mu = 0$ and $D_\mu = 0$. The direction of the harpoon indicates the position of vector bosons, and both the $X^{\alpha,ab}$ and $X^{\nu,ab}$ are defined to act on the right side. For the $SU(2)$ effective Lagrangian, the related quantities are defined as

$$\sigma^{\alpha\beta,ab} = 2 \tilde{W}_G^{\alpha\beta,ab} + \frac{1}{2} \tilde{W}_G^{\alpha\beta,ab}$$

$$- d^2 g^2 \left( W_G^{\alpha\beta,ab} + F_G^{\alpha\beta,ab} \right),$$

$$- d^2 g^2 \left( W_G^{\alpha\beta,ab} + F_G^{\alpha\beta,ab} \right),$$

$$(33)$$

$$\sigma^{\alpha\beta,ab} = X_{03}^{\alpha,ab} + \frac{1}{2} \partial_\xi A^a_{\alpha\beta},$$

$$- S_{\alpha\beta,ab} \gamma_{\xi\xi} \gamma_{\xi\alpha},$$

$$(35)$$

$$X^{\alpha,ab} = X_{03}^{\alpha,ab} + \partial_\alpha X_{03}^{\alpha,ab} + 2 S_{\alpha\beta,ab} \gamma_{\xi\beta},$$

$$(36)$$

$$X^{\alpha,ab} = X_{01}^{\alpha,ab},$$

$$X^{\nu,ab} = X_{03}^{\nu,ab} - \frac{1}{2} A_{\alpha\beta},$$

$$(37)$$

$$X^{\nu,ab} = X_{01}^{\nu,ab},$$

$$X^{\nu,ab} = X_{03}^{\nu,ab} + \frac{1}{2} \gamma_{\xi\xi} \gamma_{\xi\alpha},$$

$$+ \frac{1}{2} A_{\alpha\beta},$$

$$(41)$$
To get the above form, we have normalized the vector quantum gauge field by using
\[ \hat{X}_{\mu,ab} = \frac{1}{\xi} \tilde{X}_{\mu,ab}, \]
\[ \hat{X}_{\nu,ab} = \frac{1}{\xi} \tilde{X}_{\nu,ab}, \] (42)
where
\[ \tilde{X}_{\mu,ab} = X_{\mu,ab} - \delta^\beta_{\alpha} \tilde{X}_{\alpha,ba} + 2\tilde{S}_{\alpha\beta} \Gamma^{\beta cb} g^{a' a}, \]
\[ \tilde{X}_{\nu,ab} = X_{\nu,ab} - \delta^\beta_{\alpha} \tilde{X}_{\alpha,ba} + 2\tilde{S}_{\alpha\beta} \Gamma^{\beta cb} g^{a' a}, \] (43)
\[ X = \tilde{X}_2 - \tilde{X}_1 - \delta^\beta_{\alpha} \tilde{X}_{\alpha,ba} + 2\tilde{S}_{\alpha\beta} \Gamma^{\beta cb} g^{a' a'}, \]
\[ X_{01} = X_{01}, \]
\[ X_{03Z} = \tilde{X}_2 - \tilde{X}_1 - \delta^\beta_{\alpha} \tilde{X}_{\alpha,ba} + 2\tilde{S}_{\alpha\beta} \Gamma^{\beta cb} g^{a' a}, \]
\[ X_{03Y} = -\tilde{X}_{01}, \]
(44)
(45)
(46)
where
\[ \tilde{F}_{\mu\nu} = f^{abc} \tilde{W}_{\mu} \tilde{W}^{bc}_{\nu}, \]
\[ W_{\mu} = i f^{abc} \tilde{W}^{bc}_{\mu}, \]
\[ \Gamma_{\mu\nu} = -i a_{\mu} \tilde{W}^{ab}_{\mu G}, \]
\[ \Gamma_{ab} = -i a_{W} \tilde{W}^{ab}_{\mu G}, \]
with \( a_{\xi} = 1/2 \) and \( a_{W} = (1 + d_1 g^2/2) \) (which can be regarded as the effective charge).

To get the above form, we have normalized the vector quantum gauge field by using \( \tilde{W} \rightarrow \tilde{W}/g \).

When we take the limit \( d_i \rightarrow 0 \), the \( \sigma_{\mu\nu}^{ab} \) reaches to its usual form \( 2iW_{\mu}^{\mu\nu,ab} \),
as given in the gauge theory without symmetry breaking mechanism.

As in the \( U(1) \) case, an auxiliary dimension counting rule is introduced to extract relevant terms up to \( O(p^4) \), which reads
\[ [W_{\mu}]_a = [\partial_\mu]_a = [D_\mu]_a = 1, [v]_a = 0. \] (48)
From this rule, we know
\[ [\tilde{X}_{\mu,ab}]_a = [X_{\mu,ab}]_a = [X_{01}]_a = [X_{01}]_a = 1, \]
\[ [\sigma_{\mu,ab}^{\mu\nu}]_a = [\sigma_{2,\xi\xi}^{ab}]_a = [X_{\mu,ab}]_a = [X_{\mu,ab}]_a = [X_{03Y}]_a = [X_{03Y}]_a = 2, \]
\[ [X_{\mu,ab}]_a = [X_{03Z}]_a = [X_{03Z}]_a = 3, [\sigma_{a,\xi\xi}^{ab}]_a = 4. \] (49)
(50)
(51)
We would like to mention that this auxiliary dimension counting rule is to extract those terms with the two, three and four external fields. In the limit that all anomalous couplings equal to zero, only the \( X_{01}, X_{01}, \sigma_{\mu,ab}^{\mu\nu}, \) and \( \sigma_{a,\xi\xi}^{ab} \) do not vanish.

The tilded quantities are determined from the following pre-standard form \[ [10] \]
\[ \xi^a \partial^a \xi^b = -\xi^a \left( d^{2,ab} + \delta^{ab} m_1^2 \right) \xi^b + \xi^a \sigma_{2,\xi\xi}^{ab} + \tilde{X}^{ab}_{\mu,ab} \xi^b, \]
\[ + \xi^a \tilde{X}^{\mu,ab} \partial_\mu \xi^b + \partial_\alpha \xi^a \tilde{X}^{\alpha,ab} \partial_\beta \xi^b, \] (52)
\[ \tilde{W}^{a,\mu,ab}_{\mu} = \xi^a \tilde{W}^{a,\mu,ab}_{\mu}, \]
\[ = \partial_\alpha \tilde{W}^{a,\mu,ab}_{\mu} \partial_\beta \xi + \tilde{W}^{a,\mu,ab}_{\mu} \tilde{X}^{\mu,ab}_{\mu} \partial_\alpha \xi^b + \partial_\alpha \tilde{W}^{a,\mu,ab}_{\mu} \tilde{X}^{\mu,ab}_{\mu} \xi^b, \]
\[ + \tilde{W}^{a,\mu,ab}_{\mu} \tilde{X}^{\mu,ab}_{\mu} \xi^b + \tilde{W}^{a,\mu,ab}_{\mu} \tilde{X}^{\mu,ab}_{\mu} \xi^b. \] (53)
and from the effective Lagrangian given in Eqn. (10), we get
\[ \widetilde{X}^{\alpha\beta,ab} = \widetilde{S}^{\alpha,ab} + \widetilde{A}^{\alpha,ab} , \]
\[ \widetilde{S}^{\alpha,ab} = \frac{d_2}{v^2} \left( W^{s\alpha,b} W^{s\mu,c}_\gamma + W^{s\alpha,c} W^{s\mu,b}_\gamma + \frac{1}{2} H^{\alpha,ab}_W \right) \]
\[ + \frac{d_3}{v^2} \left( W^{s\alpha,c} \cdot W^{s\mu,c} g^{\mu\nu} \delta_{ab} + H^{\alpha,ab}_W \right) , \]
\[ \widetilde{A}^{\alpha,ab} = -i \frac{d_1}{2v^2} W^{s\alpha,ab}_G + \frac{2d_3 - d_2}{2v^2} (W^{s\alpha,a} W^{s\mu,b} - W^{s\alpha,b} W^{s\mu,a} ) , \]
\[ \widetilde{X}^{a,ab} = - \frac{d_1}{2v^2} \left( W^{s\alpha,ab}_G \wbar{W}^{\beta,ab}_G + W^{s\alpha,ab}_G \wbar{W}^{\beta,ab}_G \right) \]
\[ + i \frac{d_3}{v^2} W^{s\alpha,c} \wbar{W}^{s\beta,c}_G + \frac{d_3}{v^2} W^{s\alpha,c} \wbar{W}^{s\beta,c}_G , \]
\[ \widetilde{X}^{s,cb}_G = \frac{d_1}{4v^2} W^{s\alpha,ac}_G F^{s,cb}_{\alpha,G} , \]
\[ \widetilde{X}^{s,ac}_G = \widetilde{S}^{\alpha,ac} + \wbar{A}^{\alpha,ac} , \]
\[ \widetilde{S}^{\alpha,ac} = \frac{i}{4v} \left( 2 W^{s\alpha,ab}_G g_{\alpha,ab} - W^{s\alpha,ab}_G g_{\beta} - W^{s\alpha,ab}_G g_{\alpha} \right) , \]
\[ \widetilde{A}^{\alpha,ac} = - \frac{i d_1}{4v} (W^{s\alpha,ab}_G \wbar{W}^{s\alpha,ab}_G \wbar{W}^{s\alpha,ab}_G) , \]
\[ \widetilde{X}^{\mu,ab} = \frac{d_1}{2v} \left( W^{s\alpha,ac}_G \cdot W^{s\mu,cb}_G g^{\mu} - W^{s\alpha,ac}_G W^{s\mu,cb}_G - i W^{s\mu,ab}_G \right) \]
\[ - \frac{d_2}{v} (W^{s\alpha,b} \cdot W^{s\mu,a} g^{\mu} + W^{s\alpha,b} W^{s\mu,a} \delta_{ab} + W^{s\mu,b} W^{s\alpha,a} ) \]
\[ - \frac{d_3}{v} (W^{s\alpha,c} \cdot W^{s\mu,c} g^{\mu} \delta_{ab} + 2 W^{s\alpha,b} W^{s\mu,a} ) , \]
\[ \widetilde{X}^{s,ac}_G = \frac{d_1}{2v} F^{s,ac}_G , \]
\[ \widetilde{X}^{s,ab}_G = -igv(1 + \frac{d_1 g^2}{2}) W^{s,ab}_G , \]
\[ \widetilde{X}^{s,ac}_G = \frac{d_1}{2v} (i W^{s,ac}_G W^{s,cd}_G + i W^{s,ac}_G W^{s,cd}_G + W^{s,ac}_G W^{s,cd}_G) \]
\[ \widetilde{X}^{s,ac}_G , \]

The $H^{\alpha,ab}_W$ is defined as $H^{\alpha,ab}_W = W^{s\alpha,a} W^{s\beta,b} + W^{s\alpha,b} W^{s\beta,a}$, which is symmetric on its Lorentz (group) indices.

To get the above form, we have utilized the equation of motion of the vector bosons $W^s$, which can be formulated as
\[ \partial_{\mu} W^{\mu\nu,a} - \frac{d_1 g^2}{2} \partial_{\mu} F^{\mu\nu,a} = m^2 W^{s\nu,a} - f^{abc} (1 - \frac{d_1 g^2}{2}) W^{s\mu,b} W^{s\nu,c} + \frac{d_1 g^2}{2} f^{abc} W^{s\mu,b} F^{s\nu,c} \]
\[ - d_2 g^2 W^{s\mu,b} W^{s\nu,c} W^{s\beta} W^{s\mu,b} W^{s\beta} W^{s\mu,b} . \]

From the equation of motion given in Eqn. (65), we can get
\[ \partial_\nu W^{s\nu,a} = \frac{1}{m^2_W} \partial_\nu \left[ f^{abc} \left( 1 - \frac{d_1 g^2}{2} \right) W^b_{\mu} W^{s\mu,c} - f^{abc} \frac{d_1 g^2}{2} W^c_{\mu} S^{s\mu,c} + d_2 g^2 W^{s\mu,a} W^b_{\mu} + d_3 g^2 W^{s\mu,a} W^b_{\mu} W^{s\mu,b} \right] . \]  

(66)

Then we know that the \((\partial_\mu W^{s\mu,a})^2\) can only contribute to terms at most up to \(O(p^6)\). Therefore we simply set \(\partial_\mu W^{s\mu,a} = 0\) when only considering the renormalization up to \(O(p^4)\).

We have also used the following relations about the Lie algebra:

\[ f^{abc} f^{cde} + f^{adc} f^{ceb} + f^{aec} f^{cbd} = 0 , \]

(67)

to simplify the related expressions.

**B. The calculation of logarithm and traces**

The quadratic terms can be directly calculated in the functional integral. Then after integrating out all quantum fields, the \(\mathcal{L}_{1\text{-loop}}\) reads

\[ \int_x \mathcal{L}_{1\text{-loop}} = i \frac{1}{2} \left[ Tr \ln \Box_W + Tr \ln \Box_\xi \right. \]

\[ \left. + Tr \ln \left( 1 - X^{\mu} \Box^{-1}_W X^{\mu} \Box^{-1}_\xi \right) \right] - i Tr \Box_\xi , \]

(68)

where the contribution of the ghost has a different sign due to its anticommutator relation. The \(Tr\) is to sum over the Lorentz indices, \(\mu\nu\), group indices, \(ab\), and the coordinate space points, \(x\). The operators in the term \(Tr \ln \left( 1 - X^{\mu} \Box^{-1}_W X^{\mu} \Box^{-1}_\xi \right)\) are all defined to act on the right side, and such a form reflects that the order of integrating-out the quantum vector boson and Goldstone fields is unphysical.

The expansion of logarithm is simply expressed as

\[ \langle x | \ln(1 - X)|y \rangle = -\langle x | X|y \rangle - \frac{1}{2} \langle x | XX|y \rangle - \frac{1}{3} \langle x | XXX|y \rangle - \frac{1}{4} \langle x | XXXX|y \rangle + ... , \]

(69)

and here the \(X\) should be understood as an operator (a matrix) which acts on the quantum states of the right side.

To evaluate the trace, we will use the Schwinger proper time and heat kernel method \([11]\) in the coordinate space. In this method, the standard propagators can be expressed as

\[ \langle x | \Box^{-1}_W \Box^{ab}|y \rangle = \int_0^\infty \frac{d\tau}{(4\pi\tau)^{\frac{d}{2}}} \exp \left( -m^2_W \tau \right) \exp \left( -\frac{z^2}{4\tau} \right) H^\mu_{\nu,ab}(x, y; \tau) , \]

(70)

\[ \langle x | \Box^{-1}_\xi \Box^{ab}|y \rangle = \int_0^\infty \frac{d\tau}{(4\pi\tau)^{\frac{d}{2}}} \exp \left( -m^2_\xi \tau \right) \exp \left( -\frac{z^2}{4\tau} \right) H^\mu_{\nu,ab}(x, y; \tau) , \]

(71)

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where \( z = y - x \). The integral over the proper time \( \tau \) and the factor \( \exp \left[ -z^2/(4\tau) \right] / (4\pi \tau)^{\frac{d}{2}} \) conspire to separate the quadratic divergent part of the propagator. And the \( H(x, y; \tau) \) is analytic with reference to \( z \) and \( \tau \), which means that \( H(x, y; \tau) \) can be analytically expanded with reference to both \( z \) and \( \tau \). Then we have

\[
H(x, y; \tau) = H_0(x, y) + H_1(x, y)\tau + H_2(x, y)\tau^2 + \cdots, \tag{72}
\]

where \( H_0(x, y) \), \( H_1(x, y) \), and \( H_2(x, y) \) are the Silly-De Witt coefficients. The coefficient \( H_0(x, y) \) is the pure Wilson phase factor, which indicates the phase change of a quantum state when moving from the point \( x \) to the point \( y \) and reads

\[
H_0(x, y) = C \exp \left( - \int_y^x \Gamma(z) \cdot dz \right), \tag{73}
\]

where \( \Gamma(z) \) is the affine connection (dependent on the group representation of the quantum states) defined on the coordinate point \( z \). And the coefficient \( C \) is related with the spin of the states, for vector bosons, \( C = -g_{\mu\nu} \), and for scalar bosons, \( C = 1 \).

The divergence counting rule of the integral over the coordinate space \( x \) and the proper time \( \tau \) can be established as

\[
[z^\mu]_d = 1, \quad [\tau]_d = -2. \tag{74}
\]

Using Eqn. (69), the two propagators defined in Eqn. (70) and Eqn. (71), and the divergence and momentum counting rule given in Eqn. (48) and (74), up to \( O(p^4) \), we can get the following divergent terms

\[
\bar{\epsilon} \text{Tr} \ln \Box_{WW} = \int_x \left[ -m_{\mu\nu}^2 tr[\sigma_{WW}^{\mu\nu}] + \frac{8}{3}(\frac{1}{4}G_{W,\mu\nu}^{\mu\nu,a} + \frac{1}{2} tr[\sigma_{WW}^{\mu\nu}(-g_{\mu\nu})\sigma_{WW}^{\mu\nu}(-g_{\mu\nu})] \right], \tag{75}
\]

\[
\bar{\epsilon} \text{Tr} \ln \Box_{cc} = \int_x \left[ \frac{2}{3}(\frac{1}{4}G_{W,\mu\nu}^{\mu\nu,a}) \right], \tag{76}
\]

\[
\bar{\epsilon} \text{Tr} \ln \Box_{\xi\xi} = \int_x \left[ m_{\mu\nu}^2 tr[\sigma_{\xi\xi}] + m_{\mu\nu}^2 tr[\sigma_{\xi\xi}] + \frac{2}{3}(\frac{1}{4}G_{\xi,\mu\nu}^{\mu\nu,a} + \frac{1}{2} tr[\sigma_{\xi\xi}\sigma_{\xi\xi}] + \frac{1}{2}m_{\mu\nu}^2 tr[X^{\alpha\beta}\sigma_{\xi\xi}] \right], \tag{77}
\]

\[
\text{Tr} \ln \left( 1 - \tilde{X}^{\mu}_{W,\mu\nu} \tilde{X}^{\mu}_{W\mu\nu} \Box_{\xi\xi}^{-1} \right) = \frac{1}{\bar{\epsilon}}(p4t + p3t + p2t), \tag{78}
\]

where \( 1/\bar{\epsilon} = i(2/\epsilon - \gamma_E + \ln 4\pi^2)/(16\pi^2) \), \( \gamma_E \) is the Euler constant, and \( \epsilon = 4 - d \). The \( \Gamma_{\mu\nu} \) is the field strength tensor corresponding to the affine connection \( \Gamma_{\mu} \). We have used the dimensional regularization scheme and the modified minimal substraction scheme to
extract the divergent structures in this step. The $p_{4t}$ represents the contributions of four propagators $tr(\tilde{X}^\uparrow \tilde{X}^\downarrow \tilde{X}^\uparrow \tilde{X}^\downarrow)$, which reads

$$p_{4t} = g_{\mu\nu} g_{\rho\sigma} \left[ \frac{g^{\alpha\beta\gamma\delta}}{4} tr[2X_{\alpha\beta}X_{\alpha'\gamma}X_{\mu_1}X_{\nu_1}] + g^{\alpha\beta} g^{\rho\sigma} g^{\gamma\delta} tr[X_{\alpha\beta}X_{\alpha'\gamma}X_{\mu_1}X_{\nu_1}] \right].$$ (79)

The $p_{3t}$ represents the contributions of three propagators $tr(\tilde{X}^\uparrow \tilde{X}^\downarrow \tilde{X}^\uparrow \tilde{X}^\downarrow)$, which reads

$$p_{3t} = \frac{m_W^2}{4} g^{\alpha\beta\rho\sigma} g_{\mu\nu} tr[X_{\alpha\beta}X_{\alpha'\rho}X_{\mu_1}X_{\nu_1}] + \frac{1}{2} g^{\alpha\beta} g_{\mu\nu} tr[X_{\alpha\beta}X_{\mu_1}X_{\nu_1}].$$ (80)

The $p_{2t}$ represents the contributions of two propagators $tr(\tilde{X}^\uparrow \tilde{X}^\downarrow)$, which can be further divided into six groups:

$$p_{2t} = t_{AA} + t_{AB} + t_{AC} + t_{BB} + t_{BC} + t_{CC},$$ (81)

$$t_{AA} = \frac{m_W^2}{4} g^{\alpha\beta\rho\sigma} g_{\mu\nu} \left[ \frac{g^{\alpha\beta\gamma\delta}}{2} - \frac{g^{\alpha\beta\rho\sigma}}{2} + g^{\alpha\beta} g^{\rho\sigma} g^{\gamma\delta} tr[X_{\alpha\beta}D_{\gamma}X_{\alpha'\delta}] \right] + \frac{m_W^2}{8} g^{\alpha\beta\rho\sigma} g_{\mu\nu} tr[X_{\alpha\beta}X_{\alpha'\rho}X_{\mu_1}X_{\nu_1}],$$ (82)

$$t_{AB} = \frac{m_W^2}{2} g^{\alpha\beta\rho\sigma} g_{\mu\nu} tr[X_{\alpha\beta}X_{\alpha'\rho}X_{\mu_1}X_{\nu_1}] + g_{\mu\nu} g_{\rho\sigma} tr[X_{\alpha\beta}D_{\gamma}X_{\alpha'\delta}],$$ (83)

$$t_{AC} = \frac{m_W^2}{4} g^{\alpha\beta\rho\sigma} g_{\mu\nu} tr[X_{\alpha\beta}X_{\alpha'\rho}X_{\mu_1}X_{\nu_1}] - \frac{1}{4} g^{\alpha\beta} tr[g_{\rho\sigma} g_{\mu\nu} X_{\alpha\beta} X_{\mu_1} X_{\nu_1}] + g_{\mu\nu} g_{\rho\sigma} tr[X_{\alpha\beta}D_{\gamma}X_{\alpha'\delta}],$$ (84)

$$t_{BB} = \frac{m_W^2}{2} g^{\alpha\beta\rho\sigma} g_{\mu\nu} tr[X_{\alpha\beta}X_{\alpha'\rho}X_{\mu_1}X_{\nu_1}],$$ (85)

$$t_{BC} = \frac{m_W^2}{2} g^{\alpha\beta\rho\sigma} g_{\mu\nu} tr[X_{\alpha\beta}X_{\alpha'\rho}X_{\mu_1}X_{\nu_1}],$$ (86)

where the trace is to sum over the group indices and points of coordinate space, and the covariant differentials is defined as
\[
\bar{X} \partial \bar{X} = \bar{X} \partial \bar{X} + \bar{X} \Gamma_W \bar{X} - \bar{X} \bar{X} \Gamma_\xi, \tag{87}
\]
\[
\bar{X} \partial \bar{X} = \bar{X} \partial \bar{X} + \bar{X} \Gamma_W \bar{X} + \bar{X} \bar{X} \Gamma_\xi \Gamma_\xi - 2 \bar{X} \Gamma_W \bar{X} \Gamma_\xi \\
+ 2 \bar{X} \Gamma_W \bar{X} \partial \Gamma_\xi + \bar{X} \partial \Gamma_W \bar{X} - \bar{X} \bar{X} \partial \Gamma_\xi. \tag{88}
\]

And the tensors \( g^{\alpha \beta \gamma \delta} \) and \( g^{\alpha \beta \gamma \delta \mu \nu} \) are symmetric on all indices and defined as
\[
g^{\alpha \beta \gamma \delta} = g^{\alpha \beta} g^{\gamma \delta} + g^{\alpha \gamma} g^{\beta \delta} + g^{\alpha \delta} g^{\beta \gamma}, \tag{89}
\]
\[
g^{\alpha \beta \gamma \delta \mu \nu} = g^{\alpha \beta} g^{\gamma \delta \mu \nu} + g^{\alpha \gamma} g^{\beta \delta \mu \nu} + g^{\alpha \delta} g^{\beta \gamma \mu \nu} + g^{\alpha \mu} g^{\beta \gamma \delta \nu} + g^{\alpha \nu} g^{\beta \gamma \delta \mu}. \tag{90}
\]

To get the \( p_4t, p_3t, \) and \( p_2t \), we have used the covariant short-distance expansion technology [12] and the integral over the proper time and coordinate space. We would like to comment on the the covariant short-distance expansion technology: to formulate the quadratic form into the standard form can simplify the labor to extract the divergences, while the form given in [12] is not easy to use. The equivalence of these two forms can be easily proved by using the partial integral. As we have pointed out, the standard form given by us has the advantage to reflect the fact that the order of integrating out the quantum vector boson and Goldstone fields has no any dynamic significance, and is unphysical.

**C. The renormalization group equations**

Substituting Eqs. (33—64) to Eqs. (75—86), with somewhat tedious algebraic manipulation, we construct the counter terms and extract the renormalization constants. The renormalization constants yield the following RGEs

\[
\frac{dg^2}{dt} = \frac{g^4}{8\pi^2} \left\{ -\frac{29}{4} + 6d_1 g^2 - \frac{17d_1^2 g^4}{8} \right\}, \tag{91}
\]
\[
\frac{dv}{dt} = \frac{1}{16\pi^2} v \left\{ -\frac{3g^2}{2} - (5d_1 + 8d_2 + 14d_3) g^4 - \frac{7d_1^2 g^6}{2} \right\}, \tag{92}
\]
\[
\frac{dd_1}{dt} = \frac{1}{8\pi^2} \left\{ -\frac{1}{12} + \left( \frac{11d_1}{2} - \frac{11d_2}{2} + 11d_3 \right) g^2 \right. \\
\left. + \left[ -\frac{69d_1^2}{8} + d_1 (3d_2 - 6d_3) \right] g^4 + \frac{23d_1^3 g^6}{48} \right\}, \tag{93}
\]
\[
\frac{dd_2}{dt} = \frac{1}{8\pi^2} \left\{ -\frac{1}{12} + \left( -\frac{119d_1}{48} + 6d_2 \right) g^2 + \left( \frac{173d_1^2}{24} + 20d_1 d_2 \right) g^4 \right. \\
\left. + \left[ \frac{207d_1^3}{16} + d_1^2 \left( \frac{91d_2}{8} - \frac{5d_3}{4} \right) \right] g^6 + \frac{545d_1^4 g^8}{96} \right\}, \tag{94}
\]
\[
\frac{dd_3}{dt} = \frac{1}{8\pi^2} \left\{ -\frac{1}{24} + \left( \frac{89d_1}{48} - \frac{19d_2}{2} - 11d_3 \right) g^2 \\
+ \left[ \frac{251d_1^2}{48} - d_1 \left( 28d_2 + \frac{61d_3}{2} \right) \right] g^4 \\
+ \left[ \frac{239d_1^3}{16} - d_1^2 \left( \frac{53d_2}{4} + \frac{43d_3}{2} \right) \right] g^6 + \frac{463d_1^4 g^8}{96} \right\}. 
\tag{95}
\]

About the RGEs given in Eqs. (91—95), it is remarkable that the direct method will only get part of the result of the RGE method, which is contributed by the Goldstone boson and indicated by the constant terms independent of \(d_i, i = 1, 2, 3\) in the rhs of RGEs of \(d_i, i = 1, 2, 3\). While the rest terms of the RGEs take into account not only the effect of Goldstone boson \(\xi\), but also that of vector bosons \(\hat{W}\) and that of their mixing terms.

In order to compare and contrast, we formulate the results of the direct method in the RGE form, which read

\[
\frac{dg^2}{dt} = \frac{g^4}{8\pi^2} \left[ -\frac{29}{4} \right],
\tag{96}
\]

\[
\frac{dv}{dt} = \frac{v}{16\pi^2} \left[ -\frac{3g^2}{2} \right],
\tag{97}
\]

\[
\frac{dd_1}{dt} = \frac{1}{8\pi^2} \left[ -\frac{1}{12} \right],
\tag{98}
\]

\[
\frac{dd_2}{dt} = \frac{1}{8\pi^2} \left[ -\frac{1}{12} \right],
\tag{99}
\]

\[
\frac{dd_3}{dt} = \frac{1}{8\pi^2} \left[ -\frac{1}{24} \right].
\tag{100}
\]

The underlying reason to represent the contributions of Higgs in the RGE form might be related with the fact that the full theory is a renormalizable one and the divergences generated by the Higgs loop should be cancelled out exactly by those generated by the Goldstone bosons. To extract the divergent structures, we have used the following relations of the \(SU(2)\) gauge group

\[
tr[W^a_{\mu\nu} W^a_{\mu\nu} W^s_{\mu,G} W^s_{\nu,G}] = 2W^a_{\mu} \cdot W^a_{\nu} W^s_{\mu} \cdot W^s_{\nu},
\tag{101}
\]

\[
tr[W^a_{\mu\nu} W^a_{\mu,G} W^s_{\nu,G} W^a_{\nu,G}] = W^a_{\mu} \cdot W^s_{\mu} W^s_{\mu} \cdot W^a_{\nu} + (W^a_{\mu} \cdot W^s_{\mu})^2,
\tag{102}
\]

\[
H^a_{\mu\nu} H^{\mu\nu,a} = W^a_{\mu\nu} W^a_{\mu\nu} - 2f^{abc} W^a_{\mu\nu} W^b_{\mu\nu} c^c W^c_{\mu\nu} + (W^a_{\mu} \cdot W^s_{\mu})^2 - W^s_{\mu} \cdot W^s_{\mu} W^a_{\mu} \cdot W^a_{\mu},
\tag{103}
\]

where the variable \(H^a_{\mu\nu}\) is symmetric when exchanging its Lorentz indices \(\mu\) and \(\nu\), and is defined as \(H^a_{\mu\nu} = \partial_\mu W^a_{\nu} + \partial_\nu W^a_{\mu}\).
The term $11d_3g^2$ in the right hand side of the RGE of $d_1$ is quite remarkable: the coefficient 11 mainly comes from the fact that the gauge bosons have 3 physical components as a vector field, and have 3 components as an adjoint representation of the $SU(2)$. When the Higgs is not too heavy, the coupling $d_3$ can reach order 0.1 or 0.01, then this term can switch the sign of $d_1(m_W)$ from positive to negative. Another remarkable fact is that it is the radiative corrections from the vector bosons which mostly contribute to the linear terms of $d_i$ in the rhs of RGEs and dominate the running of RGEs when the nonlinear effects of the RGEs is still small. In the following section we will comparatively study the solutions of these two groups of RGEs.

V. NUMERICAL ANALYSIS

We concentrate on the Higgs scalar’s effects to the effective couplings $d_i, i = 1, 2, 3$. To simplify the analysis, we mimic the standard model by choosing the mass of vector boson $m_W$ to be 91 GeV. The Higgs scalar is assumed to be heavier than the vector bosons $W$. The initial condition for the coupling $g$ and the vacuum expectation value $v$ is fixed at the lower boundary point, $\mu = m_W$. The coupling $g(m_W)$ is chosen to satisfy

$$\alpha_g = \frac{g^2}{4\pi} = \frac{1}{30},$$

which gives $g(m_W) = 0.65$. and the vacuum expectation value is then fixed by $m_W = gv$, which gives $v(m_W) = 140.6$. While the initial condition for $d_i, i = 1, 2, 3$ is chosen to be fixed at the matching scale, $\mu = m_0$, as given in Eqn (18).

Below we will compare the results gotten from the direct method and the RGE method. As we know, the scalar’s effect includes both the decoupling mass square suppressed part as shown in Eqn. (18) and the nondecoupling logarithm part. So we consider the following three cases to trace the change of roles of these two competing parts: 1) the light scalar case, with $m_0 = 160$ GeV, where the decoupling mass square suppressed part dominates; 2) the not too heavy scalar case, with $m_0 = 500$ GeV, where both contributions are important; 3) the very massive scalar case, with $m_0 = 1.2$ TeV, where the nondecoupling logarithm part dominates.

The figure 1. is devoted to the first case. It is obvious that the difference of $d_i(m_W), i = 1, 2$ in two methods is quite large, and the $d_1(m_W)$’s have different signs in these two method. Figure 2. is devoted to the second case. For the $d_1$, these two methods predict different sign with the same magnitudes. Figure 3. is devoted to the third case. And the difference between the results of these two methods is relatively small.
In the all three cases, the magnitude of the $d_2(m_W)$ is about $10^{-3}$ in both methods, and the difference between these two methods is negligible.

While the $d_1(m_W)$ can reach $10^{-2}$ in the RGE method, one order larger than in the direct method, when the Higgs scalar is quite small. Even when the Higgs is medium heavy, the results of these two method have the same magnitude and different signs. Near the decoupling limit, the prediction of RGE method improves that of the direct method up to 40%—70%.

Due to its initial values at the matching scale, the $d_3(m_W)$ could have different magnitudes in these three cases, $10^{-1}$, $10^{-2}$, and $10^{-3}$, respectively. The differences of these two methods are small when the Higgs is relative light, and in the third case the relative difference can reach 5%—15%.

The difference of the running of $g$ is neglectable in these two methods so we have not depicted it.

From these figures, we can read out the tendency that the difference of $\delta d_1$ between these two methods is larger when the Higgs scalar is further below its decoupling limit. The underlying reason for this behavior is related with the initial value $d_3$ at the matching scale.
scale, and the related terms dependent on $d_3$ in the RGEs given in Eq. (91—95).

![Graphs showing the varying of $v$, $d_1$, $d_2$, and $d_3$ with the running scale $t$ ($t = \ln(m_0/m_W)$). The matching scale is the mass of Higgs scalar, which is taken to be $m_0 = 500$ GeV. The solid lines are the results of the RGE method, while the dashed ones are those of the direct method.](image)

**VI. DISCUSSIONS AND CONCLUSIONS**

In this paper, we have studied the renormalization of the nonlinear effective $SU(2)$ Lagrangian with spontaneous symmetry breaking and derived its RGEs. Compared with the $U(1)$ case, the non-Abelian case is much more complicated. And quite differently, in the $SU(2)$ case, the gauge coupling and the anomalous couplings up to $O(p^4)$ are driven to develop by the quantum fluctuations low energy DOFs. We also have comparatively studied the results of the direct method and the RGE method. From the numerical analysis, we see that the results of two methods are very different when the Higgs scalar is far below its decoupling limit. The underlying reason is related with the initial value of $d_3$ at the matching scale and with the radiative correction of all low energy degrees of freedom (both the Goldstone and vector bosons) which contributes to the $d_3$ terms in Eq. (91—95).

Normally, when the Higgs is so light, the higher dimension operators, for instance those
belong to the $O(p^6)$ order, might play some considerable parts and it might be not good to use the effective theory to describe the full theory, since the Wilsonian renormalization [6] and the surface theorem given by [7] require that the low energy scale $\mu_{IR}$ is lower enough than the UV cutoff. While we see, for the medium heavy Higgs (say, $m_0 = 400$ GeV or $m_0 = 800$ GeV), it is still appropriate to use it, and the difference between these two method is quite considerable for some anomalous coupling(s).

In the RGE method, it becomes quite manifest that the effects of heavy DOF to the low energy dynamics are related with two factors, 1) the mass of the heavy particle, which determines the matching scale $\mu_{UV}$, and 2) the initial values of anomalous couplings at the matching scale determined by integrating out the heavy particle, which are related with the spin of the heavy particle and the strength of its couplings to the low energy DOFs. If a heavy field doesn’t participate in the process of symmetry breaking, it will not contribute to the anomalous couplings up to the $O(p^4)$ order and its effects can be estimated by the decoupling theorem [8].

As we know, there are several ways for the $SU(2)$ breaking to its subgroups, $SU(2)$
breaks to $U(1)$ \cite{14}, for instance. In this paper, we only assume that the symmetry is broken from a local one to a global one, where all components of vector boson have the same mass. For the way of $SU(2)$ breaking to $U(1)$, the effective Lagrangian will be more complicated. Several of the patterns of symmetry breaking will be discussed in our next paper \cite{16} when we consider the renormalization of electroweak chiral Lagrangian.

Meanwhile, for the sake of simplicity, no fermion field is taken into account, which might introduce terms of anomaly. Also, we have not included all of terms breaking the charge, parity, and both symmetries. If included, the above procedure will be more complicated due to the properties of the complete antisymmetric tensor $\epsilon^{\mu\nu\delta\gamma}$. However, in principle, we can still make the renormalization procedure order by order even for the complicity.

The renormalization procedure in this paper can easily be extended to study the renormalization of the nonlinear sigma model with $SU(N_f)$ symmetry \cite{15}, which has a very imporant role to describe the low energy hadronic physics. We will apply the related conceptions and methods to the renormalization of the electroweak chiral effective Lagrangian and the QCD chiral Lagrangian \cite{16}.

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