We investigate the effect of the noncommutative geometry on the classical orbits of particles in a central force potential. The relation is implemented through the modified commutation relations \([x_i, x_j] = i\theta_{ij}\). Comparison with observation places severe constraints on the value of the noncommutativity parameter.

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Recently, remotivated by string theory arguments, noncommutative spaces have been studied extensively (for a review see [1,2]). One might postulate noncommutativity for a number of reasons, perhaps the simplest is the long-held belief that in quantum theories including gravity, space-time must change its nature at short distances. Quantum gravity has an uncertainty principle which prevents one from measuring positions to better accuracies than the Planck length: the momentum and energy required to make such a measurement will itself modify the geometry at these scales [3]. While motivation for this kind of space with noncommuting coordinates is mainly theoretical, it is possible to look experimentally for departures from the usually assumed commutativity among the space coordinates, e.g. see [4,5]. In this paper we consider the effects of the deformation of the canonical commutation relations on the orbits of classical particles in a central force potential. Usual quantum mechanics is formulated on commutative spaces satisfying the following commutation relations,

\[
[\hat{x}_i, \hat{x}_j] = 0, \ [\hat{p}_i, \hat{p}_j] = 0, \ [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}
\]  

Then in order to describe a noncommutative space, the above commutation relations should be changed as,

\[
[\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \ [\hat{p}_i, \hat{p}_j] = 0, \ [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}
\]

where \(\theta_{ij}\)'s are c-numbers with the dimensionality ( length )\(^2\). In the classical limit, the quantum mechanical commutator is replaced by the Poisson bracket via

\[
\frac{1}{i\hbar}[\hat{A}, \hat{B}] \longrightarrow \{\hat{A}, \hat{B}\}
\]
So the classical limit of Eqs. (3) reads

\[ \{\tilde{x}_i, \tilde{x}_j\} = \alpha_{ij}, \quad \{\tilde{p}_i, \tilde{p}_j\} = 0, \quad \{\tilde{x}_i, \tilde{p}_j\} = \delta_{ij} \]  

(4)

We are keeping the parameters \( \alpha_{ij} = \frac{\theta_{ij}}{\hbar} \) fixed as \( \hbar \to 0 \), for similar arguments see [6]. The Poisson bracket must possess the same properties as the quantum mechanical commutator, namely, it must be bilinear, anti-symmetric and must satisfy the Leibniz rules and the Jacobi Identity. The general form of the poisson brackets for this deform version of classical mechanics can be written as [6,7],

\[ \{A, B\} = \left(\frac{\partial A}{\partial \tilde{x}_i} \frac{\partial B}{\partial \tilde{p}_j} - \frac{\partial A}{\partial \tilde{p}_i} \frac{\partial B}{\partial \tilde{x}_j}\right)\{\tilde{x}_i, \tilde{p}_j\} + \frac{\partial A}{\partial \tilde{x}_i} \frac{\partial B}{\partial \tilde{x}_j}\{\tilde{x}_i, \tilde{x}_j\} \]

(5)

where repeated indices are summed. The Hamiltonian of a particle in a central force potential is given by,

\[ H = \frac{\tilde{P}^2}{2m} + V(\tilde{r}) \quad r = \sqrt{\tilde{x}_i \tilde{x}_i} \]

(6)

To make this situation tractable, one may choose a new coordinate system,

\[ x_i = \tilde{x}_i + \frac{1}{2} \alpha_{ij} \tilde{p}_j, \quad p_i = \tilde{p}_i \]

(7)

where the new variables satisfy the usual canonical brackets.

\[ \{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} \]

(8)

By replacing the new variables in the potential, one has
\[ V(\vec{r}) = V(\sqrt{(x_i - \alpha_{ij}p_j/2)(x_i - \alpha_{ik}p_k/2)}) \]
\[ = V(r) + \frac{\left(\vec{\alpha} \times \vec{p}\right)}{2} \cdot \vec{\nabla}V(r) + O(\alpha^2) \]
\[ = V(r) - \frac{\left(\vec{\alpha} \cdot \vec{L}\right) \partial V}{2r} + O(\alpha^2) \quad (9) \]

where \(\alpha_{ij} = \epsilon_{ijk}\alpha_k\), \(\vec{L} = \vec{r} \times \vec{p}\). For the Coulomb potential the Hamiltonian up to the first order in \(\alpha\) becomes,

\[ H = \frac{\vec{p}^2}{2m} - \frac{k}{r} - \frac{k}{r^3}(\vec{\alpha} \cdot \vec{L}) \quad (10) \]

The new term in the Hamiltonian is small and its effects can be obtained by standard perturbation theory, however it causes a time dependent angular momentum. We assume that the time variation of the vector \(\vec{L}\) is so small that in a short time interval (for example one century) it could be taken as a constant of motion. Our aim is to put a bound on the value of \(\alpha\) by comparing the results of the perturbing term with the experimental value of the precession of the perihelion of Mercury. For the Kepler problem it is known that the bounded orbits are closed, which is a result of the following integral,

\[ \Delta \phi^{(0)} = -2\frac{\partial}{\partial L} \int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{2m(E - V(r)) - \frac{L^2}{r^2}} \, dr \quad (11) \]
\[ = 2\pi \quad (12) \]

where

\[ E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) \quad , \quad L = mr^2\dot{\phi} \quad (13) \]
By perturbing the potential with a small term $V \rightarrow V + \delta V$ and expanding the integral up to the first order in $\delta V$, one has

$$\Delta \varphi = \Delta \varphi^{(0)} + \Delta \varphi^{(1)}$$

where

$$\Delta \varphi^{(1)} = \frac{\partial}{\partial L} \int_0^\pi \left( \frac{2mr^2}{L} \right) \delta V \, d\varphi$$

After a straightforward calculation, we arrive at

$$\Delta \varphi^{(1)} = \left( \frac{2\pi k^2 m^2 \cos(\gamma)}{L^3} \right) \alpha$$

where $\gamma$ is an angle between $\bar{\alpha}$ and $\bar{L}$. According to Ref. [8], the observed advance of the perihelion of Mercury that is unexplained by Newtonian planetary perturbations or solar oblateness is

$$\Delta \varphi_{\text{obs}} = 42.980 \pm 0.002 \text{ arc seconds per century}$$

$$= 2\pi(7.98734 \pm 0.00037) \times 10^{-8} \text{ radians/revolution}$$

This advance is usually explained by General Relativity which predicts,

$$\Delta \varphi_{\text{GR}} = \frac{6\pi GM}{c^2 a(1 - e^2)}$$

For Mercury, the parameters are [9,10],
\[
\frac{2GM}{c^2} = 2.95325008 \times 10^3 \text{ m}
\]

\[
m = 3.3022 \times 10^{23} \text{ kg}
\]

\[
e = 0.20563069
\]

\[
a = \frac{r_{\text{max}} + r_{\text{min}}}{2}
\]

\[
= 5.7909175 \times 10^{10} \text{ m}
\]  \(20\)

and

\[
\triangle \varphi_{GR} = 2\pi(7.98744 \times 10^{-8}) \text{ radians/revolution}
\]  \(20\)

Comparison of Eqs. (17) and (20) yields

\[
\triangle \varphi_{GR} - \triangle \varphi_{\text{obs}} = 2\pi(0.00010 \pm 0.00037) \times 10^{-8} \text{ radians/revolution}
\]  \(21\)

So we can assume that in this scenario,

\[
\left| \triangle \varphi(1) \right| \leq \left| \triangle \varphi_{GR} - \triangle \varphi_{\text{obs}} \right|
\]  \(22\)

Thus,

\[
\alpha \cos(\gamma) \leq 1.2 \times 10^{-29} \left( \frac{m^2}{J \cdot \text{s.}} \right)
\]  \(23\)

Considering a scenario in which the angle \(\gamma\) takes values from 0 to a few seconds less than \(\frac{\pi}{2}\). So we can place a constraint on the value of the noncommutative parameter \(\theta\)
\[ \hbar \alpha = \theta \leq 10^{-62} \text{ m}^2 \]  

and

\[ \sqrt{\theta} \approx 10^3 \ell_{\text{planck}} \]  

This limit is much smaller than the limits which have already been obtained [4,5]. For \( \gamma \) equal to \( \frac{\pi}{2} \) the perturbing term vanishes and one has to consider higher orders in \( \alpha \). By a similar calculation with \( \gamma = \frac{\pi}{2} \) and up to the second order in \( \alpha \) we obtain an even smaller number for \( \alpha \) in this special scenario.

As we have already mentioned, the perturbing term in Eq. (10) causes a time dependent angular momentum.

\[
\frac{d\vec{L}}{dt} = \{ \vec{L}, H \} \\
= -\frac{k}{r^3} \left( \vec{\alpha} \times \frac{\vec{L}}{2} \right) 
\]  

Eq. (26) has a simple physical interpretation, the angular momentum vector \( \vec{L} \) rotates around \( \vec{\alpha} \) with a frequency which is about,

\[
w_{\text{rot}} \simeq \left( \frac{GMm}{2r^3\hbar} \right) \theta \]  

It means that in this scenario, the two dimensional plane which contains Mercury and the sun and is perpendicular to the angular momentum vector has a rotation about \( \vec{\alpha} \) with a
period of about ten billion \(10^{10}\) years which is indeed a long time and comparable to the age of the solar system.

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**References**

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